SOME SPECIFIC PROBABILITY DISTRIBUTIONS

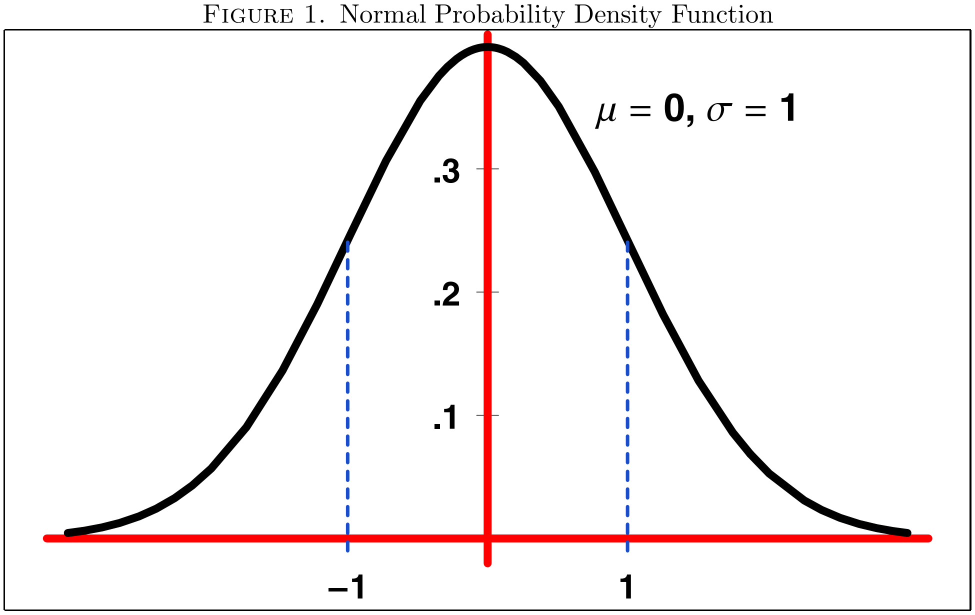
1. Normal random variables

1.1. **Probability Density Function.** The random variable X is said to be normally distributed with mean μ and variance σ^2 (abbreviated by x $\sim$ N[μ, σ^2] if the density function of x is given by

$$f\,(x\,;\;\mu,\sigma^2)\;=\;\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{1}$$

The normal probability density function is bell-shaped and symmetric. The figure below shows the probability distribution function for the normal distribution with a $\mu = 0$ and σ =1. The areas between the two lines is 0.68269. This represents the probability that an observation lies within one standard deviation of the mean.

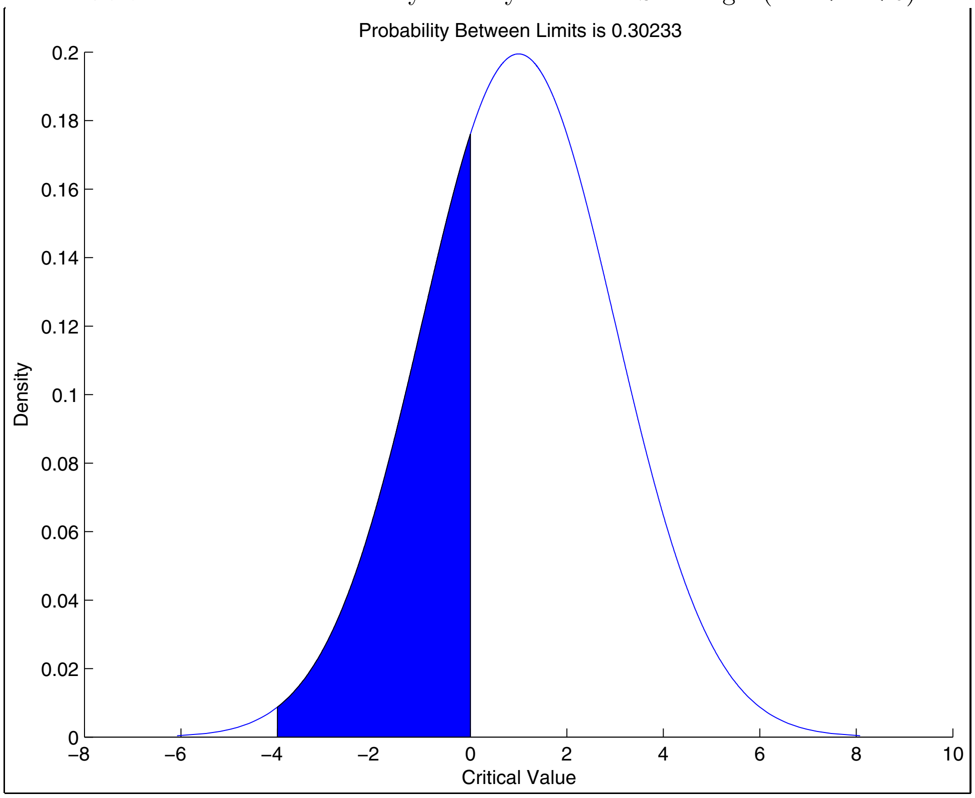
Figure 1. Normal Probability Density Function



Date: August 9, 2004.

The next figure below shows the portion of the distribution between -4 and 0 when the mean is one and σ is equal to two.

FIGURE 2. Normal Probability Density Function Showing $\mathrm{P}(-4 < x < 0)$



1.2. Properties of the normal random variable.

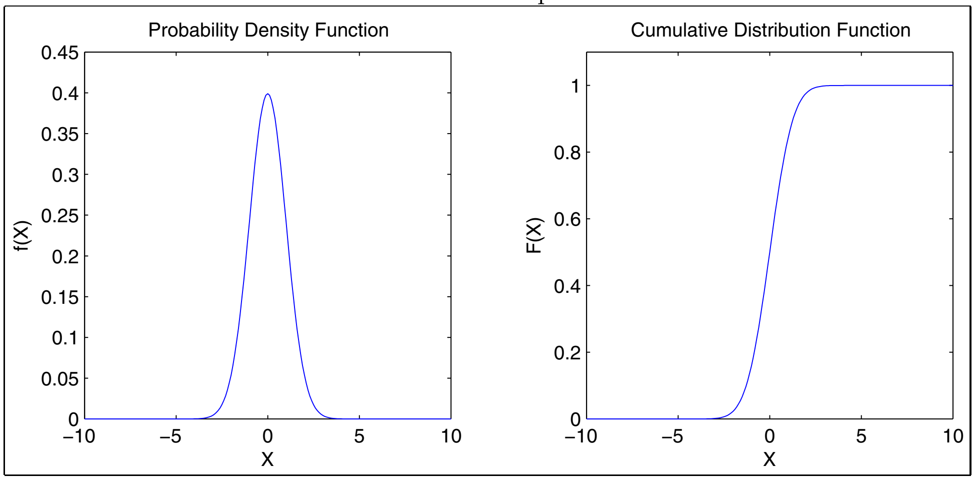
a: $\mathrm{E(x)} = \mu$, $\mathrm{Var(x)} = \sigma^2$.
b: The density is continuous and symmetric about μ.
c: The population mean, median, and mode coincide.
d: The range is unbounded.
e: There are points of inflection at $\mu \pm \sigma$.
f: It is completely specified by the two parameters μ and σ^2.
g: The sum of two independently distributed normal random variables is normally distributed. If $\mathrm{Y} = \alpha \mathrm{X}_1 + \beta \mathrm{X}_2 + \gamma$ where $\mathrm{X}_1 \sim \mathrm{N}(\mu_1, {\sigma_1}^2)$ and $\mathrm{X}_2 \sim \mathrm{N}(\mu_2, {\sigma^2}_2)$ and X_1 and X_2 are independent, then $\mathrm{Y} \sim \mathrm{N}(\alpha\mu_1 + \beta\mu_2 + \gamma;\ \alpha^2 {\sigma^2}_1 + \beta^2 {\sigma^2}_2)$.

1.3. Distribution function of a normal random variable.

$$F(x\,;\ \mu,\, \sigma^2) \ = \ Pr\,(X \ \le \ x) \ = \int_{-\infty}^{x} f\,(s\,;\,\mu,\,\sigma^2\,)ds \tag{2}$$

Here is the probability density function and the cumulative distribution of the normal distribution with $\mu = 0$ and $\sigma = 1$.

FIGURE 3. Normal pdf and cdf



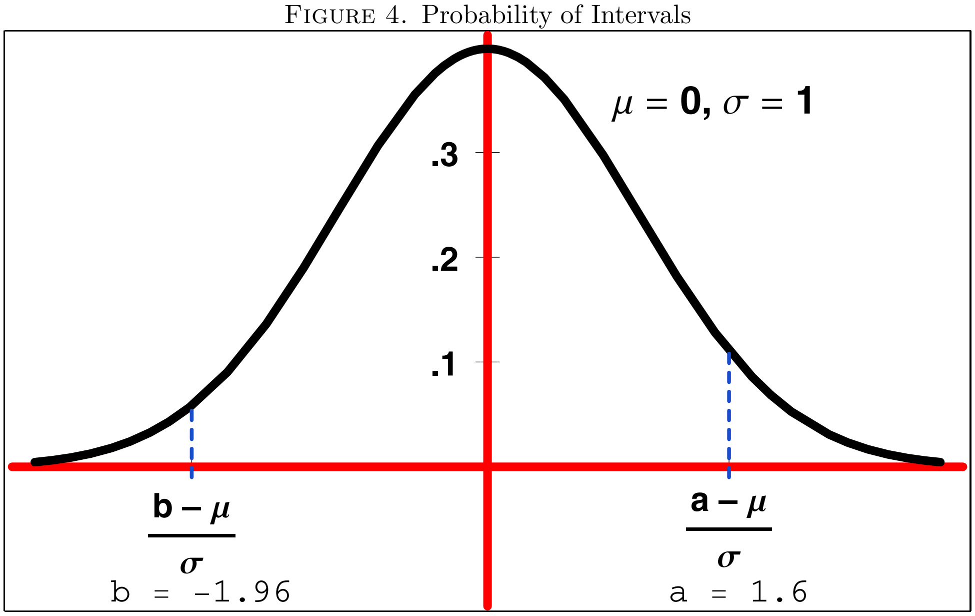
1.4. **Evaluating probability statements with a normal random variable.** If x $\sim$ N(μ,σ^2) then,

$$\begin{aligned} Z &= \tfrac{X-\mu}{\sigma} \ \sim \ N(0,1) \\ E\,(Z) &= E\,\left(\tfrac{X-\mu}{\sigma}\right) \ = \ \tfrac{1}{\sigma}\cdot(E(X)-\mu) \ = \ 0 \\ Var\,(Z) &= Var\,\left(\tfrac{X-\mu}{\sigma}\right) \ = \ \tfrac{1}{\sigma^2}Var(X-\sigma) \\ &= \tfrac{\sigma^2}{\sigma^2} \ = \ 1 \end{aligned} \tag{3}$$

Consequently,

$$\begin{aligned} Pr(a \;\le\; x \;\le\; b) \;&= Pr\,(a \,-\, \mu \;\le\; x \,-\, \mu \;\le\; b \,-\, \mu) \\ &= Pr\left[\tfrac{a\,-\,\mu}{\sigma} \;\le\; \tfrac{x\,-\,\mu}{\sigma} \;\le\; \tfrac{b\,-\,\mu}{\sigma}\right] \\ &= F\;\left(\tfrac{b\,-\,\mu}{\sigma};\; 0,\, 1\right) - F\;\left(\tfrac{a\,-\,\mu}{\sigma};\; 0,\, 1\right) \\ &= \textit{area below} \end{aligned} \tag{4}$$

FIGURE 4. Probability of Intervals



We can then merely look in tables for the distribution function of a N(0,1) variable.

1.5. **Moment generating function of a normal random variable.** The moment generating function for the central moments is as follows

$$M_X(t) \;=\; e^{\frac{t^2\,\sigma^2}{2}}\,. \tag{5}$$

The first central moment is

$$\begin{aligned} E\,(X \;-\; \mu\,) \;&= \tfrac{d}{dt}\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,|_{t\,=\,0} \\ &= t\;\sigma^2\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,|_{t\,=\,0} \\ &= 0 \end{aligned} \tag{6}$$

The second central moment is

$$\begin{aligned} E(X - \mu)^2 &= \frac{d^2}{dt^2} \left(e^{\frac{t^2 \sigma^2}{2}} \right) |_{t=0} \\ &= \frac{d}{dt} \left(t\, \sigma^2 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= \left(t^2 \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + \sigma^2 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= \sigma^2 \end{aligned} \tag{7}$$

The third central moment is

$$\begin{aligned} E(X - \mu)^3 &= \frac{d^3}{dt^3} \left(e^{\frac{t^2 \sigma^2}{2}} \right) |_{t=0} \\ &= \frac{d}{dt} \left(t^2 \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + \sigma^2 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= \left(t^3 \sigma^6 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 2\, t\, \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + t\, \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= \left(t^3 \sigma^6 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 3\, t\, \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= 0 \end{aligned} \tag{8}$$

The fourth central moment is

$$\begin{aligned} E(X - \mu)^4 &= \frac{d^4}{dt^4} \left(e^{\frac{t^2 \sigma^2}{2}} \right) |_{t=0} \\ &= \frac{d}{dt} \left(t^3 \sigma^6 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 3\, t\, \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= \left(t^4 \sigma^8 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 3\, t^2 \sigma^6 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 3\, t^2 \sigma^6 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 3\, \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= \left(t^4 \sigma^8 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 6\, t^2 \sigma^6 \left(e^{\frac{t^2 \sigma^2}{2}} \right) + 3\, \sigma^4 \left(e^{\frac{t^2 \sigma^2}{2}} \right) \right) |_{t=0} \\ &= 3\, \sigma^4 \end{aligned} \tag{9}$$

2. Chi-square random variable

2.1. **Probability Density Function.** The random variable X is said to be a chi-square random variable with ν degrees of freedom [abbreviated $\chi^2(\nu)$] if the density function of X is given by

$$\begin{aligned} f\ (x\,;\ \nu) &= \frac{1}{2^{\frac{\nu}{2}}\ \Gamma\left(\frac{\nu}{2}\right)}\ x^{\frac{\nu-2}{2}} e^{\frac{-x}{2}}\ \ 0 < x \\ &= 0\ otherwise \end{aligned} \tag{10}$$

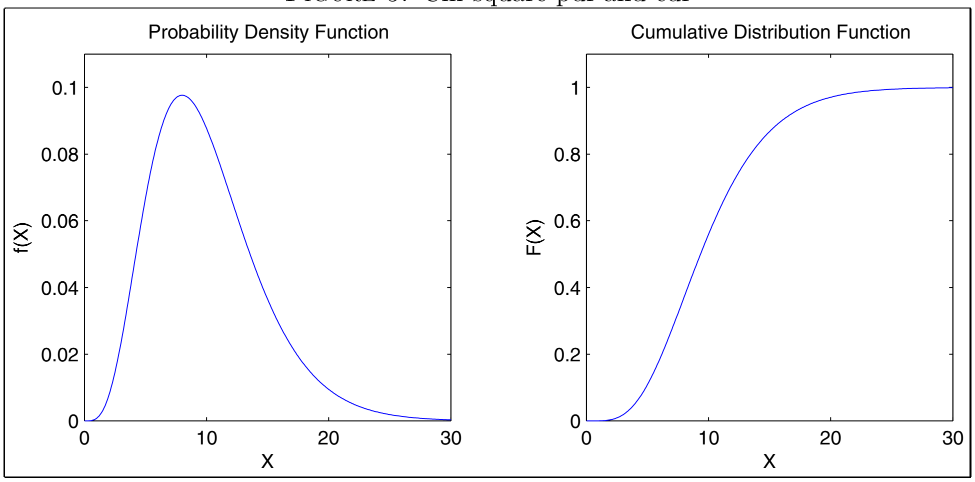
where $\Gamma\ (\,\cdot\,)$ is the gamma function defined by

$$\Gamma\ (r\,) = \int_0^\infty u^{\,r\,-\,1}\, e^{\,-u}\ du\ \ r > 0 \tag{11}$$

Note that for positive integer values of r, Γ(r) = (r - 1)!

The following diagram shows the pdf and cdf for the chi-square distribution with parameters ν =10.

FIGURE 5. Chi-square pdf and cdf



2.2. **Properties of the chi-square random variable.**

2.2.1. χ^2 *and N(0,1).* Consider n independent random variables.

$$\begin{array}{c} If\ X_i \sim N(0,\,1)\ i\ =\ 1,\,2,\,\ldots,\,n \\ then\ \sum_{i=1}^{n}\ X_i^2\ \sim\ \chi^2(n) \end{array} \tag{12}$$

It can also be shown that

$$\begin{array}{c} If\ X_i \sim N(0,\,1)\ i\ =\ 1,\,2,\,\ldots,\,n \\ then\ \sum_{i=1}^{n}\ (X_i - \bar{X})^2\ \sim\ \chi^2(n-1) \end{array} \tag{13}$$

because this is the sum of (n-1) independent random variables given that $\bar{X}$ and (n-1) of the x's are independent.

2.2.2. χ^2 *and N(μ,σ^2).*

$$\begin{array}{c} If\ X_i\ \sim\ N(\mu,\ \sigma^2)\ i\ =\ 1,\,2,\,\ldots,\,n \\ then\ \sum_{i=1}^{n}\left(\frac{X_i-\ \mu}{\sigma}\right)^2\ \sim\ \chi^2(n) \\ and\ \sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2\ \sim\ \chi^2\ (n-1) \end{array} \tag{14}$$

2.2.3. *Sums of chi-square random variables.* If y_1 and y_2 are independently distributed as $\chi^2(\nu_1)$ and $\chi^2(\nu_2)$, respectively, then

$$y_1\ +\ y_2\ \sim\ \chi^2(\nu 1 + \nu 2). \tag{15}$$

2.2.4. *Moments of chi-square random variables.*

$$\begin{aligned} Mean\ (\chi^2\,(\nu)) &= \nu = degrees\ of\ freedom \\ Var\ (\chi^2\,(\nu)) &= 2\,\nu \\ Mode\ (\chi^2\,(\nu)) &= \nu - 2 \end{aligned} \tag{16}$$

2.3. The distribution function of $\chi^2(\nu)$.

$$F(x; \nu) = \int_0^x f\ (s; \nu) ds \tag{17}$$

is tabulated in most statistics and econometrics texts.

2.4. Moment generating function. The moment generating function is as follows

$$M_X(t) = \frac{1}{(1 - 2\,t)^{\,\upsilon/2}}, t < \frac{1}{2} \tag{18}$$

The first moment is

$$\begin{aligned} E\,(\,X\,) &= \tfrac{d}{dt}\left(\tfrac{1}{(\,1\,-\,2\,t\,)^{\,\upsilon/2}}\right)\,|_{t=0} \\ &= \left(\tfrac{\upsilon}{(\,1\,-\,2\,t\,)^{\,(\,\upsilon\,+\,1\,)/2}}\right)\,|_{t=0} \\ &= \upsilon \end{aligned} \tag{19}$$

3. The Student's t random variable

This distribution was published by William Gosset in 1908. His employer, Guinness Breweries, required him to publish under a pseudonym, so he chose "Student."

3.1. Relationship of Student's t-Distribution to Normal Distribution. The ratio

$$t = \frac{N(0,1)}{\sqrt{\frac{\chi^2(\nu)}{\nu}}} \tag{20}$$

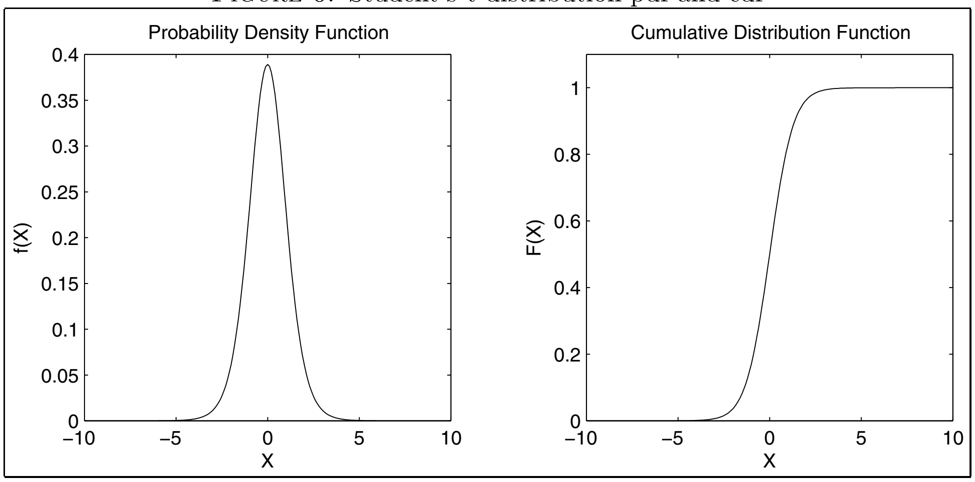
has the Student's t density function with ν degrees of freedom where the standard normal variate in the numerator is distributed independently of the χ^2 variate in the denominator. Tabulations of the associated distribution function are included in most statistics and econometrics books. Note that it is symmetric about origin.

3.2. Probability Density Function. The density of Student's t distribution is given by:

$$f\,(\,t\,;\,\nu\,) = \frac{\Gamma\left(\frac{\nu\,+\,1}{2}\right)}{\sqrt{\pi\nu}\;\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{t^2}{\nu}\right)^{\frac{-(\,\nu\,+\,1\,)}{2}} \quad -\infty < t < \infty \tag{21}$$

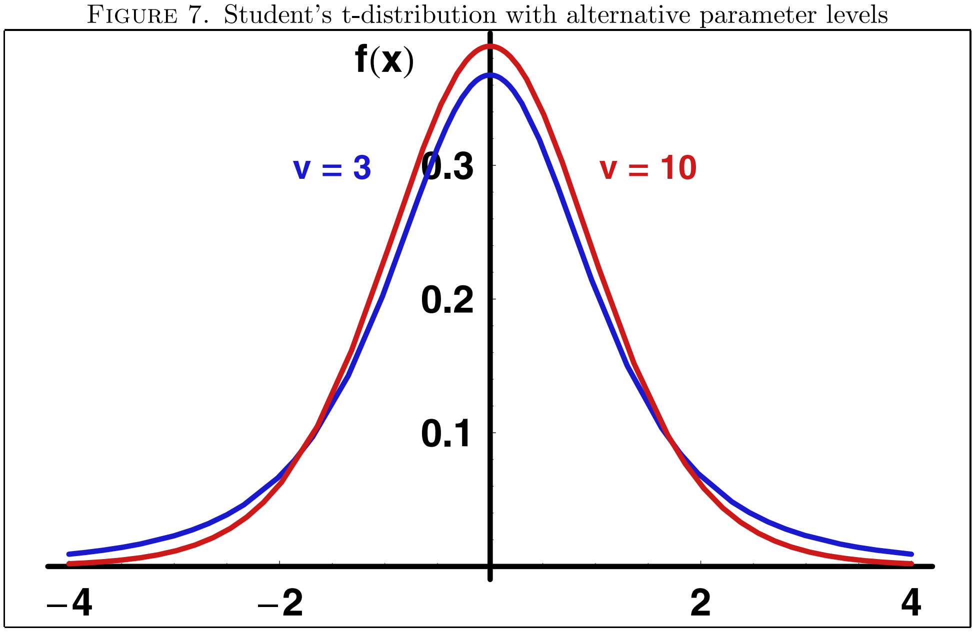
The following diagram shows the pdf and cdf for the Student's t-distribution with parameter $\nu = 10$.

FIGURE 6. Student's t distribution pdf and cdf



The following diagram shows the cdf for the Student's t-distribution with parameters $\nu = 10$ and $\nu = 3$.

FIGURE 7. Student's t-distribution with alternative parameter levels



3.3. **Moments of Student's t-distribution.**

$$\begin{aligned} Mean\ (t\,(\nu)) \;&=\; 0 \\ Var\ (t\,(\nu)) \;&=\; \frac{\nu}{\nu\,-\,2} \end{aligned} \tag{22}$$

4. The F (Fisher variance ratio) statistic

4.1. **Distribution Function.** If ${\chi^2}_1(\nu_1)$ and ${\chi^2}_2(\nu_2)$ are independently distributed chi-square variates, then

$$F(\nu_1, \nu_2\,) \;=\; \frac{\frac{\chi^2_1(\nu_1)}{\nu_1}}{\frac{\chi^2_2(\nu_2)}{\nu_2}} \;=\; \frac{\nu_2}{\nu_1} \cdot \frac{\chi^2_1(\nu_1)}{\chi^2_2(\nu_2)} \tag{23}$$

has the F density with ν_1 and ν_2 degrees of freedom.

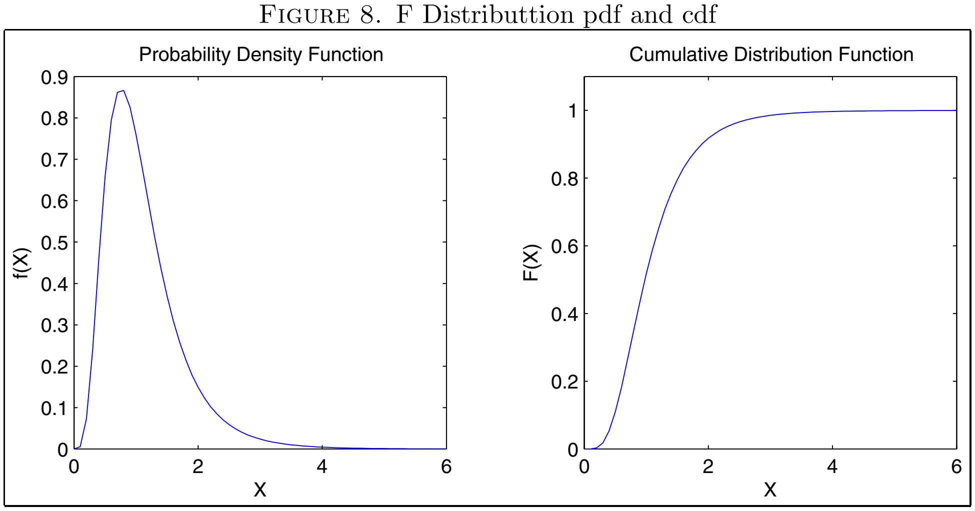
4.2. **Probability Density Function.** The density of the F distribution is

$$\begin{aligned} f\,(\,F;\,\nu_1,\,\nu_2) \;&=\; \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\,\left(\frac{\nu_1}{2}\right)\,\Gamma\,\left(\frac{\nu_2}{2}\right)} \cdot \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \cdot F^{\frac{\nu_1}{2}-1} \cdot \left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{-(\nu_1+\nu_2)}{2}} \quad F \;>\; 0 \\ &= 0 \ otherwise \end{aligned} \tag{24}$$

Tabulations of the distribution of F(ν_1,ν_2) are widely available. Note that $F_{\nu_1,\,\nu_2} \;\sim\; \left(\frac{1}{F_{\nu_2,\,\nu_1}}\right)$ and therefore the critical values can be found from $f_{\alpha\ \nu_1\,,\,\nu_2} \;=\; \left(\frac{1}{f_{1-\alpha\ \ \nu_2,\,\nu_1}}\right)$.

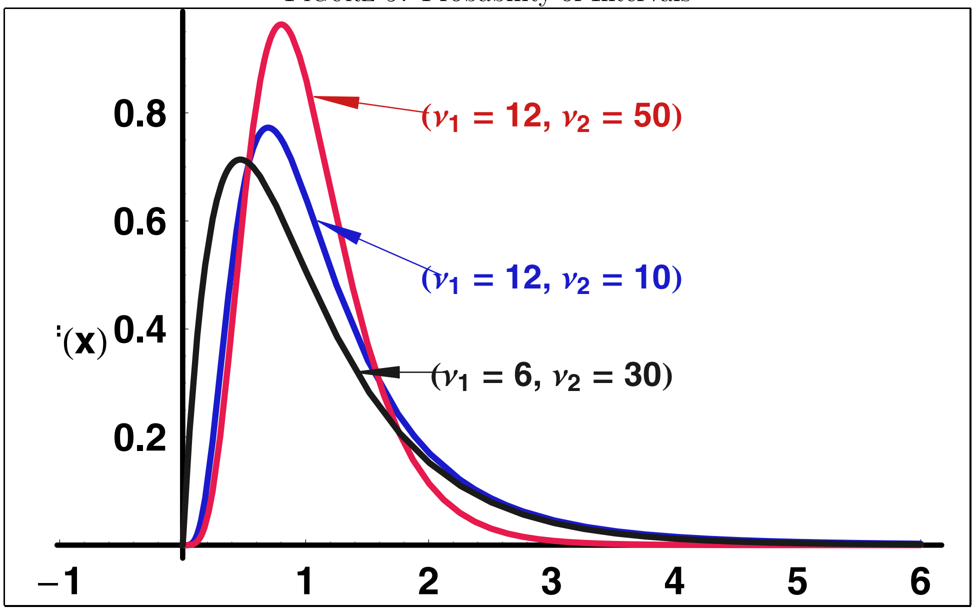
The following diagram shows the pdf and cdf for the F distribution with parameters $\nu_1 = 12$ and $\nu_2 = 20$.

Figure 8. F Distributtion pdf and cdf



Here is the pdf of the F distribution for some alternative values of pairs of values (ν_1 and ν_2).

Figure 9. Probability of Intervals



4.3. **moments of the F distribution.**

$$E(F) = \frac{\nu_2}{\nu_2 - 2} \tag{25}$$

$$Var(F) = \frac{2\nu_2^2(\nu 1 + \nu 2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)} \tag{26}$$