高维微分学——逆映照定理

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1　知识要素

1.1　由压缩映照定理获得逆映照定理

定理 1.1 (逆映照定理/微分同胚局部存在性定理). 设有映照

$$\boldsymbol{f}(\boldsymbol{x}):\mathbb{R}^m\supset D_x\ni\boldsymbol{x}\mapsto\boldsymbol{f}(\boldsymbol{x})\in\mathbb{R}^m$$

满足:

1. $\boldsymbol{f}(\boldsymbol{x})\in\mathscr{C}^1(D_x;\mathbb{R}^m)$,
2. $\exists\,\boldsymbol{x}_0\in D_x$ 使得 $Df(\boldsymbol{x}_0)\in\mathbb{R}^{m\times m}$ 可逆,

则有

1. $\exists\,U(\boldsymbol{x}_0)\subset D_x, V(\boldsymbol{f}(\boldsymbol{x}_0))\subset\boldsymbol{f}(D_x)$, 使得 $\boldsymbol{f}(\boldsymbol{x})$ 在 $U(\boldsymbol{x}_0)$ 和 $V(\boldsymbol{f}(\boldsymbol{x}_0))$ 上为双射, 即 $\exists\,\boldsymbol{f}^{-1}(\boldsymbol{y})$ 满足:
$$\boldsymbol{f}^{-1}(\boldsymbol{y}):V(\boldsymbol{f}(\boldsymbol{x}_0))\ni\boldsymbol{y}\mapsto\boldsymbol{f}^{-1}(\boldsymbol{y})\in D_x;$$
2. $\boldsymbol{f}^{-1}(\boldsymbol{y})\in\mathscr{C}^1(V(\boldsymbol{f}(\boldsymbol{x}_0));\mathbb{R}^m)$.

证明 直接利用有界闭集上的压缩映照定理.

首先, 作 $B_{\lambda_0}(\boldsymbol{x}_0)\subset D_x$, $B_{\mu_0}(\boldsymbol{y}_0)\subset\mathbb{R}^m$ 以及辅助映照

$$\boldsymbol{\psi}_y(\boldsymbol{x}):B_{\lambda_0}(\boldsymbol{x}_0)\ni\boldsymbol{x}\mapsto\boldsymbol{\psi}_y(\boldsymbol{x})=\boldsymbol{x}+(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)(\boldsymbol{y}-\boldsymbol{f}(\boldsymbol{x}))\in\mathbb{R}^m,\quad\forall\,\boldsymbol{y}\in B_{\mu_0}(\boldsymbol{y}_0).$$

如果对 $\forall\,\boldsymbol{y}\in B_{\mu_0}(\boldsymbol{y}_0)$, $\exists\,\boldsymbol{x}_y\in B_{\lambda_0}(\boldsymbol{x}_0)$, 满足 $\boldsymbol{\psi}_y(\boldsymbol{x}_y)=\boldsymbol{x}_y$ 等价于 $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x}_y)$.

现设定 $\boldsymbol{\psi}_y(\boldsymbol{x})$ 的不动点唯一存在, 则可作

$$\boldsymbol{\eta}(\boldsymbol{y}):B_{\mu_0}(\boldsymbol{y}_0)\ni\boldsymbol{y}\mapsto\boldsymbol{\eta}(\boldsymbol{y})\in\mathbb{R}^m,$$

满足

$$\begin{cases}\boldsymbol{\eta}(\boldsymbol{y})\in B_{\lambda_0}(\boldsymbol{x}_0)\\ \boldsymbol{y}=\boldsymbol{f}(\boldsymbol{\eta}(\boldsymbol{y})),\quad\forall\,\boldsymbol{y}\in B_{\mu_0}(\boldsymbol{y}_0)\end{cases}.$$

藉此作

$$U\triangleq\{\boldsymbol{x}\in B_{\lambda_0}(\boldsymbol{x}_0)|\boldsymbol{f}(\boldsymbol{x})\in B_{\mu_0}(\boldsymbol{y}_0)\}\,,$$

具有如下性质.

1. U 为开集. 考虑到 $\boldsymbol{f}(\boldsymbol{x})$ 在 $B_{\lambda_0}(\boldsymbol{x}_0)$ 上的连续性, 故对: $\boldsymbol{x} \in U, \forall \varepsilon > 0, \exists \delta_\varepsilon > 0$, 成立: $\boldsymbol{f}(B_{\delta_\varepsilon}(\boldsymbol{x})) \subset B_\varepsilon(\boldsymbol{f}(\boldsymbol{x})) \subset B_{\mu_0}(\boldsymbol{y}_0)$, 即有 $B_{\delta_\varepsilon}(\boldsymbol{x}) \subset U$.

2. $\boldsymbol{f}(\boldsymbol{x})$ 在 U 上为单射. 利用反证法, 设

$$\begin{cases} \boldsymbol{x}_1, \boldsymbol{x}_2 \in U\,,\, \boldsymbol{x}_1 \neq \boldsymbol{x}_2, \\ \boldsymbol{f}(\boldsymbol{x}_1) = \boldsymbol{f}(\boldsymbol{x}_2) \in B_\mu(\boldsymbol{y}_0). \end{cases}$$

   记 $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_1) = \boldsymbol{f}(\boldsymbol{x}_2)$ 则按上述求解的唯一性, $\exists ! \boldsymbol{x}_y \in B_{\lambda_0}(\boldsymbol{x}_0)$, 满足 $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_y)$, 而有 $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{x}_y$, 同假设矛盾.

3. $\boldsymbol{f}(U) = B_{\mu_0}(\boldsymbol{y}_0)$. 显然 $\boldsymbol{f}(U) \subset B_{\mu_0}(\boldsymbol{y}_0)$. 考虑 $\forall \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0)$, 则 $\exists ! \boldsymbol{x}_y \in B_{\lambda_0}(\boldsymbol{x}_0)$. 满足 $\boldsymbol{f}(\boldsymbol{x}_y) = \boldsymbol{y} \in B_{\mu_0}(\boldsymbol{y}_0)$. 由此有 $B_\mu(\boldsymbol{y}_0) \subset \boldsymbol{f}(U)$.

综上, $\boldsymbol{f}(\boldsymbol{x})$ 实现 $B_{\mu_0}(\boldsymbol{x}_0) \supset U$ 同 $\boldsymbol{f}(U) = B_{\mu_0}(\boldsymbol{y}_0)$ 之间的双射, 故存在逆映照. 又由对 $\forall \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0)$ 有 $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{\eta}(\boldsymbol{y}))$, 所以对 $\forall \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0)$ 有 $\boldsymbol{f}^{-1}(\boldsymbol{y}) = \boldsymbol{f}^{-1} \circ \boldsymbol{f} \circ \boldsymbol{\eta}(\boldsymbol{y}) = \boldsymbol{\eta}(\boldsymbol{y})$. 因此有

$$\boldsymbol{f}^{-1}(\boldsymbol{y}) = \boldsymbol{\eta}(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0).$$

以下基于压缩映照定理, 证明不动点的唯一存在性.

第一步, 估计

$$\begin{aligned} \left|\boldsymbol{\psi}_y(\boldsymbol{x}) - \boldsymbol{x}_0\right|_{\mathbb{R}^m} &= \left|\boldsymbol{\psi}_y(\boldsymbol{x}) - \boldsymbol{\psi}_{y_0}(\boldsymbol{x}_0)\right|_{\mathbb{R}^m} = |\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}) - \boldsymbol{\psi}(\boldsymbol{y}_0,\boldsymbol{x}_0)|_{\mathbb{R}^m} \\ &\leqslant |\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}) - \boldsymbol{\psi}(\boldsymbol{y}_0,\boldsymbol{x})|_{\mathbb{R}^m} + |\boldsymbol{\psi}(\boldsymbol{y}_0,\boldsymbol{x}) - \boldsymbol{\psi}(\boldsymbol{y}_0,\boldsymbol{x}_0)|_{\mathbb{R}^m} \\ &\leqslant \sup_{\theta_2 \in (0,1)} |D_y\boldsymbol{\psi}(\boldsymbol{y}_0 + \theta_2(\boldsymbol{y}-\boldsymbol{y}_0),\boldsymbol{x})|_{\mathbb{R}^{m\times m}} \cdot |\boldsymbol{y}-\boldsymbol{y}_0|_{\mathbb{R}^m} \\ &\quad + \sup_{\theta_1 \in (0,1)} |D_x\boldsymbol{\psi}(\boldsymbol{y}_0,\boldsymbol{x}_0 + \theta_1(\boldsymbol{x}-\boldsymbol{x}_0))|_{\mathbb{R}^{m\times m}} \cdot |\boldsymbol{x}-\boldsymbol{x}_0|_{\mathbb{R}^m}\,, \end{aligned}$$

此处

$$\begin{cases} \boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}) \triangleq \boldsymbol{\psi}_y(\boldsymbol{x}) = \boldsymbol{x} + (D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x})), \quad \forall \boldsymbol{y} \in B_{\mu_0}(\boldsymbol{y}_0),\, \boldsymbol{x} \in B_{\lambda_0}(\boldsymbol{x}_0), \\ D_y\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}) = (D\boldsymbol{f})^{-1}(\boldsymbol{x}_0), \\ D_x\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}) = I_{\mathbb{R}^m} - (D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)D\boldsymbol{f}(\boldsymbol{x}) = (D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\cdot[D\boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x})], \end{cases}$$

故

$$\begin{aligned} \left|\boldsymbol{\psi}_y(\boldsymbol{x}) - \boldsymbol{x}_0\right|_{\mathbb{R}^m} \leqslant{} & \left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}} |\boldsymbol{y}-\boldsymbol{y}_0|_{\mathbb{R}^m} \\ & + \left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}} \cdot |D\boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x})|_{\mathbb{R}^{m\times m}} \cdot |\boldsymbol{x}-\boldsymbol{x}_0|_{\mathbb{R}^m}\,. \end{aligned}$$

考虑到 $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{C}^1(D_x;\mathbb{R}^m)$, 故可缩小 λ_0 到 λ_1, 使得

$$\left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}} \cdot |D\boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x})|_{\mathbb{R}^{m\times m}} < \frac{1}{2}.$$

再缩小 μ_0 到 μ_1, 使得

$$\left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}} \cdot |\boldsymbol{y}-\boldsymbol{y}_0|_{\mathbb{R}^m} < \frac{1}{2}\lambda_1.$$

综上, 有

$$\left|\boldsymbol{\psi}_y(\boldsymbol{x})-\boldsymbol{x}_0\right|_{\mathbb{R}^m}<\frac{1}{2}\lambda_1+\frac{1}{2}\left|\boldsymbol{x}-\boldsymbol{x}_0\right|_{\mathbb{R}^m}<\lambda_1, \quad \forall\, \boldsymbol{x}\in\overline{B_{\lambda_1}}(\boldsymbol{x}_0),\, \boldsymbol{y}\in B_{\mu_1}(\boldsymbol{y}_0),$$

即有 $\forall\, \boldsymbol{y}\in B_{\mu}(\boldsymbol{y}_0)$, 有 $\boldsymbol{\psi}_y(\overline{B_{\lambda_1}}(\boldsymbol{x}_0))\subset B_{\lambda_1}(\boldsymbol{x}_0)\subset\overline{B_{\lambda_1}}(\boldsymbol{x}_0)$.

第二步, 估计压缩性, 即

$$\begin{aligned}\left|\boldsymbol{\psi}_y(\boldsymbol{x}_1)-\boldsymbol{\psi}_y(\boldsymbol{x}_2)\right|_{\mathbb{R}^m}&=\left|\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}_1)-\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}_2)\right|_{\mathbb{R}^m}\\&\leqslant\sup_{\theta_1\in(0,1)}\left|D_x\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}_1+\theta_1(\boldsymbol{x}_2-\boldsymbol{x}_1))\right|_{\mathbb{R}^{m\times m}}\cdot\left|\boldsymbol{x}_1-\boldsymbol{x}_2\right|_{\mathbb{R}^m}.\\&<\frac{1}{2}\left|\boldsymbol{x}_1-\boldsymbol{x}_2\right|_{\mathbb{R}^m}.\end{aligned}$$

综上有: $\boldsymbol{\psi}_y(\boldsymbol{x})$ 为 $\overline{B_{\lambda_1}}(\boldsymbol{x}_0)$ 上的压缩映照, 此处 $\forall\,\boldsymbol{y}\in B_{\mu_1}(\boldsymbol{y}_0)$. 故 $\exists!\,\boldsymbol{x}_y\in\overline{B_{\lambda_1}}(\boldsymbol{x}_0)$, 满足

$$\boldsymbol{\psi}_y(\boldsymbol{x}_y)=\boldsymbol{x}_y\in B_{\lambda_1}(\boldsymbol{x}_0).$$

故可作

$$\boldsymbol{\eta}(\boldsymbol{y}):B_{\mu_1}(\boldsymbol{y}_0)\ni\boldsymbol{y}\mapsto\boldsymbol{\eta}(\boldsymbol{y})=\boldsymbol{x}_y\in\mathbb{R}^m,$$

满足

$$\begin{cases}\boldsymbol{\eta}(\boldsymbol{y})\in B_{\mu_1}(\boldsymbol{y}_0),\\ \boldsymbol{y}=\boldsymbol{f}\circ\boldsymbol{\eta}(\boldsymbol{y}), \quad \forall\,\boldsymbol{y}\in B_{\mu_1}(\boldsymbol{y}_0).\end{cases}$$

以下证 $\boldsymbol{\eta}(\boldsymbol{y})$ 的连续性, 即 $\boldsymbol{\eta}(\boldsymbol{y})\in C(B_{\mu_1}(\boldsymbol{y}_0);\mathbb{R}^m)$. 估计

$$\begin{aligned}&\left|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})\right|_{\mathbb{R}^m}=\left|\boldsymbol{\psi}_{\boldsymbol{y}+\Delta\boldsymbol{y}}(\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y}))-\boldsymbol{\psi}_y(\boldsymbol{\eta}(\boldsymbol{y}))\right|_{\mathbb{R}^m}\\&\quad=\left|\boldsymbol{\psi}(\boldsymbol{y}+\Delta\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y}))-\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}))\right|_{\mathbb{R}^m}\\&\quad\leqslant\left|\boldsymbol{\psi}(\boldsymbol{y}+\Delta\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y}))-\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y}))\right|_{\mathbb{R}^m}+\left|\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y}))-\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}))\right|_{\mathbb{R}^m}\\&\quad\leqslant\sup_{\theta_2\in(0,1)}\left|D_y\boldsymbol{\psi}(\boldsymbol{y}+\theta_2\Delta\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y}))\right|_{\mathbb{R}^{m\times m}}\left|\Delta\boldsymbol{y}\right|_{\mathbb{R}^m}\\&\qquad+\sup_{\theta_1\in(0,1)}\left|D_x\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y})+\theta_1(\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})))\right|_{\mathbb{R}^{m\times m}}\left|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})\right|_{\mathbb{R}^m}\\&\quad<\left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}}\left|\Delta\boldsymbol{y}\right|_{\mathbb{R}^m}+\frac{1}{2}\left|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})\right|_{\mathbb{R}^m},\end{aligned}$$

即有

$$\left|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})\right|_{\mathbb{R}^m}<2\left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}}\left|\Delta\boldsymbol{y}\right|_{\mathbb{R}^m},$$

所以

$$\boldsymbol{\eta}(\boldsymbol{y})\in C(B_{\mu_1}(\boldsymbol{y}_0);\mathbb{R}^m).$$

以下证

$$\boldsymbol{\eta}(\boldsymbol{y})\in\mathscr{C}^1(B_{\mu_1}(\boldsymbol{y}_0);\mathbb{R}^m),$$

考虑

$$\boldsymbol{f}(\boldsymbol{x})\in\mathscr{C}^1(D_x;\mathbb{R}^m),$$

故有

$$\boldsymbol{f}(\boldsymbol{x}+\Delta\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{x})=D\boldsymbol{f}(\boldsymbol{x})\cdot\Delta\boldsymbol{x}+o(|\Delta\boldsymbol{x}|_{\mathbb{R}^m}),\quad \forall\,\boldsymbol{x}\in D_x.$$

现取

$$\begin{cases}\boldsymbol{x}=\boldsymbol{\eta}(\boldsymbol{y}), & \forall\,\boldsymbol{y}\in B_{\mu_0}(\boldsymbol{y}_0),\\ \Delta\boldsymbol{x}=\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y}),\end{cases}$$

则

$$\begin{aligned}\Delta\boldsymbol{x}&=\boldsymbol{f}\circ\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{f}\circ\boldsymbol{\eta}(\boldsymbol{y})=\boldsymbol{y}+\Delta\boldsymbol{y}-\boldsymbol{y}=\Delta\boldsymbol{y},\\ \Delta\boldsymbol{y}&=D\boldsymbol{f}(\boldsymbol{x})\cdot(\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y}))+o(|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m}).\end{aligned}$$

此处要求:

$$\exists\,(D\boldsymbol{f})^{-1}(\boldsymbol{x})\in\mathbb{R}^{m\times m},\quad \forall\,\boldsymbol{x}\in B_{\lambda_1}(\boldsymbol{x}_0).$$

由于 $\exists\,D\boldsymbol{f}(\boldsymbol{x}_0)\in\mathbb{R}^{m\times m}$ 可逆, 故在选定 $B_{\mu_0}(\boldsymbol{x}_0)$ 时, 就可使得对 $\forall\,\boldsymbol{x}\in B_{\mu_0}(\boldsymbol{x}_0)$, 都有 $\exists\,(D\boldsymbol{f})^{-1}(\boldsymbol{x})$. 由于

$$|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m}<2\left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}}|\Delta\boldsymbol{y}|_{\mathbb{R}^m},$$

所以

$$\begin{aligned}\frac{|o(|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m})|_{\mathbb{R}^m}}{|\Delta\boldsymbol{y}|_{\mathbb{R}^m}}&=\frac{|o(|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m})|_{\mathbb{R}^m}}{|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m}}\cdot\frac{|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m}}{|\Delta\boldsymbol{y}|_{\mathbb{R}^m}}\\ &<\frac{|o(|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m})|_{\mathbb{R}^m}}{|\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})|_{\mathbb{R}^m}}\cdot 2\left|(D\boldsymbol{f})^{-1}(\boldsymbol{x}_0)\right|_{\mathbb{R}^{m\times m}}\to 0\\ &\quad(\Delta\boldsymbol{y}\to 0).\end{aligned}$$

由于 $\boldsymbol{\eta}(\boldsymbol{y})$ 为 $B_{\lambda_1}(\boldsymbol{y}_0)$ 上单射, 故在 $\Delta\boldsymbol{y}\neq 0\in\mathbb{R}^m$ 时,$\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})\in\mathbb{R}^m$ 满足非接触性条件.

综上, 有

$$\boldsymbol{\eta}(\boldsymbol{y}+\Delta\boldsymbol{y})-\boldsymbol{\eta}(\boldsymbol{y})=(D\boldsymbol{f})^{-1}(\boldsymbol{x})\Delta\boldsymbol{y}+o(\Delta\boldsymbol{y}),\quad \boldsymbol{x}=\boldsymbol{f}^{-1}(\boldsymbol{y}).$$ □

1.2 由隐映照定理获得逆映照定理

定理 1.2 (逆映照定理/微分同胚局部存在性定理). 设有映照

$$\boldsymbol{f}(\boldsymbol{x}):\mathbb{R}^m\supset D_x\ni\boldsymbol{x}\mapsto\boldsymbol{f}(\boldsymbol{x})\in\mathbb{R}^m$$

满足:

1. $\boldsymbol{f}(\boldsymbol{x})\in\mathscr{C}^1(D_x;\mathbb{R}^m)$,

2. $\exists\,\boldsymbol{x}_0\in D_x$ 使得 $Df(\boldsymbol{x}_0)\in\mathbb{R}^{m\times m}$ 可逆,

则有

1. $\exists U(\boldsymbol{x}_0) \subset D_x, V(\boldsymbol{f}(\boldsymbol{x}_0)) \subset \boldsymbol{f}(D_x)$, 使得 $\boldsymbol{f}(\boldsymbol{x})$ 在 $U(\boldsymbol{x}_0)$ 和 $V(\boldsymbol{f}(\boldsymbol{x}_0))$ 上为双射, 即 $\exists\, \boldsymbol{f}^{-1}(\boldsymbol{y})$ 满足:
$$\boldsymbol{f}^{-1}(\boldsymbol{y}) : V(\boldsymbol{f}(\boldsymbol{x}_0)) \ni \boldsymbol{y} \mapsto \boldsymbol{f}^{-1}(\boldsymbol{y}) \in D_x;$$

2. $\boldsymbol{f}^{-1}(\boldsymbol{y}) \in \mathscr{C}^1(V(\boldsymbol{f}(\boldsymbol{x}_0)); \mathbb{R}^m)$.

证明 利用隐映照定理, 作
$$F(\boldsymbol{y},\boldsymbol{x}) : \mathbb{R}^m \times D_x \ni \{\boldsymbol{y},\boldsymbol{x}\} \mapsto F(\boldsymbol{y},\boldsymbol{x}) \triangleq \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^m,$$
由 $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{C}^1(D_x;\mathbb{R}^m)$, 有 $F(\boldsymbol{y},\boldsymbol{x}) \in \mathscr{C}^1(\mathbb{R}^m \times D_x;\mathbb{R}^m)$. 由于 $Df(\boldsymbol{x}_0) \in \mathbb{R}^{m\times m}$ 可逆, 有
$$D_xF(\boldsymbol{y}_0,\boldsymbol{x}_0) = -Df(\boldsymbol{x}_0) \in \mathbb{R}^{m\times m}\text{可逆},$$
此处 $\boldsymbol{y}_0 := \boldsymbol{f}(\boldsymbol{x}_0) \in \mathbb{R}^m$. 按隐映照定理, $\exists\, B_\mu(\boldsymbol{y}_0) \subset \mathbb{R}^m, B_\lambda(\boldsymbol{x}_0) \subset D_x$, 满足
$$\forall\, \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0),\ \exists!\, \boldsymbol{x}_y \in B_\lambda(\boldsymbol{x}_0)\text{ 满足 } F(\boldsymbol{y},\boldsymbol{x}_y) = 0 \in \mathbb{R}^m.$$
由此可作 $\boldsymbol{\eta}(\boldsymbol{y}) : B_\mu(\boldsymbol{y}_0) \ni \boldsymbol{y} \mapsto \boldsymbol{\eta}(\boldsymbol{y}) \in \mathbb{R}^m$, 满足
$$\begin{cases} \boldsymbol{\eta}(\boldsymbol{y}) \in B_\lambda(\boldsymbol{x}_0);\ F(\boldsymbol{y},\boldsymbol{\eta}(\boldsymbol{y})) = 0 \in \mathbb{R}^m, \\ \boldsymbol{\eta}(\boldsymbol{y}) \in \mathscr{C}^1(B_\mu(\boldsymbol{y}_0);\mathbb{R}^m). \end{cases}$$
作 $U \triangleq \{\boldsymbol{x} \in B_\lambda(\boldsymbol{x}_0) | \boldsymbol{f}(\boldsymbol{x}) \in B_\mu(\boldsymbol{y}_0)\} \subset B_\lambda(\boldsymbol{x}_0)$, 具有如下性质.

1. U 为开集. 考虑 $\forall\, \boldsymbol{x} \in U$, 则 $\boldsymbol{f}(\boldsymbol{x}) \in B_\mu(\boldsymbol{y}_0)$, 由 $\boldsymbol{f}(\boldsymbol{x})$ 在点 $\boldsymbol{x}$ 连续, 则 $\forall\, \varepsilon > 0, \exists\, \delta_\varepsilon > 0$, 满足 $\boldsymbol{f}(B_{\delta_\varepsilon}(\boldsymbol{x})) \in B_\varepsilon(\boldsymbol{f}(\boldsymbol{x}))$, 自然可使 $B_\varepsilon(\boldsymbol{f}(\boldsymbol{x})) \subset B_\mu(\boldsymbol{y}_0), B_{\delta_\varepsilon}(\boldsymbol{x}) \subset B_\lambda(\boldsymbol{x}_0)$, 故有 $B_{\delta_\varepsilon}(\boldsymbol{x}) \subset U$, 即证得 U 为开集.

2. $\boldsymbol{f}(\boldsymbol{x})$ 在 U 上为单射. 设有 $\widetilde{\boldsymbol{x}}, \widehat{\boldsymbol{x}} \in U$, 满足 $\boldsymbol{f}(\widetilde{\boldsymbol{x}}) = \boldsymbol{f}(\widehat{\boldsymbol{x}}) \in B_\mu(\boldsymbol{y}_0)$, 则按隐映照定理, 可得 $\widetilde{\boldsymbol{x}} = \widehat{\boldsymbol{x}}$.

3. $\boldsymbol{f}(U) = B_\mu(\boldsymbol{y}_0)$. 显然 $\boldsymbol{f}(U) \subset B_\mu(\boldsymbol{y}_0)$, 则只需证明 $B_\mu(\boldsymbol{y}_0) \subset \boldsymbol{f}(U)$, 即 $\forall\, \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0), \exists\, \boldsymbol{x}_y \in U$, 满足 $\boldsymbol{f}(\boldsymbol{x}_y) = \boldsymbol{y}$. 按隐映照定理, $\exists\, \boldsymbol{\eta}(\boldsymbol{y}) \in B_\lambda(\boldsymbol{x}_0)$, 使得 $\boldsymbol{f}(\boldsymbol{\eta}(\boldsymbol{y})) = \boldsymbol{y}$. 考虑到 $\boldsymbol{y} \in B_\mu(\boldsymbol{y}_0)$, 则 $\boldsymbol{\eta}(\boldsymbol{y}) \in U$.

综上, 有 $\boldsymbol{f}(\boldsymbol{x})$ 实现 $U(B_\lambda(\boldsymbol{x}_0))$ 同 $\boldsymbol{f}(U) = B_\mu(\boldsymbol{y}_0)$ 之间的双射. 因此 $\exists\, \boldsymbol{f}^{-1}(\boldsymbol{y})$
$$\boldsymbol{f}^{-1}(\boldsymbol{y}) : B_\mu(\boldsymbol{y}_0) \ni \boldsymbol{y} \mapsto \boldsymbol{f}^{-1}(\boldsymbol{y}) \in \mathbb{R}^m,$$
满足
$$\begin{cases} \boldsymbol{f}^{-1}(\boldsymbol{y}) \in U, \\ \boldsymbol{f} \circ \boldsymbol{f}^{-1}(\boldsymbol{y}) = \boldsymbol{y}, \forall\, \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0). \end{cases}$$
考虑到 $\boldsymbol{f} \circ \boldsymbol{\eta}(\boldsymbol{y}) = \boldsymbol{y}, \forall\, \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0)$, 所以 $\boldsymbol{f}^{-1} \circ \boldsymbol{f} \circ \boldsymbol{\eta}(\boldsymbol{y}) = \boldsymbol{f}^{-1}(\boldsymbol{y}), \forall\, \boldsymbol{y} \in B_\mu(\boldsymbol{y}_0)$. 因此在 $B_\mu(\boldsymbol{y}_0)$ 上, 有 $\boldsymbol{\eta} = \boldsymbol{f}^{-1}$, 由 $\boldsymbol{\eta}(\boldsymbol{y}) \in \mathscr{C}^1(B_\mu(\boldsymbol{y}_0);\mathbb{R}^m)$, 则有 $\boldsymbol{f}^{-1}(\boldsymbol{y}) \in \mathscr{C}^1(\boldsymbol{f}(U);\mathbb{R}^m)$.

综上所述, 有 $\boldsymbol{f}(\boldsymbol{x})$ 实现 U 同 $\boldsymbol{f}(U)$ 之间的双射, $U, \boldsymbol{f}(U)$ 均为开集, 而且 $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{C}^1(U;\mathbb{R}^m), \boldsymbol{f}^{-1}(\boldsymbol{y}) \in \mathscr{C}^1(\boldsymbol{f}(U);\mathbb{R}^m)$, 即 $\boldsymbol{f}(\boldsymbol{x})$ 为 U 同 $\boldsymbol{f}(U)$ 之间的 $\mathscr{C}^1$ 微分同胚, 记作 $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{C}^1(U;\boldsymbol{f}(U))$. □

1.3 微分同胚

2 应用事例

3 建立路径