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Transverse vibrations of a thin loaded rod: theory and experiment

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Abstract
The general formulation of a determinate solution problem is deduced for the transverse vibrations of a thin loaded rod. The vibration frequencies of a thin homogeneous rod carrying a concentrated mass as a function of the load's position and mass are exactly solved. The dynamic measurement method of Young's modulus of the rods is presented within this theory. Our measurements of Young's modulus in the dynamic method agree with those in the traditional bending method, therefore the theory is verified by our experiments.

Keywords: transverse vibrations of a load rod, Young's modulus, dynamic measurement method

(Some figures may appear in colour only in the online journal)

1. Introduction

In college-level instruction on vibration and waves, the model is usually simplified as vibrations of a single particle even for a tuning fork. Actually, the tuning fork consists of two elastic rods and we need to analyze the transverse vibrations of these rods. The textbooks on the method of mathematical physics have only the longitudinal vibrations of a rod [1–3] and rarely refer to its transverse vibrations [4–6] with and without a load. Even in engineering mathematics [4] and applied mathematics [5], the equation of motion on the vertical vibrations of a thin long rod is only directly given and its derivation is not present. In the course of the method of mathematical physics, the transverse vibrations of a rod were first introduced by [6], but its deduction on the solving problem is too simple to follow by students. Even in the mechanics of vibration, a fundamental course for a mechanics major, the instruction on transverse vibrations of a rod is still simple [7, 8] although its longitudinal vibration is discussed with a load [7]. Earlier papers have addressed the problem of vibrating rods [9, 10]. In special textbooks [11], the exact solutions of the rod's transverse vibrations are put forward systematically, but only a few solutions at special conditions are given [12, 13]. It is necessary to derive a general formulation of the determinant solution problem for the transverse vibrations of a thin loaded rod, and find the exact solutions in general conditions. Much work has been done on the more complicated case of anisotropic vibrations of rods. For example, the greatest use of piezoelectric resonators is the famous 32.768 kHz quartz tuning fork, including a review paper [14], the fabrication [15], and the analysis of frequency [16]. Although this work treats only the isotropic case, which is a limiting work in the mechanics major, we emphasize the physical foundation for undergraduates in the physics major.

Knowledge of Young's modulus is fundamentally important to understand the mechanical behaviour of materials, such as metals, ceramic grinding stones, dental composites, and polymers [17–20]. Young's moduli are determined traditionally by the static and dynamic methods. In static measurements [21, 22], such as the classical tensile or compressive test, a uniaxial stress is exerted on the material, and the elastic modulus is calculated from the transverse and axial deformations as the slope of the stress-strain curve at the origin. The static methods include the three-point bending [17–20], four-point bending [23], clamped beam, and compression/tension stress, etc. A dynamic method of measuring Young's modulus of stalloy was described in an earlier paper [24] using a loaded fixed-free bar vibrating in flexure and developed to measure Young's modulus of elasticity of a solid [25], carrying a heavy mass of precise finite dimensions at the free end and giving the derivation of the equation of motion. Dynamic methods [26–29] are more precise since they use very small strains, far below the elastic limit and therefore are virtually nondestructive and allow repeated testing of the same sample. These include the ultrasonic pulse-echo [27], bar resonance methods [22, 28, 29], travelling or standing wave, bending/transversal or longitudinal wave, transient pulse generation, etc. Recently, a new vibration beam technique [30] for the determination of the dynamic Young's modulus has been developed, but without the added loads. The dynamic methods redeem the defects that the static bending method cannot be applied to the measurement of fragile materials. Our method has added a variable with the loads at different positions and different masses.

For didactic purposes undergraduates majoring in physics, in this paper we derive a general formulation of the determinant solution problem for the transverse vibrations of a thin loaded rod, obtain an exact solution of the problem, and deduce a general relationship between eigenfrequencies and the load's position and mass. Different resonance frequencies are measured by adding both the same mass to different positions of the rod and different masses to the same position of the rod. We deal with Young's modulus measurement method based on the vibration of a thin long rod with added point mass. The elastic modulus is calculated from the least square fit of frequency versus the square of the wave number calculated from the characteristic equation. According to this model, a new kind of dynamic measurement method of Young's modulus is presented. This method is more comprehensive and is advanced when it is not convenient to change the length of samples.

The paper is organized as follows. The mechanical model and the solution for the eigenfrequencies as a function of the load's position and mass are illustrated in section 2. A practical implementation and the results of the experiments are described in section 3, showing results in agreement with model predictions. A summary is given in section 6.

2. The mechanical model and the solutions

We consider a rod of length l along the x-axis in equilibrium. The mass of the rod is $m = \int_0^l \mathrm{d}x\rho(x)s(x)$ with dx being the line element, $\rho(x)$ being the volume density, and $s(x) = w(x)h(x)$ being the cross-section area. Here the width of the rod is $y = w(x)$ and the thickness is $z = h(x)$. The turning radius of this cross-section $r(x)$ satisfies

$$r^2(x) = \frac{2}{h(x)}\int_0^{\frac{1}{2}h(x)} z^2 \mathrm{d}z = \frac{h(x)^2}{12}. \tag{1}$$

Figure 1 shows the diagram and cross-section of the thin homogenous rod. Let the mass element be $\mathrm{d}m = \rho(x)\mathrm{d}\tau$ with the volume element $\mathrm{d}\tau = s(x)\mathrm{d}x$, one obtains the rotational inertia to the oy-axis as

$$\mathrm{d}I_{yy} = \frac{h(x)}{12}\rho(x)\mathrm{d}\tau. \tag{2}$$

When the rod deforms transversely, its every cross-section should produce shearing forces. Let the shearing force on the left of volume element be $Q(x, t)$ (down direction), and the right one be $Q' = Q(x, t) + \partial_x Q(x, t)\mathrm{d}x$ (up direction) with ∂_x being an abbreviation of $\partial/\partial x$. These two shearing forces form a force couple, which bends the rod. Figure 2 shows an element of a thin homogenous rod in bending.

We consider the characteristic quantity of the transverse vibrations of a thin loaded rod, the displacement of the rod away from the equilibrium position along the z-direction at space-time point (x,t), is $u(x, t)$. The curvature radius of the bending is

$$R = \left[1 + \left(\partial_x u\right)^2\right]^{\frac{3}{2}} \Big/ \partial_{xx}^2 u. \tag{3}$$

When the rod bends, the central line length dx remains unchanged. However, the upper part of the central line that suffers the tension of the nearby elements is prolonged; the lower part that suffers the pressure is compressed. Consequently, the force couple consists of tension and pressure, the so-called bending moment. Let the bending moment on the left of the volume element be $M(x, t)$ (clockwise), and the bending moment on the right be $M' = M(x, t) + \partial_x M(x, t)\mathrm{d}x$ (counter-clockwise). The bending moments act as resistance to the bending of the rod, and lead the system to the dynamic equilibrium states. Figure 3 shows the force acting on an element of a thin homogenous rod.

As shown in figure 2, we take a lamina with thickness dz at z position and width $w(x)$. We recall that the length is dx at the center line of the volume element. The length of the lamina is $(R + z)\mathrm{d}\theta = \mathrm{d}x + z\mathrm{d}x/R$ and the relative extension is z/R. So that the tensile stress is $P = -Yz/R$ with Y being Young's modulus. The tension element is $\mathrm{d}G = Pw\mathrm{d}z$, the bending moment element is $\mathrm{d}M = z\mathrm{d}G = -Yz^2 w\mathrm{d}z/R$, and the bending moment is

$$M = -YJ/R, \tag{4}$$

with $J = \int z^2 w \mathrm{d}z = s(x)r^2(x)$ the inertia moment per mass for the cross-section $s(x)$ to the center oy-axis.

For the mass element $\rho(x)\mathrm{d}\tau$ of the bending rod, the inertia force is $-\rho(x)\partial_{tt}^2 u\mathrm{d}\tau$, the external force is $f(x, t)\mathrm{d}\tau$, and the external bending moment is $m(x, t)\mathrm{d}x$. The equilibrium equation of moment to the left center C is

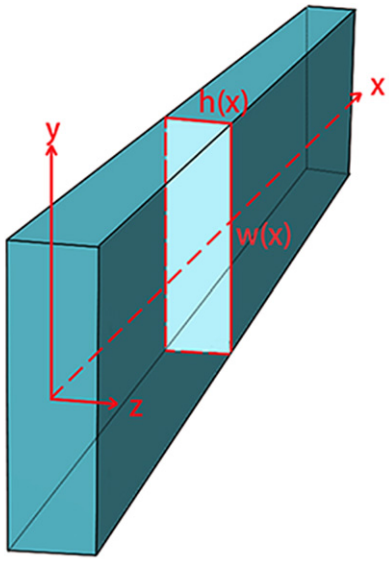

Figure 1. Diagram and cross-section of a homogenous rod.

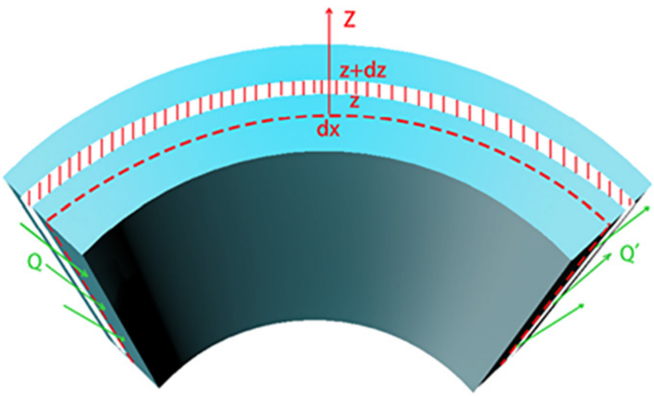

Figure 2. An element of thin homogenous rod in bending.

$$(M + M') - M = Q'\mathrm{d}x - Q\mathrm{d}x_c + m(x, t)\mathrm{d}x + \rho(x)\partial_{tt}^2 u\mathrm{d}x\frac{1}{2}\mathrm{d}x. \tag{5}$$

Omitting the higher order ($\geqslant$2 order) small quantities, the reciprocal of the curvature radius is simplified as $R^{-1} \simeq \partial_{xx}^2 u(x, t)$ and equation (5) becomes

$$\partial_x M(x, t) = Q(x, t) + m(x, t). \tag{6}$$

Without the gravity, the equilibrium equation of force acting in the transverse directions is

$$\rho(x)\partial_{tt}^2 u(x, t) = \frac{1}{s(x)}\partial_x Q(x, t) + f(x, t). \tag{7}$$

Substituting $M = -Ys(x)r^2(x)\partial_{xx}^2 u(x, t)$ into equation (6) and combining equations (6) and (7), one obtains the general equation of motion as

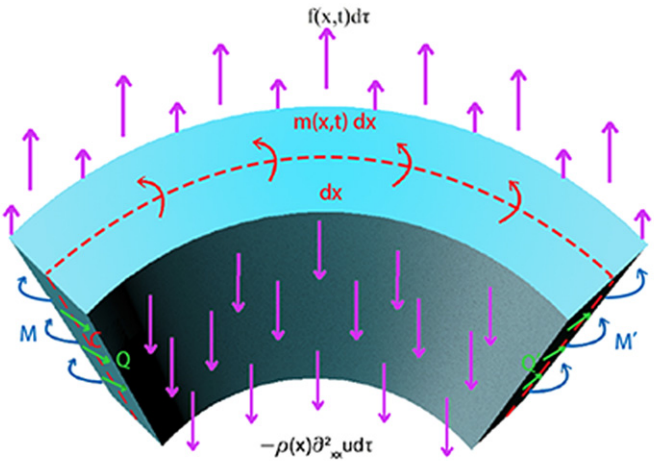

Figure 3. The forces acting on an element of a thin homogenous rod.

$$\rho(x)\partial_{tt}^2 u(x,\, t) + \frac{Y}{s(x)}\partial_{xx}^2\left[s(x)r^2(x)\partial_{xx}^2 u(x,\, t)\right] = f(x,\, t) - \partial_x \frac{m(x,\, t)}{s(x)}. \tag{8}$$

This is just the Euler–Bernoulli equation [7, 8, 31]. For the thin homogenous rod with a load of mass m' at x', equation (8) is simplified as

$$\partial_{xxxx}^4 u(x,\, t) + \frac{1}{Yr^2}\left[\rho + \frac{m'}{s}\delta(x - x')\right]\partial_{tt}^2 u(x,\, t) = 0,\ (0 \leqslant x \leqslant l,\ 0 < x' < l,\ 0 \leqslant t < \infty). \tag{9}$$

We take the variables separation method for $u(x,\, t) = \tilde{X}(x)\mathrm{e}^{\mathrm{i}\omega t}$ with the normal vibration circular frequency ω, and introduce three dimensionless lengths: space position $\bar{x} = x/l\,(0 \leqslant \bar{x} \leqslant 1)$, load position $\bar{x}' = x'/l$, and displacement function $X(\bar{x}) = \tilde{X}(x)/l$, $X(\bar{x})$ satisfies the following eigenvalue equation

$$X''''(\bar{x}) = k^4\left[1 + \frac{m'}{m}\delta\left(\bar{x} - \bar{x}'\right)\right]X(\bar{x}), \tag{10}$$

with k being the dimensionless momentum. This parameter satisfies $k^4 = \frac{\rho l^4}{Yr^2}\omega^2$. From this eigenvalue, the normal vibration circular frequency is $\omega = rl^{-2}\sqrt{Y/\rho}\,k^2$.

For the boundary conditions, if the end of $x = 0$ is fixed, then

$$X(0) = X'(0) = 0, \tag{11}$$

with the notation of $X'(0) = (\mathrm{d}X/\mathrm{d}\bar{x})|_{\bar{x}=0}$, while the end of $x = l$ is free,

$$X''(1) = X'''(1) = 0,\ \left(\bar{x}' < 1\right). \tag{12}$$

We divide the finite region $0 \leqslant \bar{x} \leqslant 1$ into two parts: $0 \leqslant \bar{x} \leqslant \bar{x}'$ and $\bar{x}' \leqslant \bar{x} \leqslant 1$ with $0 < \bar{x}' < 1$. In $\bar{x} = \bar{x}'$, we have the following connection conditions. Integrating equation (10) over $\bar{x}$ on interval $[\bar{x}'_-,\, \bar{x}'_+]$ with $\bar{x}'_\pm = \bar{x}' \pm \varepsilon$, and then taking $\varepsilon \to 0^+$, we get

$$X'''\left(\bar{x}'_{+}\right) - X'''\left(\bar{x}'_{-}\right) = \frac{m'}{m} k^4 X\left(\bar{x}'\right), \left(0 < \bar{x}' < 1\right). \tag{13}$$

There is a jump at $X'''(\bar{x}')$ due to the rod carrying a concentrated mass at $\bar{x}' \neq 1$. If the load is located at the endpoint of $x' = l$, equations (11) and (12) become

$$X''(1) = 0, \text{ and } - X'''(1) = \frac{m'}{m} k^4 X(1). \tag{14}$$

For $0 < \bar{x}' < 1$, the continuous connection conditions at the both sides of the jump are

$$X\left(\bar{x}'_{+}\right) = X\left(\bar{x}'_{-}\right), X'\left(\bar{x}'_{+}\right) = X'\left(\bar{x}'_{-}\right), \text{ and } X''\left(\bar{x}'_{+}\right) = X''\left(\bar{x}'_{-}\right). \tag{15}$$

We use a linear combination of such allowed solutions, expressed as the sum of the cosine wave, hyperbolic cosine wave, sinusoids, and hyperbolic sinusoids of $k\bar{x}$, to describe the transverse vibrations of a rod of length l clamped at end of $x = 0$. The solutions of equation (10) are

$$X_{<}(\bar{x}) = A\left(\cosh k\bar{x} - \cos k\bar{x}\right) + B\left(\sinh k\bar{x} - \sin k\bar{x}\right), \left(0 \leqslant \bar{x} \leqslant \bar{x}'_{-}\right). \tag{16}$$

$$X_{>}(\bar{x}) = C \cosh k\bar{x} + D \cos k\bar{x} + F \sinh k\bar{x} + G \sin k\bar{x}, \left(\bar{x}'_{+} \leqslant \bar{x} \leqslant 1\right). \tag{17}$$

They satisfy the conditions (11). From the conditions (12), we have

$$k^{-2} X''_{>}(1) = C \cosh k - D \cos k + F \sinh k - G \sin k. \tag{18}$$

$$k^{-3} X''_{>}{}'(1) = C \sinh k + D \sin k + F \cosh k - G \cos k. \tag{19}$$

From the continuous connection conditions in equation (15) with $k' \equiv k\bar{x}'$, we have

$$\begin{aligned} &A(\cosh k' - \cos k') + B(\sinh k' - \sin k') \\ &= C \cosh k' + D \cos k' + F \sinh k' + G \sin k', \end{aligned} \tag{20}$$

$$\begin{aligned} &A(\sinh k' + \sin k') + B(\cosh k' - \cos k') \\ &= C \sinh k' - D \sin k' + F \cosh k' + G \cos k', \end{aligned} \tag{21}$$

$$\begin{aligned} &A(\cosh k' + \cos k') + B(\sinh k' + \sin k') \\ &= C \cosh k' - D \cos k' + F \sinh k' - G \sin k'. \end{aligned} \tag{22}$$

The jump at equation (13) leads to

$$\begin{aligned} &- A(\sinh k' - \sin k') - B(\cosh k' + \cos k') \\ &+ C \sinh k' + D \sin k' + F \cosh k' - G \cos k' \\ &= k\frac{m'}{m}[A(\cosh k' - \cos k') + B(\sinh k' - \sin k')]. \end{aligned} \tag{23}$$

Six equations (18)–(23) comprise a set of linear homogeneous equations with six coefficient variables (A, B, C, D, F, G), in which the physical variables are m'/m and x'/l with the parameter k to be determined. The condition of non-zero solution of equations (18)–(23) is the vanishing determinant. So the eigenvalue equation of k is

$$
\begin{aligned}
&\frac{2\,m}{m'k}(1+\cos k\cosh k)\\
&=\sin k'\cosh k'-\cos k'\sinh k'+\sinh(k-k')\cos(k-k')\\
&\quad-\sin(k-k')\cosh(k-k')+\sin k\cosh k'\cosh(k-k')\\
&\quad-\sinh k\cos k'\cos(k-k').
\end{aligned}
\tag{24}
$$

From this equation we can obtain the eigenvalues k_n as a function of x'/l, m'/m for $n = 0, 1, 2,...$. They determine the available frequencies of the rod

$$
\omega_n=\frac{r}{l^2}\sqrt{\frac{Y}{\rho}}k_n^2\left(\frac{x'}{l},\frac{m'}{m}\right).
\tag{25}
$$

In some special cases, equation (24) can be simplified. For example, it becomes equation (6) of [13] for $c = \infty$ (the end of $x = l$ is hinged by a rotational spring of constant stiffness c; in our work, it is easy to include the finite c). It also becomes $1+\cos k\cosh k = k\frac{m'}{m}(\sin k\cosh k-\cos k\sinh k)$ for $\bar{x}'=1$. It can further be simplified as $1+\cos k\cosh k=0$ for $m'=0$ without a load. In this case, Young's modulus is $Y=38.32\,l^4\rho f^2/h^2$ after solving the fundamental eigenvalue k_0 with $f=\omega/(2\pi)$. This is the theoretical basis of Young's modulus measurement in the traditional dynamic method [17]. And now we extend it by adding two degrees of freedom of x'/l and m'/m. The traditional dynamic method is only a special case of our model.

3. Experiments and results

Experimental setup consists of the following apparatus: two thin homogenous rods, several magnet loads, an oscilloscope, two supports and bases, and a photometer. We do not draw the schematic diagram of the experimental apparatus and setups. The heavy base is used to fix the homogenous rod. The photometer records the time intervals of the rod blocking light and outputs the voltage signal to the oscilloscope. There is a linear light source on one side of the photometer and a photo-resistor on the other side. The light intensity changes the resistance of photometer, which changes the voltage of the photo-resistor. Without obstruction in front of the photometer, the light irradiates the photo-resistor to lower its resistance; otherwise, the resistance increases. Thus, the vibration frequencies of the rod can be read by observing the periods of the voltage signal.

In the experiments, we put two magnet loads at x' of the rod on the y-direction symmetrically, and clap the endpoint of $x = l$ to start the transverse vibrations. In the beginning, there is a fundamental frequency and higher harmonic frequencies. After a moment, the higher harmonic frequencies decay quickly, and the rod vibrates at the fundamental frequency. At this time, the fundamental frequency can be read out by observing the wave shape on the oscilloscope. By adding two magnet loads at the same position symmetrically, we measure the fundamental frequency and obtain the mass-dependent fundamental frequency at the fixed x'. For a given x', on the other hand, the eigenvalues k, which correspond to the different mass of loads, are obtained by solving equation (24) numerically. From equation (25), it is clear that the fundamental frequency is proportional to k^2. Thus Young's modulus can be fitted by the least square method according to the linear relationship between frequencies ω and eigenvalues k^2.

In the similar approach, keeping the masses of loads and changing the positions of loads, the eigenvalues k that correspond to the different positions of loads are solved from equation (24) numerically and Young's modulus can be fit by the least square method.

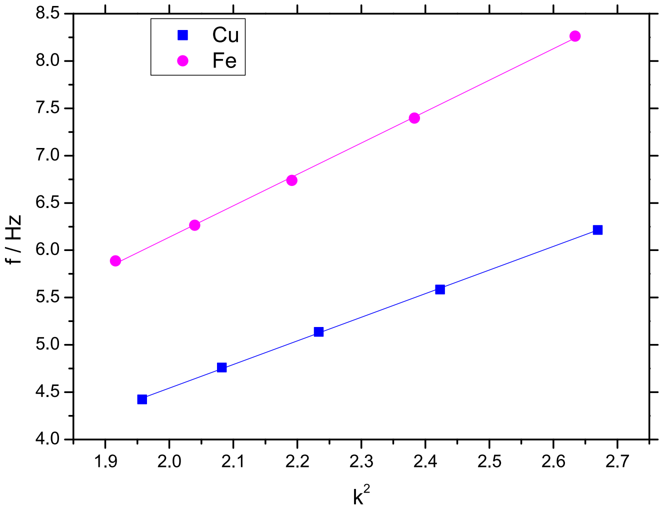

Figure 4. The vibration frequencies f in the units of Hz versus the dimensionless eigenvalues k^2 with the different load masses for two kinds of thin homogenous Fe (red) and Cu (blue) rods.

From equation (24), we can see that $f = \zeta k^2$ with the slope $\zeta = \frac{eq1}{2\pi l^2}\sqrt{\frac{Y}{\rho}}$. For two kinds of thin homogeneous rods, iron (Fe) and copper (Cu), the experimental parameters are $l = 0.26$ m, $h = 1.020 \times 10^{-3}$ m, $r = 2.945 \times 10^{-4}$ m, and $\rho = 7.90 \times 10^3$ kg m^{-3}, and $l = 0.26$ m, $h = 1.027 \times 10^{-3}$ m, $r = 2.965 \times 10^{-4}$m, and $\rho = 8.34 \times 10^3$ kg m^{-3}, respectively. Our observations for f and calculation on k^2 versus the different masses of the loads are shown in figure 4. The red dots (blue squares) are the experimental data for the Fe (Cu) rod; while the red line (blue line) is the theoretical result for the Fe (Cu) rod. By using the least square method and data from figure 4, we obtain the slope $\zeta_{Fe} = (3.31 \pm 0.05)$Hz and Young's modulus $Y_{Fe} = (1.81 \pm 0.03) \times 10^{11}$ N m^{-2} for the iron rod, and the slope $\zeta_{Cu} = (2.49 \pm 0.05)$Hz and Young's modulus $Y_{Cu} = (1.06 \pm 0.03) \times 10^{11}$ N m^{-2} for the copper rod. The speeds of sound for the shear wave $v = \sqrt{Y/\rho}$ are $v_{Fe} = 4.787 \times 10^3$ m s^{-1} and $v_{Cu} = 3.565 \times 10^3$ m s^{-1} in our experiments.

With a similar approach, our observations for f and calculation on k^2 versus the different positions of the loads are shown in figure 5. We obtain $\zeta_{Fe} = (3.27 \pm 0.05)$Hz and $Y_{Fe} = (1.75 \pm 0.02) \times 10^{11}$ N m^{-2} for the iron rod, and $\zeta_{Cu} = (2.43 \pm 0.05)$ Hz and $Y_{Cu} = (1.00 \pm 0.01) \times 10^{11}$ N m^{-2} for the copper rod.

To compare our method with other methods, we independently determine Young's modulus of the above rods with the three-point bending method [17–20]. This is a traditional method in experimental physics that teaches measuring of Young's modulus of materials. The displacement Δz at the position of loads $x = l/2$ is linear with the total mass $M' = m' + M$, i.e., $\Delta z = \xi M'$ with the slope $\xi = \frac{d^3 g}{4 h^3 w Y}$, the load mass m' and the tray mass M. Here $d = 0.23$m is the distance between the support, g is the gravitational acceleration, and $w = 0.023$m is the width of the rod.

Our observations for Δz as a function of m' are shown in figure 6. The red dots (blue squares) are the experimental data for the above Fe (Cu) rod, while the red line (blue line) is the linear fitting for the same Fe (Cu) rod. Here, ξ is the slope of the straight line and ξM is its ordinate at the origin. From figure 6, we can fit the slopes $\xi_{Fe} = 0.0066$ m kg^{-1} and

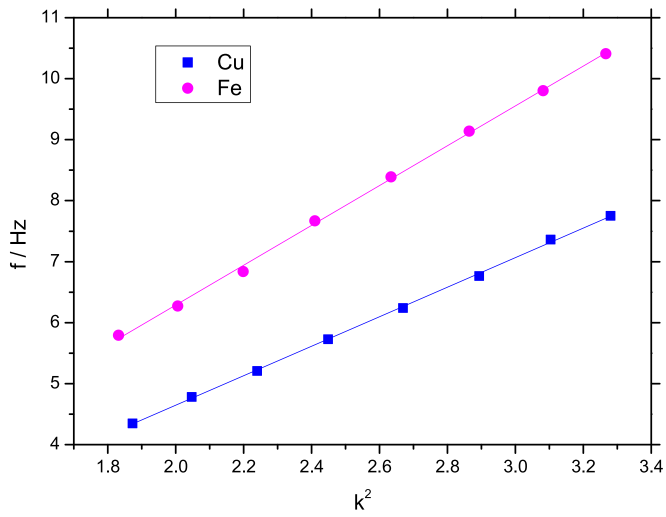

Figure 5. The vibration frequencies f in the units of Hz versus the dimensionless eigenvalues k^2 with the different load positions for two kinds of thin homogenous Fe (red) and Cu (blue) rods.

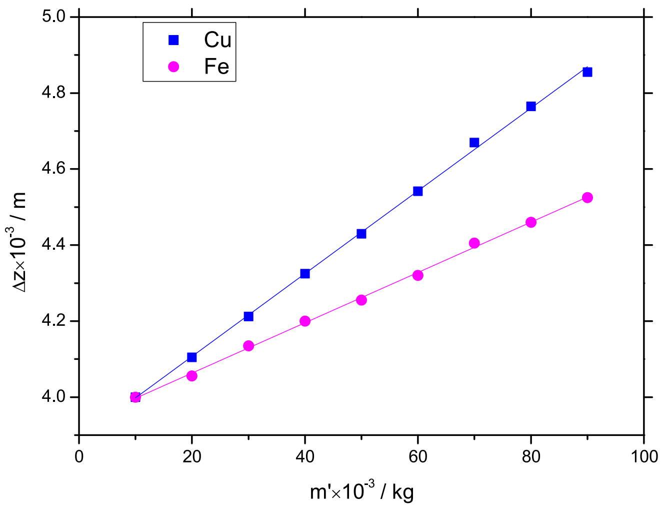

Figure 6. The displacement Δz in the units of 10^{-3}m versus the mass m' of loads in the units of 10^{-3}kg for two kinds of thin homogenous Fe (red) and Cu (blue) rods.

$\xi_{\mathrm{Cu}} = 0.0109$ m kg^{-1}, and Young's moduli $Y_{\mathrm{Fe}} = (1.79 \pm 0.02) \times 10^{11}$ N m^{-2} and $Y_{\mathrm{Cu}} = (1.07 \pm 0.02) \times 10^{11}$ N m^{-2}.

4. Discussions and conclusions

The results of Young's moduli are all essentially the same for three methods and the relative error is about 2%. From the effective radius of our rod $\sqrt{wh/\pi} = 0.0027$ m $\ll l = 0.26$ m, it is indeed a thin rod and enables us to appreciate the validity of the linear approximations. The amplitude of vibration dependence (maximum level of used stress/strain compare to the elasticity limits) is also in the linear regime. The mechanical quality of all bars used for the measurement was high enough not to influence the measurement method's accuracy substantially. So this is a new dynamical method to measure Young's modulus of a rod, in which we extend the frequency to the load's position and mass dependent eigenvalue.

The method that changes mass of loads is better than the one that changes positions of loads. This is because (1) the mass of the rod $m = 48.12 \times 10^{-3}$ kg is comparable to the mass of the loads $m' = (10 \sim 90) \times 10^{-3}$ kg and (2) the load is approximated to a point mass and described by Dirac function $\delta(x - x')$. However, the added magnets in the practical case have finite sizes. Since the higher harmonic frequencies decay quickly due to the vibration damping, the rod vibrates at the fundamental frequency.

The results illustrated in this article are potentially interesting for educational purposes for undergraduate physics students. For graduate students, the elastic plates, which have rotatory inertia and shear [32], and the rods, which have nonlinear elastic effects [31, 33], are especially interesting in a didactic nature.

In summary, we have constructed the general formulation of a determinant solution problem for the transverse vibrations of a thin loaded rod. We have exactly solved the vibration frequencies of a thin homogeneous rod carrying a point mass as a function of loads position and mass. Based on this model, we have presented a new kind of dynamic measurement method of Young's modulus. Young's modulus of the rod is determined by recombining the model and measurements. This method avoids the disadvantages of the bending method that cannot measure fragile materials. Compared to the traditional dynamic method, our method does not need to change the length of the sample and only needs to change the load's mass or position, so it has advantages when the length of the rod is difficult to change. The theory on fast determination of Young's modulus has been verified by our experiments and calculations in this new dynamic method. Theoretical analysis and experimental results suggest that the proposed method is a useful tool to study the dynamics of the rods.

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