曲面上张量场沿坐标线的二阶偏导数

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1　知识要素

1.1　Riemann-Christoffel 张量

定义 1.1 (Riemann-Christoffel 张量)**.** Riemann-Christoffel 张量是四阶张量, 其分量定义为

$$R_{ijpq} = b_{ip}b_{jq} - b_{jp}b_{iq}.$$

性质 1.1 (Riemann-Christoffel 张量的基本性质)**.**

1. 指标升降关系: $R^{i\cdot p}_{\cdot j\cdot q} = b^{ip}b_{jq} - b^p_j b^i_q$;

2. $R^{i\cdot p}_{\cdot j\cdot q} = -R^{\cdot ip\cdot}_{j\cdot\cdot q}$,　$R^{i\cdot p}_{\cdot j\cdot q} = -R^{i\cdot\cdot p}_{\cdot jq\cdot}$;

3. $R^{i\cdot p}_{\cdot j\cdot q} = R^{p\cdot i}_{\cdot q\cdot j}$.

证明 按 Riemann-Christoffel 张量的定义和指标升降关系, 可有

1.
$$R^{i\cdot p\cdot}_{\cdot j\cdot q} = g^{is}g^{pt}R_{sjtq} = g^{is}g^{pt}(b_{st}b_{jq} - b_{sq}b_{jt}) = b^{ip}b_{jq} - b^p_j b^i_q.$$

2.
$$\begin{aligned} R^{i\cdot p}_{\cdot j\cdot q} =& b^{ip}b_{jq} - b^p_j b^i_q = -(b^p_j b^i_q - b^{ip}b_{jq}) = -R^{\cdot ip\cdot}_{j\cdot\cdot q}; \\ R^{i\cdot p}_{\cdot j\cdot q} =& b^{ip}b_{jq} - b^p_j b^i_q = -(b^p_j b^i_q - b^{ip}b_{jq}) = -R^{i\cdot\cdot p}_{\cdot jq\cdot}. \end{aligned}$$

3.
$$R^{i\cdot p}_{\cdot j\cdot q} = b^{ip}b_{jq} - b^p_j b^i_q = b^{pi}b_{qj} - b^i_q b^p_j = R^{p\cdot i}_{\cdot q\cdot j}.$$ □

另外, 对于二维曲面成立下述定理表述的关系.

定理 1.2 (二维曲面上的 Riemann-Christoffel 张量同度量张量之间的关系)**.**

$$R_{ijpq} = b_{ip}b_{jq} - b_{jp}b_{iq} = K_G(g_{ip}g_{jq} - g_{jp}g_{iq}).$$

证明 根据 Gauss 曲率 K_G 的定义, 有

$$K_G = \det\left(g^{ik} b_{kj}\right) = \frac{\det\left(b_{ij}\right)}{\det\left(g_{ij}\right)},$$

亦即有

$$K_G \det\left(g_{ij}\right) = \det\left(b_{ij}\right).$$

在二维的情况下, 上式即为

$$K_G \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.$$

即有

$$K_G \begin{vmatrix} g_{ip} & g_{iq} \\ g_{jp} & g_{jq} \end{vmatrix} = \begin{vmatrix} b_{ip} & b_{iq} \\ b_{jp} & b_{jq} \end{vmatrix}.$$

亦即

$$K_G(g_{ip}g_{jq} - g_{jp}g_{iq}) = b_{ip}b_{jq} - b_{jp}b_{iq}. \qquad \square$$

定义 1.2 (Ricci 张量)**.** Ricci 张量是二阶张量, 其分量定义为

$$R_{ij} = R^{s}_{\cdot isj} = R_{i\cdot js}^{\cdot s}.$$

定义 1.3 (数量曲率)**.** 数量曲率是一个标量, 定义为

$$R = R^{i}_{\cdot i} = \mathrm{tr}\boldsymbol{R}.$$

定理 1.3. 对于二维曲面, 有

$$R_{ij} = \frac{1}{2} R g_{ij} = K_G g_{ij}.$$

证明 首先有

$$R_{ij} = R^{s}_{\cdot isj} = K_G(\delta^s_s g_{ij} - g_{is}\delta^s_j) = K_G g_{ij},$$

而

$$\begin{aligned} \frac{1}{2} R g_{ij} &= \frac{1}{2} R^{s\cdot}_{\cdot s} g_{ij} = \frac{1}{2} R^{s\cdot\cdot t}_{\cdot ts\cdot} g_{ij} = \frac{1}{2} K_G(\delta^s_s \delta^t_t - g_{ts}g^{st}) g_{ij} \\ &= \frac{1}{2} K_G(4-2) g_{ij} = K_G g_{ij}. \end{aligned} \qquad \square$$

1.2 Ricci 等式与 Codazzi 方程

现在, 研究曲面上的向量场 $\boldsymbol{A}(\boldsymbol{x}_\Sigma) = A^i(\boldsymbol{x}_\Sigma)\boldsymbol{g}_i(\boldsymbol{x}_\Sigma) \in T\Sigma$. 考虑该张量场在某坐标线上的变化率:

$$\begin{aligned} \frac{\partial \boldsymbol{A}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) &= \frac{\partial}{\partial x^p_\Sigma}(A^s \boldsymbol{g}_s)(\boldsymbol{x}_\Sigma) = \frac{\partial A^s}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\boldsymbol{g}_s + A^s(\varGamma^t_{ps}\boldsymbol{g}_t + b_{ps}\boldsymbol{n}) \\ &= \left(\frac{\partial A^t}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) + \varGamma^t_{ps}A^s\right)\boldsymbol{g}_t + A^s b_{ps}\boldsymbol{n} = \nabla_p A^t \boldsymbol{g}_t + A^s b_{ps}\boldsymbol{n}. \end{aligned}$$

此表达式利用了针对曲面张量分量的曲面协变导数, 对任意曲面上仿射量场 $\boldsymbol{\Theta} = \Theta^{i\cdot}_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ (向量场同理), 其定义如下:

$$\nabla_l \Theta^i_{\cdot j} \triangleq \frac{\partial \Theta^i_{\cdot j}}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma) + \Gamma^i_{ls}\Theta^s_{\cdot j} - \Gamma^s_{lj}\Theta^i_{\cdot s},$$

所以

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{A}}{\partial x^q_\Sigma \partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) &= \frac{\partial}{\partial x^q_\Sigma}(\nabla_p A^t)(\boldsymbol{x}_\Sigma)\boldsymbol{g}_t + \nabla_p A^t(\Gamma^s_{qt}\boldsymbol{g}_s + b_{qt}\boldsymbol{n}) + \frac{\partial}{\partial x^q_\Sigma}(A^s b_{ps})\boldsymbol{n} + A^s b_{ps}(-b^t_q)\boldsymbol{g}_t \\
&= \left[\frac{\partial}{\partial x^q_\Sigma}(\nabla_p A^s)(\boldsymbol{x}_\Sigma) + \Gamma^s_{qt}\nabla_p A^t\right]\boldsymbol{g}_s + (\nabla_p A^t)b_{qt}\boldsymbol{n} + \frac{\partial}{\partial x^q_\Sigma}(A^s b_{ps})\boldsymbol{n} \\
&\quad - A^s b_{ps} b^t_q \boldsymbol{g}_t \\
&= \left[\frac{\partial}{\partial x^q_\Sigma}(\nabla_p A^s)(\boldsymbol{x}) + \Gamma^s_{qt}\nabla_p A^t - A^t b_{pt} b^s_q\right]\boldsymbol{g}_s + \left[b_{qt}\nabla_p A^t + \frac{\partial}{\partial x^q_\Sigma}\left(A^t b_{pt}\right)\right]\boldsymbol{n}.
\end{aligned}$$

按有限维 Euclid 上的微分学, 一定有关系式

$$\frac{\partial^2 \boldsymbol{A}}{\partial x^q_\Sigma \partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) = \frac{\partial^2 \boldsymbol{A}}{\partial x^p_\Sigma \partial x^q_\Sigma}(\boldsymbol{x}_\Sigma).$$

按 $\boldsymbol{g}_s$ 项的系数平衡, 有

$$\frac{\partial}{\partial x^q_\Sigma}(\nabla_p A^s)(\boldsymbol{x}_\Sigma) + \Gamma^s_{qt}\nabla_p A^t - A^t b_{pt} b^s_q = \frac{\partial}{\partial x^p_\Sigma}(\nabla_q A^s)(\boldsymbol{x}_\Sigma) + \Gamma^s_{pt}\nabla_q A^t - A^t b_{qt} b^s_p.$$

在上式两端加上 $-\Gamma^t_{qp}\nabla_t A^s$, 将有

$$\nabla_q\nabla_p A^s - b_{pt}b^s_q A^t = \nabla_p\nabla_q A^s - b_{qt}b^s_p A^t,$$

即

$$\nabla_q\nabla_p A^s - \nabla_p\nabla_q A^s = (b_{pt}b^s_q - b_{qt}b^s_p)A^t.$$

根据 Riemann-Christoffel 张量的定义, 有

$$\nabla_q\nabla_p A^s - \nabla_p\nabla_q A^s = R^s_{\cdot tqp}A^t,$$

称为 **Gauss 方程**.

下面考虑曲面上仿射量场 $\boldsymbol{\Phi}(\boldsymbol{x}_\Sigma) = \Phi^i_{\cdot j}(\boldsymbol{x}_\Sigma)\boldsymbol{g}_i(\boldsymbol{x}_\Sigma) \otimes \boldsymbol{g}^j(\boldsymbol{x}_\Sigma)$, 则有

$$\begin{aligned}
\frac{\partial \boldsymbol{\Phi}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) &= \frac{\partial \Phi^i_{\cdot j}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\boldsymbol{g}_i \otimes \boldsymbol{g}^j + \Phi^i_{\cdot j}\frac{\partial \boldsymbol{g}_i}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) \otimes \boldsymbol{g}^j + \Phi^i_{\cdot j}\boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}^j}{\partial x^p_\Sigma}(\boldsymbol{x}) \\
&= \frac{\partial \Phi^i_{\cdot j}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\boldsymbol{g}_i \otimes \boldsymbol{g}^j + \Phi^i_{\cdot j}\left(\Gamma^k_{pi}\boldsymbol{g}_k + b_{pi}\boldsymbol{n}\right) \otimes \boldsymbol{g}^j + \Phi^i_{\cdot j}\boldsymbol{g}_i \otimes \left(-\Gamma^j_{pk}\boldsymbol{g}^k + b^j_p\boldsymbol{n}\right) \\
&= \left[\frac{\partial \Phi^i_{\cdot j}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) + \Gamma^i_{pk}\Phi^k_{\cdot j} - \Gamma^k_{pj}\Phi^i_{\cdot k}\right]\boldsymbol{g}_i \otimes \boldsymbol{g}^j + \Phi^i_{\cdot j}b^j_p\boldsymbol{g}_i \otimes \boldsymbol{n} + \Phi^i_{\cdot j}b_{pi}\boldsymbol{n} \otimes \boldsymbol{g}^j \\
&= \nabla_p\Phi^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j + \Phi^i_{\cdot j}b^j_p\boldsymbol{g}_i \otimes \boldsymbol{n} + \Phi^i_{\cdot j}b_{pi}\boldsymbol{n} \otimes \boldsymbol{g}^j.
\end{aligned}$$

进一步计算, 可有

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^q_\Sigma \partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) = \left[\frac{\partial}{\partial x^q_\Sigma}(\nabla_p \Phi^i_{\cdot j}) + \Gamma^i_{sq}\nabla_p \Phi^s_{\cdot j} - \Gamma^s_{qj}\nabla_p \Phi^i_{\cdot s} - \Phi^i_{\cdot s} b^s_p b_{qj} - \Phi^s_{\cdot j} b_{ps} b^i_q\right] \boldsymbol{g}_i \otimes \boldsymbol{g}^j$$
$$+ \left[b^j_q \nabla_p \Phi^i_{\cdot j} + \nabla_q (\Phi^i_{\cdot j} b^j_p)\right] \boldsymbol{g}_i \otimes \boldsymbol{n} + \left[b_{qi}\nabla_p \Phi^i_{\cdot j} + \nabla_q(\Phi^i_{\cdot j} b_{pi})\right] \boldsymbol{n} \otimes \boldsymbol{g}^j$$
$$+ \Phi^i_{\cdot j}(b_{pi} b^j_q + b^j_p b_{qi}) \boldsymbol{n} \otimes \boldsymbol{n}.$$

按张量赋范线性空间上微分学, 可有

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^q_\Sigma \partial x^p_\Sigma}(\boldsymbol{x}_\Sigma) = \frac{\partial^2 \boldsymbol{\Phi}}{\partial x^p_\Sigma \partial x^q_\Sigma}(\boldsymbol{x}_\Sigma).$$

按 $\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ 的系数平衡, 有

$$\frac{\partial}{\partial x^q_\Sigma}(\nabla_p \Phi^i_{\cdot j}) + \Gamma^i_{sq}\nabla_p \Phi^s_{\cdot j} - \Gamma^s_{qj}\nabla_p \Phi^i_{\cdot s} - \Phi^i_{\cdot s} b^s_p b_{qj} - \Phi^s_{\cdot j} b_{ps} b^i_q$$
$$= \frac{\partial}{\partial x^p_\Sigma}(\nabla_q \Phi^i_{\cdot j}) + \Gamma^i_{sp}\nabla_q \Phi^s_{\cdot j} - \Gamma^s_{pj}\nabla_q \Phi^i_{\cdot s} - \Phi^i_{\cdot s} b^s_q b_{pj} - \Phi^s_{\cdot j} b_{qs} b^i_p.$$

在上式两端加上 $\Gamma^s_{qp}\nabla_s \Phi^i_{\cdot j}$, 将有

$$\nabla_q \nabla_p \Phi^i_{\cdot j} + \Phi^i_{\cdot s} b^s_p b_{qj} - \Phi^s_{\cdot j} b_{ps} b^i_q = \nabla_p \nabla_q \Phi^i_{\cdot j} - \Phi^i_{\cdot s} b^s_q b_{pj} - \Phi^s_{\cdot j} b_{qs} b^i_p.$$

根据 Riemann-Christoffel 张量的定义, 有

$$\nabla_q \nabla_p \Phi^i_{\cdot j} - \nabla_p \nabla_q \Phi^i_{\cdot j} = R^{i\cdot\cdot\cdot}_{\cdot tqp}\Phi^t_{\cdot j} + R^{\cdot t\cdot\cdot}_{j\cdot qp}\Phi^i_{\cdot t}.$$

上式为 Gauss 方程的推广, 称为 **Ricci 恒等式**.

按 $\boldsymbol{n} \otimes \boldsymbol{g}^j$ 的系数平衡有

$$b_{qi}\nabla_p \Phi^i_{\cdot j} + \nabla_q(\Phi^i_{\cdot j} b_{pi}) = b_{pi}\nabla_q \Phi^i_{\cdot j} + \nabla_p(\Phi^i_{\cdot j} b_{qi}).$$

因为

$$b_{qi}\nabla_p \Phi^i_{\cdot j} + \nabla_q(\Phi^i_{\cdot j} b_{pi}) = b_{qi}\nabla_p \Phi^i_{\cdot j} + b_{pi}\nabla_q \Phi^i_{\cdot j} + \Phi^i_{\cdot j}(\nabla_q b_{pi}),$$

所以平衡方程变为

$$\nabla_q b_{pi} = \nabla_p b_{qi},$$

称为 **Codazzi 方程**.

再按 $\boldsymbol{g}_i \otimes \boldsymbol{n}$ 的系数平衡, 有

$$b^j_q \nabla_p \Phi^i_{\cdot j} + \nabla_q(\Phi^i_{\cdot j} b^j_p) = b^j_p \nabla_q \Phi^i_{\cdot j} + \nabla_p(\Phi^i_{\cdot j} b^j_q).$$

由于

$$b^j_q \nabla_p \Phi^i_{\cdot j} + \nabla_q(\Phi^i_{\cdot j} b^j_p) = b^j_q \nabla_p \Phi^i_{\cdot j} + b^j_p \nabla_q \Phi^i_{\cdot j} + \Phi^i_{\cdot j}\nabla_q b^j_p,$$

此时平衡方程同样变为 Codazzi 方程

$$\nabla_q b_{pi} = \nabla_p b_{qi}.$$

按 $\boldsymbol{n}\otimes\boldsymbol{n}$ 的系数平衡, 有

$$\Phi^{i}_{\cdot j}(b_{pi}b^{j}_{q}+b^{j}_{p}b_{qi})=\Phi^{i}_{\cdot j}(b_{qi}b^{j}_{p}+b^{j}_{q}b_{pi}).$$

上式是恒成立的.

进一步, 考虑下面的曲面上仿射量场 $\boldsymbol{\Phi}(\boldsymbol{x}_\Sigma)=\Phi^{i}_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}\in\mathscr{T}^2(\mathbb{R}^3)$, 则有

$$\begin{aligned}
\frac{\partial\boldsymbol{\Phi}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)&=\frac{\partial\Phi^{i}_{\cdot 3}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^{i}_{\cdot 3}\frac{\partial\boldsymbol{g}_i}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\otimes\boldsymbol{n}+\Phi^{i}_{\cdot 3}\boldsymbol{g}_i\otimes\frac{\partial\boldsymbol{n}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\\
&=\frac{\partial\Phi^{i}_{\cdot 3}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^{i}_{\cdot 3}(\Gamma^j_{pi}\boldsymbol{g}_j+b_{pi}\boldsymbol{n})\otimes\boldsymbol{n}-\Phi^{i}_{\cdot 3}b_{pj}\boldsymbol{g}_i\otimes\boldsymbol{g}^j\\
&=\left[\frac{\partial\Phi^{i}_{\cdot 3}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)+\Gamma^i_{pj}\Phi^{j}_{\cdot 3}\right]\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^{i}_{\cdot 3}b_{pi}\boldsymbol{n}\otimes\boldsymbol{n}-\Phi^{i}_{\cdot 3}b_{pj}\boldsymbol{g}_i\otimes\boldsymbol{g}^j\\
&=\nabla_p\Phi^{i}_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^{i}_{\cdot 3}b_{pi}\boldsymbol{n}\otimes\boldsymbol{n}-\Phi^{i}_{\cdot 3}b_{pj}\boldsymbol{g}_i\otimes\boldsymbol{g}^j.
\end{aligned}$$

进一步计算

$$\begin{aligned}
\frac{\partial^2\boldsymbol{\Phi}}{\partial x^q_\Sigma\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)=&-\left[\nabla_q(\Phi^{i}_{\cdot 3}b_{pj})+b_{qj}\nabla_p\Phi^{i}_{\cdot 3}\right]\boldsymbol{g}_i\otimes\boldsymbol{g}^j-\Phi^{i}_{\cdot 3}(b_{pi}b_{qj}+b_{pj}b_{qi})\boldsymbol{n}\otimes\boldsymbol{g}^j\\
&+\left[\frac{\partial}{\partial x^q_\Sigma}(\nabla_p\Phi^{i}_{\cdot 3})(\boldsymbol{x}_\Sigma)+\Gamma^i_{qs}\Phi^{s}_{\cdot 3}-\Phi^{i}_{\cdot 3}b_{pj}b^j_q-\Phi^{j}_{\cdot 3}b_{pj}b^i_q\right]\boldsymbol{g}_i\otimes\boldsymbol{n}\\
&+\left[b_{qi}\nabla_p\Phi^{i}_{\cdot 3}+\frac{\partial}{\partial x^q_\Sigma}(\Phi^{i}_{\cdot 3}b_{pi})(\boldsymbol{x}_\Sigma)\right]\boldsymbol{n}\otimes\boldsymbol{n},
\end{aligned}$$

其基于微分学的恒等式为

$$\frac{\partial^2\boldsymbol{\Phi}}{\partial x^q_\Sigma\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)=\frac{\partial^2\boldsymbol{\Phi}}{\partial x^p_\Sigma\partial x^q_\Sigma}(\boldsymbol{x}_\Sigma).$$

按 $\boldsymbol{g}_i\otimes\boldsymbol{g}^j$ 的系数平衡, 有

$$\nabla_q(\Phi^{i}_{\cdot 3}b_{pj})+b_{qj}\nabla_p\Phi^{i}_{\cdot 3}=\nabla_p(\Phi^{i}_{\cdot 3}b_{qj})+b_{pj}\nabla_q\Phi^{i}_{\cdot 3},$$

将得到 Codazzi 方程

$$\nabla_q b_{pj}=\nabla_p b_{qj}.$$

按 $\boldsymbol{g}_i\otimes\boldsymbol{n}$ 的系数平衡, 有

$$\begin{aligned}
&\frac{\partial}{\partial x^q_\Sigma}(\nabla_p\Phi^{i}_{\cdot 3})(\boldsymbol{x}_\Sigma)+\Gamma^i_{qs}\Phi^{s}_{\cdot 3}-\Phi^{i}_{\cdot 3}b_{pj}b^j_q-\Phi^{j}_{\cdot 3}b_{pj}b^i_q\\
&\qquad=\frac{\partial}{\partial x^p_\Sigma}(\nabla_q\Phi^{i}_{\cdot 3})(\boldsymbol{x}_\Sigma)+\Gamma^i_{ps}\Phi^{s}_{\cdot 3}-\Phi^{i}_{\cdot 3}b_{qj}b^j_p-\Phi^{j}_{\cdot 3}b_{qj}b^i_p.
\end{aligned}$$

在上式两端加上 $-\Gamma^s_{pq}\nabla_s\Phi^{i}_{\cdot 3}$, 即可得到 Gauss 方程

$$\nabla_q\nabla_p\Phi^{i}_{\cdot 3}-\nabla_p\nabla_q\Phi^{i}_{\cdot 3}=R^i{}_{jqp}\Phi^{j}_{\cdot 3};$$

按 $\boldsymbol{n}\otimes\boldsymbol{g}^j$ 的系数平衡, 将得到恒等式

$$-\Phi^{i}_{\cdot 3}(b_{pi}b_{qj}+b_{pj}b_{qi})=-\Phi^{i}_{\cdot 3}(b_{qi}b_{pj}+b_{qj}b_{pi});$$

按 $\boldsymbol{n}\otimes\boldsymbol{n}$ 的系数平衡, 有

$$b_{qi}\nabla_p\varPhi^i_{\cdot 3}+\frac{\partial}{\partial x^q_\Sigma}(\varPhi^i_{\cdot 3}b_{pi})(\boldsymbol{x}_\Sigma)=b_{pi}\nabla_q\varPhi^i_{\cdot 3}+\frac{\partial}{\partial x^p_\Sigma}(\varPhi^i_{\cdot 3}b_{qi})(\boldsymbol{x}_\Sigma).$$

计算

$$b_{qi}\nabla_p\varPhi^i_{\cdot 3}+\frac{\partial}{\partial x^q_\Sigma}(\varPhi^i_{\cdot 3}b_{pi})(\boldsymbol{x}_\Sigma)=\frac{\partial\varPhi^i_{\cdot 3}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)b_{qi}+\frac{\partial\varPhi^i_{\cdot 3}}{\partial x^q_\Sigma}(\boldsymbol{x}_\Sigma)b_{pi}+\varGamma^i_{ps}\varPhi^s_{\cdot 3}b_{qi}+\varPhi^i_{\cdot 3}\frac{\partial b_{pi}}{\partial x^q_\Sigma}(\boldsymbol{x}_\Sigma),$$

故平衡条件为

$$\frac{\partial b_{pi}}{\partial x^q_\Sigma}(\boldsymbol{x}_\Sigma)+\varGamma^s_{pi}b_{qs}=\frac{\partial b_{qi}}{\partial x^p_\Sigma}(\boldsymbol{x}_\Sigma)+\varGamma^s_{qi}b_{ps}.$$

在上式两端加上 $-\varGamma^s_{pq}b_{si}$, 即可得到 Codazzi 方程

$$\nabla_q b_{pi}=\nabla_p b_{qi}.$$

上述分析揭示了体积上以及曲面上协变导数的本质差异——体积上的协变导数可以交换次序, 而曲面上的协变导数的次序交换需联系于 Riemann-Christoffel 张量.

2 应用事例

2.1 Levi-Civita 梯度算子

相对于曲面梯度算子, 可以形式上定义 **Levi-Civita 梯度算子** $\overset{C}{\boldsymbol{\nabla}}\equiv\boldsymbol{g}^l\nabla_{\frac{\partial}{\partial x^l_\Sigma}}$:

$$\begin{aligned}
\overset{C}{\boldsymbol{\nabla}}\circledcirc\boldsymbol{\varPhi}&\equiv\left(\boldsymbol{g}^l\nabla_{\frac{\partial}{\partial x^l_\Sigma}}\right)\circledcirc(\varPhi^i_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\varPhi^i_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\varPhi^3_{\cdot j}\boldsymbol{n}\otimes\boldsymbol{g}^j+\varPhi^3_{\cdot 3}\boldsymbol{n}\otimes\boldsymbol{n})\\
&\triangleq\boldsymbol{g}^l\circledcirc\nabla_{\frac{\partial}{\partial x^l_\Sigma}}(\varPhi^i_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\varPhi^i_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\varPhi^3_{\cdot j}\boldsymbol{n}\otimes\boldsymbol{g}^j+\varPhi^3_{\cdot 3}\boldsymbol{n}\otimes\boldsymbol{n})\\
&=\nabla_l\varPhi^i_{\cdot j}(\boldsymbol{g}^l\circledcirc\boldsymbol{g}_i)\otimes\boldsymbol{g}^j+\nabla_l\varPhi^i_{\cdot 3}(\boldsymbol{g}^l\circledcirc\boldsymbol{g}_i)\otimes\boldsymbol{n}+\nabla_l\varPhi^3_{\cdot j}(\boldsymbol{g}^l\circledcirc\boldsymbol{n})\otimes\boldsymbol{g}^j\\
&\quad+\nabla_l\varPhi^3_{\cdot 3}(\boldsymbol{g}^l\circledcirc\boldsymbol{n})\otimes\boldsymbol{n}.
\end{aligned}$$

不同于曲面梯度算子, Levi-Civita 梯度算子仅对张量分量相对于切空间的指标有效, 如上式中的指标 i 和 j. 一般而言, 对同一曲面上张量场, 其 Levi-Civita 梯度算子仅是其曲面梯度算子的一部分.

我们用 $\overset{\Sigma}{\boldsymbol{\nabla}}$ 代表曲面梯度算子, 用 $\overset{C}{\boldsymbol{\nabla}}$ 代表曲面 Levi-Civita 梯度算子. 值得指出, 曲面梯度算子基于微分学, 故有明确的极限定义及极限值; 而 Levi-Civita 算子是按曲面上 Levi-Civita 联络引入的形式上的定义. 一般而言, Levi-Civita 算子只是曲面梯度算子的一部内容.

3 建立路径

- 由于现研究的曲面位于 Euclid 空间, 故可以按极限观点定义曲面上张量场 (可以既有切空间上分量也可以有法向分量) 沿曲面上坐标线上的一阶, 二阶甚至高阶偏导数/变化率, 并且按极限分析 (基于 Landau 符号) 获得对于的极限值. 按一般赋范线性空间中的微分学, 现情形①张量场整体沿坐标线的偏导数可以交换次序.

① 设定张量场具有足够的光滑性/正则性.

- 由于张量场整体沿坐标线的二阶偏导数可以交换次序, 故对于的二个极限值需要一致, 由此引出 Ricci 等式以及 Codazzi 方程. Riemann-Christoffel 张量的引入可以是为了整理 Ricci 等式对应的结果.
- 按作者现有认识, 区分曲面梯度算子与 Levi-Civita 算子隶属力学与数学之间的关系; 基于这样的区分作者建立固定曲面上二位流动的涡量动力学理论, 主要由 Levi-Civita 算子建立主要的关系式.