复旦大学力学与工程科学学系

2012 ～ 2013 学年第二学期期末考试

□A 卷　　□B 卷

课程名称：连续介质力学基础　　　　课程代码：MECH130105

开课院系：力学与工程科学学系　　　考试形式：开卷/闭卷/课程论文

姓名：__________ 学号：__________ 专业：__________

题 号	1/(1)	1/(2)	1/(3)	2/(1)	2/(2)	2/(3)	2/(4)	3/(1)	3/(2)	3/(3)
得 分										
题 号	4/(1)	4/(2)	4/(3)	4/(4)	4/(5)	4/(6)				总 分
得 分										

Problem 1 (Fundamentals of curvilinear coordinates)

1. *To proof the following identity*

$$\nabla \times (\nabla \times \boldsymbol{\Phi})^* = \nabla \otimes (\boldsymbol{\Phi} \cdot \nabla) + (\nabla \cdot \boldsymbol{\Phi}) \otimes \nabla - \boldsymbol{\Delta\Phi} - \nabla \otimes (\nabla (tr\boldsymbol{\Phi})) \\ - \boldsymbol{I}[\nabla \cdot \boldsymbol{\Phi} \cdot \nabla - \boldsymbol{\Delta}(tr\boldsymbol{\Phi})]$$

for any symmetric affine tensor $\boldsymbol{\Phi} = \boldsymbol{\Phi}^* \in \mathscr{T}^2(\mathbb{R}^3)$ *through field analysis with respect to general curvilinear coordinates.*

2. *To consider a tensor represented in two point forms*

$$\boldsymbol{\Phi} = \Phi^{i \cdot B \cdot}_{\cdot A \cdot j}(\xi, x)\boldsymbol{g}_i(x) \otimes \boldsymbol{G}^A(\xi) \otimes \boldsymbol{G}_B(\xi) \otimes \boldsymbol{g}^j(x)$$

where $\{\boldsymbol{g}_i(x)\}_{i=1}^m$ *are covariant basis with respect to the curvilinear coordinates* $X(x) \in \mathscr{C}^p(\mathscr{D}_x, X(\mathscr{D}_x))$ *and* $\{\boldsymbol{G}_A(x)\}_{A=1}^m$ *to* $X(\xi) \in \mathscr{C}^p(\mathscr{D}_\xi, X(\mathscr{D}_\xi))$ *accompanying with the relations* $x = x(\xi) \in \mathscr{C}^p(\mathscr{D}_\xi, \mathscr{D}_x)$. *To deduce the representation of* $\frac{\partial}{\partial x^l}\boldsymbol{\Phi}(x)$ *through covariant differentiations with respect to different curvilinear coordinates.*

3. *The so-called transfer tensor is defined as*

$$\mathbf{I} = g^i_A(\xi, x)\, \mathbf{g}_i(x) \otimes \mathbf{G}^A(\xi), \quad g^i_A := (\mathbf{g}^i, \mathbf{G}_A)_{\mathbb{R}^m}$$

To prove one of the relations

$$\frac{\partial}{\partial \xi^L}\boldsymbol{I}(\xi) = \mathbf{0}, \quad \frac{\partial}{\partial x^l}\boldsymbol{I}(x) = \mathbf{0}$$

Problem 2 (Stain tensor on an arbitrary deformable surface) *Based on the intrinsic decomposition with respect to any direction*

$$\boldsymbol{\Phi} = \begin{cases} \boldsymbol{e} \otimes (\boldsymbol{e}, \boldsymbol{\Phi})_{\mathbb{R}^3} - [\boldsymbol{e}, [\boldsymbol{e}, \boldsymbol{\Phi}]] \\ (\boldsymbol{\Phi}, \boldsymbol{e})_{\mathbb{R}^3} \otimes \boldsymbol{e} - [[\boldsymbol{\Phi}, \boldsymbol{e}], \boldsymbol{e}] \end{cases} \quad \forall\, |\boldsymbol{e}|_{\mathbb{R}^3} = 1, \quad \forall \boldsymbol{\Phi} \in \mathscr{T}^p(\mathbb{R}^3)$$

the following representation of the strain tensor on an arbitrary deformable surface can be deduced

$$\begin{aligned} \boldsymbol{D} &\triangleq \frac{1}{2}(\boldsymbol{V} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \boldsymbol{V}) \\ &= \left(\theta - \overset{\Sigma}{\boldsymbol{\nabla}} \cdot \boldsymbol{V}\right) \boldsymbol{n} \otimes \boldsymbol{n} + \frac{1}{2}[(\boldsymbol{\omega} + \boldsymbol{W}) \times \boldsymbol{n}] \otimes \boldsymbol{n} + \frac{1}{2}\boldsymbol{n} \otimes [(\boldsymbol{\omega} + \boldsymbol{W}) \times \boldsymbol{n}] + \overset{\Sigma}{\boldsymbol{D}} \end{aligned}$$

where $\overset{\Sigma}{\boldsymbol{D}} \triangleq \left(\boldsymbol{V} \otimes \overset{\Sigma}{\boldsymbol{\nabla}} + \overset{\Sigma}{\boldsymbol{\nabla}} \otimes \boldsymbol{V}\right)/2$ *is the strain of the boundary,* $\boldsymbol{W} := -\left(\overset{\Sigma}{\nabla} V^3 + \boldsymbol{V} \cdot \boldsymbol{K}\right) \times \boldsymbol{n}$ *is purely determined by the boundary, and* $\theta := \boldsymbol{\nabla} \cdot \boldsymbol{V}$ *is the dilation.*

1. *To give the relation between the full dimensional gradient operator* $\boldsymbol{\nabla} \triangleq \boldsymbol{i}_\alpha \frac{\partial}{\partial X^\alpha}$ *and the surface gradient operator* $\overset{\Sigma}{\boldsymbol{\nabla}} \triangleq \boldsymbol{g}^l \frac{\partial}{\partial x^l_\Sigma}$*. The reason should be indicated.*

2. *To proof the following identity*

$$\boldsymbol{V} \otimes \boldsymbol{\nabla} = \left(\theta - \overset{\Sigma}{\boldsymbol{\nabla}} \cdot \boldsymbol{V}\right) \boldsymbol{n} \otimes \boldsymbol{n} + (\boldsymbol{\omega} \times \boldsymbol{n}) \otimes \boldsymbol{n} + (\boldsymbol{W} \times \boldsymbol{n}) \otimes \boldsymbol{n} - [[\boldsymbol{V} \otimes \boldsymbol{\nabla}, \boldsymbol{n}], \boldsymbol{n}]$$

3. *To fulfill the deduction.*

4. *To give the component matrix of* $\boldsymbol{D}$ *on a sphere which does the radical oscillation with fixed amplitude and frequency.*

Problem 3 (Some studies based on the intrinsic Stokes formulas) *We have attained the Stokes formula of the following forms*

$$\oint_C \boldsymbol{\tau} \circ - \boldsymbol{\Phi}\, dl = \int_\Sigma \left(\boldsymbol{n} \times \overset{\Sigma}{\boldsymbol{\nabla}}\right) \circ - \boldsymbol{\Phi}\, d\sigma$$

$$\oint_C (\boldsymbol{\tau} \times \boldsymbol{n}) \circ - \boldsymbol{\Phi}\, dl = \int_\Sigma \left(\overset{\Sigma}{\boldsymbol{\nabla}} \circ - \boldsymbol{\Phi} + H\boldsymbol{n} \circ - \boldsymbol{\Phi}\right) d\sigma$$

which are termed as the intrinsic Stokes formulas.

1. *On any deformable smooth surface, the following identity is keeping valid*

$$\left(\boldsymbol{n} \times \overset{\Sigma}{\boldsymbol{\nabla}}\right) \cdot (\boldsymbol{n} \times \boldsymbol{\Phi}) = \overset{\Sigma}{\boldsymbol{\nabla}} \cdot \boldsymbol{\Phi} + H\boldsymbol{n} \cdot \boldsymbol{\Phi}, \quad \forall \boldsymbol{\Phi} \in \mathscr{T}^p(\mathbb{R}^3)$$

   To prove the identity as mentioned above by the intrinsic Stokes formula of the second kind.

2. *To deduce the differential equation of mass conservation for the two dimensional compressible steady flow on an arbitrary fixed smooth surface.*

3. *To give the differential equation of mass conservation in detail for the two dimensional compressible steady flow on a fixed sphere.*

Problem 4 (Governing equations in Lagrangian variables) *The governing equations of continuum media represented in Lagrangian arguments could be listed as follows*

$$\begin{array}{ll} \textit{mass conservation} & \rho(\xi,t)\,|\mathbf{F}|(\xi,t) = \overset{\circ}{\rho}(\xi), \quad \mathbf{F} \triangleq \dfrac{\partial x^i}{\partial \xi^A}(\xi,t)\mathbf{g}_i(x,t) \otimes \mathbf{G}^A(\xi) \\ \textit{momentum conservation} & \overset{\circ}{\rho}(\xi)\,\mathbf{a}(\xi,t) = \overset{\circ}{\rho}(\xi)\,\mathbf{f}_m(\xi,t) + \left\{ \begin{array}{l} [\mathbf{t} \cdot (|\mathbf{F}|\mathbf{F}^{-*})] \cdot \overset{\circ}{\square} =: \tau \cdot \overset{\circ}{\square} \\ (\mathbf{F} \cdot \mathbf{T}) \cdot \overset{\circ}{\square}, \quad \mathbf{T} = \mathbf{F}^{-1} \cdot \tau \end{array} \right. \\ \textit{constitution relation} & \mathbf{T} = -p\,\overset{\circ}{\mathbf{C}}^{-1} + 2\left[\dfrac{\partial \Sigma}{\partial I_1}\mathbf{I} + \dfrac{\partial \Sigma}{\partial I_2}(I_1\,\mathbf{I} - \overset{\circ}{\mathbf{C}}^{-1})\right], \quad \overset{\circ}{\mathbf{C}} := \mathbf{F}^* \cdot \mathbf{F} \end{array}$$

where the constitution relation is corresponding to the incompressible flow.

The procedure to study the finite bending deflection of cube can be divided into following steps.

1. ***Configurations and curvilinear coordinates*** *The curvilinear coordinates with respect to the initial physical configuration can be just Cartesian coordinates, namely*

$$\xi = \begin{bmatrix} \overset{\circ}{X} \\ \overset{\circ}{Y} \\ \overset{\circ}{Z} \end{bmatrix} \mapsto \overset{\circ}{X}(\xi) = \begin{bmatrix} \overset{\circ}{X} \\ \overset{\circ}{Y} \\ \overset{\circ}{Z} \end{bmatrix} \triangleq \begin{bmatrix} \overset{\circ}{X} \\ \overset{\circ}{Y} \\ \overset{\circ}{Z} \end{bmatrix}$$

   The curvilinear coordinates with respect to the current physical configuration is familiar cylindrical coordinates, namely

$$x = \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} \mapsto X(x) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \triangleq \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$

2. ***Assumption of deformation*** *One can assume the deformation in the following form*

$$\xi = \begin{bmatrix} \overset{\circ}{X} \\ \overset{\circ}{Y} \\ \overset{\circ}{Z} \end{bmatrix} \mapsto x(\xi) = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} \triangleq \begin{bmatrix} r(\overset{\circ}{X}) \\ \theta(\overset{\circ}{Y}) \\ z(\overset{\circ}{Z}) \end{bmatrix}$$

3. ***Deformation gradient tensor*** *To calculate the component matrix of deformation gradient tensor* $\mathbf{F}$ *with respect to the anholonomic orthonormal bases.*

4. ***Strain tensor*** *To calculate the component matrices of strain tensor* $\overset{\circ}{\mathbf{C}}$ *with respect to the anholonomic orthonormal bases. Subsequently, its principle invariants can be determined readily.*

5. ***Piola-Kirchhoff stress tensor*** *To calculate the component matrices of Piola-Kirchhoff stress tensor of the second kind with respect to the anholonomic orthonormal bases.*

6. ***Momentum Equation*** *To calculate the component equations of the momentum conservation.*

The questions are

1. *To deduce the component matrix of deformation gradient tensor* $\mathbf{F}$ *with respect to the anholonomic orthonormal bases.*

2. *To deduce the component matrix of strain tensor* $\overset{\circ}{\mathbf{C}}$ *with respect to the anholonomic orthonormal bases.*

3. *To deduce the representations of the principle invariants of* $\overset{\circ}{\mathbf{C}}$.

4. *To deduce the component matrix of Piola-Kirchhoff stress tensor of the second kind* $\mathbf{T}$ *with respect to the anholonomic orthonormal bases.*

5. *To deduce all of the component equations of the momentum conservations with respect to the anholonomic orthonormal bases.*

6. *To show that there exists the relation* $p = p(\overset{\circ}{X})$, *in other words one has* $\frac{\partial p}{\partial \overset{\circ}{Y}} = \frac{\partial p}{\partial \overset{\circ}{Z}} = 0$

Note: To give the deduction and calculation in detail. And as the score is considered, the reflection of the correct methodologies is oriented.