体积形态连续介质有限变形理论—变形刻画

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2015 年 4 月 2 日

1　知识要素

基于变形梯度的基本性质, 可按郭仲衡 (1980)[1]关于一般有限变形理论的处理, 将变形的全部刻画分为 4 类, 归结为如下 4 个性质.

1.1　变形梯度基本性质

性质 1.1 (变形梯度基本性质)**.**

1. $\det \boldsymbol{F} = \dfrac{\sqrt{g}}{\sqrt{G}} \det \left(\dfrac{\partial x^i}{\partial \xi^A} \right) (\boldsymbol{\xi}, t) =: |\boldsymbol{F}|$;
2. $\dot{\boldsymbol{F}} = (\boldsymbol{V} \otimes \square) \cdot \boldsymbol{F}$;
3. $\dot{\overline{|\boldsymbol{F}|}} = \theta |\boldsymbol{F}|$,　此处 $\theta \triangleq \boldsymbol{V} \cdot \square = \square \cdot \boldsymbol{V}$.

1.2　各类物质系统的向量值映照刻画

基于微分学研究变形刻画, 首先引入

1. 初始及当前物理构型中物质线的向量值映照刻画 (如图1所示):

$$\begin{aligned} \overset{\circ}{\boldsymbol{X}}(\lambda) : [a,b] \ni \lambda \ &\mapsto \ \overset{\circ}{\boldsymbol{X}}(\lambda) \triangleq \overset{\circ}{\boldsymbol{X}}(\boldsymbol{\xi}(\lambda)), \\ \overset{t}{\boldsymbol{X}}(\lambda) : [a,b] \ni \lambda \ &\mapsto \ \overset{t}{\boldsymbol{X}}(\lambda) \triangleq \boldsymbol{X}(\boldsymbol{x}(\boldsymbol{\xi}(\lambda), t), t). \end{aligned}$$

2. 初始及当前物理构型中物质面的向量值映照刻画 (如图2所示):

$$\begin{aligned} \overset{\circ}{\boldsymbol{X}}(\lambda, \mu) : D_{\lambda\mu} \ni \{\lambda, \mu\} \ &\mapsto \ \overset{\circ}{\boldsymbol{X}}(\lambda, \mu) \triangleq \overset{\circ}{\boldsymbol{X}}(\boldsymbol{\xi}(\lambda, \mu)), \\ \overset{t}{\boldsymbol{X}}(\lambda, \mu) : D_{\lambda\mu} \ni \{\lambda, \mu\} \ &\mapsto \ \overset{t}{\boldsymbol{X}}(\lambda, \mu) \triangleq \boldsymbol{X}(\boldsymbol{x}(\boldsymbol{\xi}(\lambda, \mu), t), t). \end{aligned}$$

3. 初始及当前物理构型中物质体的向量值映照刻画 (如图3所示):

$$\begin{aligned} \overset{\circ}{\boldsymbol{X}}(\lambda, \mu, \gamma) : D_{\lambda\mu\gamma} \ni \{\lambda, \mu, \gamma\} \ &\mapsto \ \overset{\circ}{\boldsymbol{X}}(\lambda, \mu, \gamma) \triangleq \overset{\circ}{\boldsymbol{X}}(\boldsymbol{\xi}(\lambda, \mu, \gamma)), \\ \overset{t}{\boldsymbol{X}}(\lambda, \mu, \gamma) : D_{\lambda\mu\gamma} \ni \{\lambda, \mu \gamma\} \ &\mapsto \ \overset{t}{\boldsymbol{X}}(\lambda, \mu, \gamma) \triangleq \boldsymbol{X}(\boldsymbol{x}(\boldsymbol{\xi}(\lambda, \mu, \gamma), t), t). \end{aligned}$$

① 郭仲衡. 非线性弹性理论. 北京: 科学出版社, 1980.

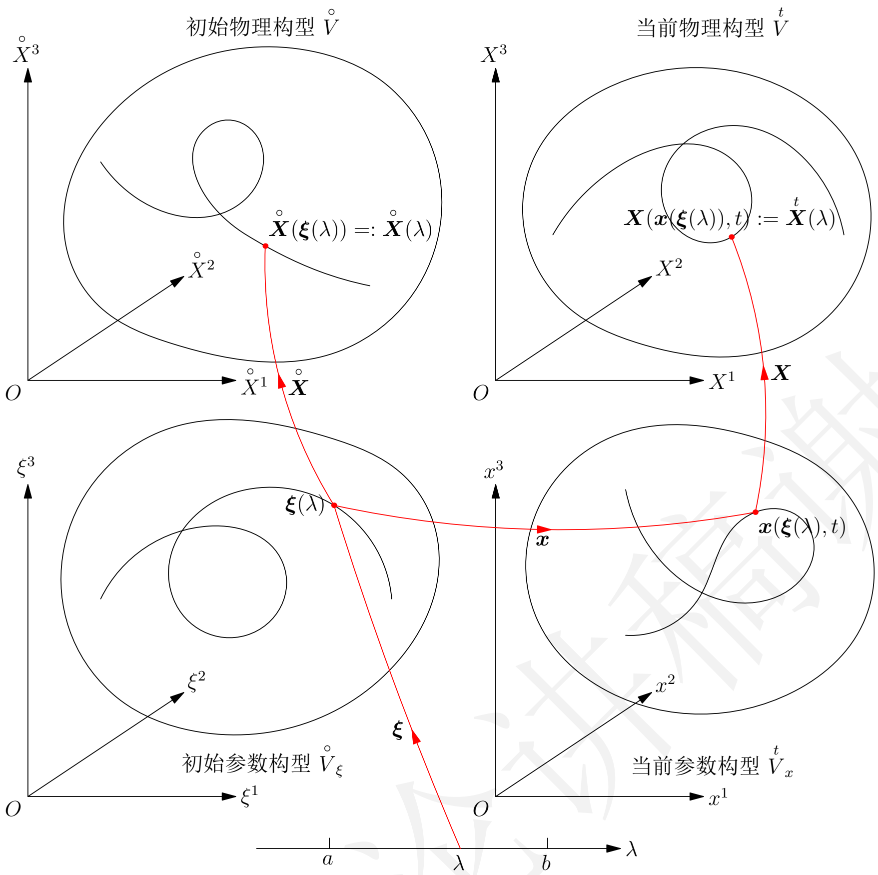

Figure 1: 物质线变形刻画示意

1.3 变形刻画

基于上述向量值映照以及微分学, 可获得变形刻画, 归类为下述 4 类性质.

1.3.1 第一类 初始物理构型与当前物理构型中有向线元面元以及体元之间的关系式

性质 1.2 (初始物理构型—当前物理构型中有向线元、面元以及体元之间的关系式).

1. $\dfrac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda) = \boldsymbol{F} \cdot \dfrac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)$;
2. $\left(\dfrac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda} \times \dfrac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}\right)(\lambda, \mu) = (|\boldsymbol{F}|\boldsymbol{F}^{-*}) \cdot \left(\dfrac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial \lambda} \times \dfrac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial \mu}\right)(\lambda, \mu)$;
3. $\left[\dfrac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda}, \dfrac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}, \dfrac{\partial \overset{t}{\boldsymbol{X}}}{\partial \gamma}\right]_{\mathbb{R}^3} (\lambda, \mu, \gamma) = |\boldsymbol{F}| \left[\dfrac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial \lambda}, \dfrac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial \mu}, \dfrac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial \gamma}\right]_{\mathbb{R}^3} (\lambda, \mu, \gamma)$.

证明 本性质证明主要应用链式求导法则及 Nanson 公式.

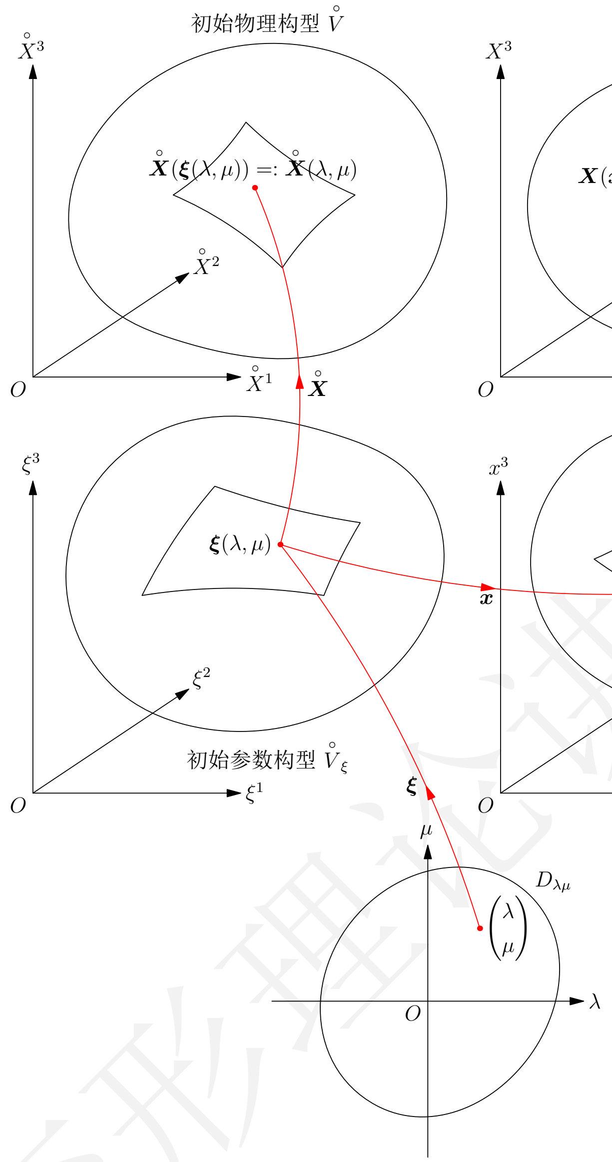

Figure 2: 物质面变形刻画示意

1. 由 $\overset{t}{\boldsymbol{X}}(\lambda) = \boldsymbol{X}(\boldsymbol{x}(\boldsymbol{\xi}(\lambda),t),t)$, 有

$$\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} = \frac{\partial \boldsymbol{X}}{\partial x^i}(\boldsymbol{x},t)\frac{\partial x^i}{\partial \xi^A}(\boldsymbol{\xi})\frac{\mathrm{d}\xi^A}{\mathrm{d}\lambda}(\lambda) = \left(\frac{\partial x^i}{\partial \xi^A}(\boldsymbol{\xi})\boldsymbol{g}_i \otimes \boldsymbol{G}^A\right) \cdot \left(\frac{\mathrm{d}\xi^B}{\mathrm{d}\lambda}(\lambda)\boldsymbol{G}_B\right)$$

$$= \boldsymbol{F} \cdot \left(\frac{\mathrm{d}\xi^B}{\mathrm{d}\lambda}\frac{\partial \mathring{\boldsymbol{X}}}{\partial \xi^B}(\boldsymbol{\xi})\right) = \boldsymbol{F} \cdot \frac{\mathrm{d}\mathring{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda).$$

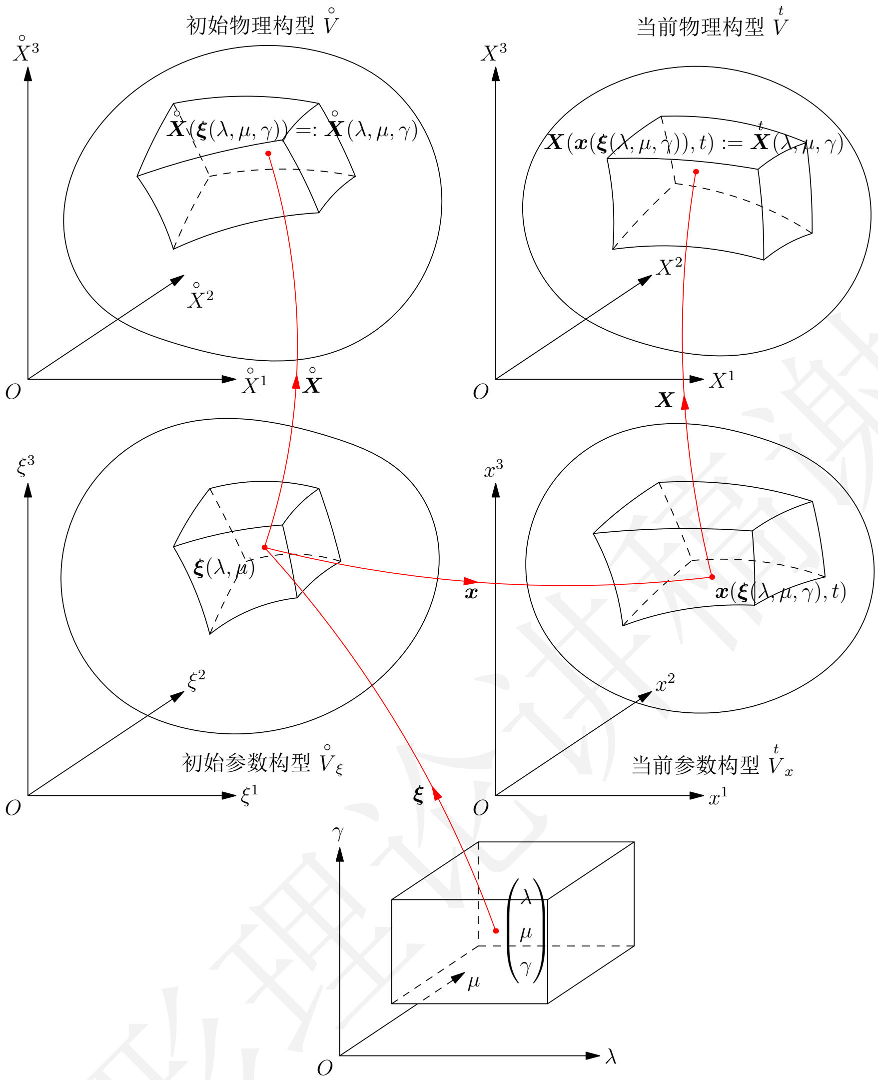

Figure 3: 物质体变形刻画示意

2. 由 $\overset{t}{\boldsymbol{X}}(\lambda,\mu) = \boldsymbol{X}(\boldsymbol{x}(\boldsymbol{\xi}(\lambda,\mu),t),t)$, 有

$$\begin{aligned}\left(\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}\right)(\lambda,\mu) &= \left[\boldsymbol{F}\cdot\frac{\partial \mathring{\boldsymbol{X}}}{\partial \lambda}(\lambda,\mu)\right] \times \left[\boldsymbol{F}\cdot\frac{\partial \mathring{\boldsymbol{X}}}{\partial \mu}(\lambda,\mu)\right] \\ &= |\boldsymbol{F}|\boldsymbol{F}^{-*}\cdot\left(\frac{\partial \mathring{\boldsymbol{X}}}{\partial \lambda} \times \frac{\partial \mathring{\boldsymbol{X}}}{\partial \mu}\right)(\lambda,\mu).\end{aligned}$$

式中最后一步利用了 Nanson 公式.

3. 由 $\overset{t}{\boldsymbol{X}}(\lambda,\mu,\gamma) = \boldsymbol{X}(\boldsymbol{x}(\boldsymbol{\xi}(\lambda,\mu,\gamma),t),t)$, 有

$$\left[\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}, \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}, \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\gamma}\right]_{\mathbb{R}^3}(\lambda,\mu,\gamma) = \left[\boldsymbol{F}\cdot\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}, \boldsymbol{F}\cdot\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}, \boldsymbol{F}\cdot\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\gamma}\right]_{\mathbb{R}^3}(\lambda,\mu,\gamma)$$
$$= |\boldsymbol{F}|\left[\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}, \frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}, \frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\gamma}\right]_{\mathbb{R}^3}(\lambda,\mu,\gamma).$$

式中最后一步直接利用了 Nanson 公式. □

1.3.2 第二类 初始物理构型与当前物理构型中有向线元面元模之间的关系式

性质 1.3 (初始物理构型－当前物理构型中有向线元、面元模之间的关系式)**.**

1. $\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right|_{\mathbb{R}^3} = \left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right|_{\mathbb{R}^3}$;

2. $\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu) = |\boldsymbol{F}|\left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-\frac{1}{2}}\cdot\left(\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)\right|_{\mathbb{R}^3}$.

证明 本性质证明主要应用链式求导法则及对称正定仿射量的幂运算.

1. 利用性质1.2中的相应结论, 有

$$\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right|^2_{\mathbb{R}^3} = \left(\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda), \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right)_{\mathbb{R}^3} = \left(\boldsymbol{F}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda), \boldsymbol{F}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right)_{\mathbb{R}^3}$$
$$= \frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\cdot(\boldsymbol{F}^*\cdot\boldsymbol{F})\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)$$
$$= \left[\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\cdot(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\right]\cdot\left[(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right]$$
$$= \left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right|^2_{\mathbb{R}^3},$$

即有

$$\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right|_{\mathbb{R}^3} = \left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)\right|_{\mathbb{R}^3}.$$

2. 利用性质1.2中的相应结论, 有

$$\begin{aligned}\left|\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right|^2_{\mathbb{R}^3}(\lambda,\mu) &= \left(\left(\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu),\left(\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)\right)_{\mathbb{R}^3}\\ &= |\boldsymbol{F}|^2\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\cdot(\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-*})\cdot\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\\ &= |\boldsymbol{F}|^2\left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-\frac{1}{2}}\cdot\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)\right|^2_{\mathbb{R}^3},\end{aligned}$$

即有

$$\left|\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu) = |\boldsymbol{F}|\left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-\frac{1}{2}}\cdot\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)\right|_{\mathbb{R}^3}.$$ □

1.3.3 第三类 当前物理构型中有向线元面元以及体元的物质导数同其之间的关系式

性质 1.4 (当前物理构型中有向线元、面元以及体元的物质导数同其之间的关系式)**.**

1. $\dot{\overline{\dfrac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}}}(\lambda) = \boldsymbol{L}\cdot\dfrac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)$;

2. $\dot{\overline{\left(\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right)}}(\lambda,\mu) = \boldsymbol{B}\cdot\left(\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)$;

3. $\dot{\overline{\left[\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda},\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu},\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\gamma}\right]}}(\lambda,\mu,\gamma) = \theta\left[\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda},\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu},\dfrac{\partial\overset{t}{\boldsymbol{X}}}{\partial\gamma}\right](\lambda,\mu,\gamma)$.

此处 $\boldsymbol{B}\triangleq\theta\boldsymbol{I}-\square\otimes\boldsymbol{V}$ 称为曲线坐标系显含时间有限变形理论的**面变形梯度**.

证明 本性质证明主要利用性质1.2及变形梯度基本性质1.1.

1. 利用性质1.2中相应结论, 有

$$\dot{\overline{\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}}}(\lambda) = \dot{\overline{\boldsymbol{F}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}}}(\lambda) = \dot{\boldsymbol{F}}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda) = \boldsymbol{L}\cdot\boldsymbol{F}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda) = \boldsymbol{L}\cdot\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda).$$

2. 利用性质1.2中相应结论, 有

$$\begin{aligned}\dot{\overline{\left(\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right)}}(\lambda,\mu) &= \dot{\overline{|\boldsymbol{F}|\boldsymbol{F}^{-*}}}\cdot\left(\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)\right)\\ &= (\dot{\overline{|\boldsymbol{F}|}}\boldsymbol{F}^{-*}+|\boldsymbol{F}|\dot{\overline{\boldsymbol{F}^{-*}}})\cdot\left(\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)(\lambda,\mu)\right),\end{aligned}$$

由于 $\boldsymbol{F}^* \cdot \boldsymbol{F}^{-*} = \boldsymbol{I}$, 可有

$$\dot{\overline{\boldsymbol{F}^*}} \cdot \boldsymbol{F}^{-*} + \boldsymbol{F}^* \cdot \dot{\overline{\boldsymbol{F}^{-*}}} = \boldsymbol{0},$$

因此有

$$\dot{\overline{\boldsymbol{F}^{-*}}} = -\boldsymbol{F}^{-*} \cdot \dot{\overline{\boldsymbol{F}^*}} \cdot \boldsymbol{F}^{-*} = -\boldsymbol{F}^{-*} \cdot (\boldsymbol{L} \cdot \boldsymbol{F})^* \cdot \boldsymbol{F}^{-*} = -\boldsymbol{L}^* \cdot \boldsymbol{F}^{-*}.$$

可得

$$\dot{\overline{|\boldsymbol{F}|}}\boldsymbol{F}^{-*} + |\boldsymbol{F}|\dot{\overline{\boldsymbol{F}^{-*}}} = (\theta \boldsymbol{I} - \boldsymbol{L}^*) \cdot (|\boldsymbol{F}|\boldsymbol{F}^{-*}).$$

综上, 可有

$$\dot{\overline{\left(\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}\right)}}(\lambda, \mu) = \boldsymbol{B} \cdot \left(\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}\right)(\lambda, \mu).$$ □

1.3.4 第四类 当前物理构型中有向线元与面元模的物质导数同其之间的关系式

性质 1.5 (当前物理构型中有向线元、面元模的物质导数同其之间的关系式).

1. $\dot{\overline{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}}}(\lambda) = (\boldsymbol{\tau} \cdot \boldsymbol{D} \cdot \boldsymbol{\tau}) \left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda)$;
2. $\dot{\overline{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}\right|_{\mathbb{R}^3}}}(\lambda, \mu) = (\theta - \boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{n}) \left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{X}}}{\partial \mu}\right|_{\mathbb{R}^3}(\lambda, \mu)$.

此处 $\boldsymbol{D} \triangleq \dfrac{\boldsymbol{L} + \boldsymbol{L}^*}{2}$ 称为曲线坐标系显含时间有限变形理论的**变形率张量**; $\boldsymbol{\tau}$ 和 $\boldsymbol{n}$ 分别表示有向线元的指向以及有向面元的单位法向量.

证明 本性质证明主要利用性质1.2, 变形梯度基本性质1.1以及张量点积求导的 Leibniz 性.

1. 考虑

$$\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|^2_{\mathbb{R}^3}(\lambda) = \frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda) \cdot \boldsymbol{F}^* \cdot \boldsymbol{F} \cdot \frac{\mathrm{d}\overset{\circ}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda),$$

以及

$$\begin{aligned}\dot{\overline{\boldsymbol{F}^* \cdot \boldsymbol{F}}} &= \dot{\boldsymbol{F}}^* \cdot \boldsymbol{F} + \boldsymbol{F}^* \cdot \dot{\boldsymbol{F}}^* = (\boldsymbol{L} \cdot \boldsymbol{F})^* \cdot \boldsymbol{F} + \boldsymbol{F}^* \cdot (\boldsymbol{L} \cdot \boldsymbol{F}) \\ &= \boldsymbol{F}^* \cdot \boldsymbol{L}^* \cdot \boldsymbol{F} + \boldsymbol{F}^* \cdot \boldsymbol{L} \cdot \boldsymbol{F} = 2\boldsymbol{F}^* \cdot \boldsymbol{D} \cdot \boldsymbol{F},\end{aligned}$$

则有

$$2\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda)\dot{\overline{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}}}(\lambda) = 2\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda) \cdot \boldsymbol{D} \cdot \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda).$$

即有

$$\frac{\overline{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}}^{\cdot}(\lambda)}{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda)} = \boldsymbol{\tau}\cdot\boldsymbol{D}\cdot\boldsymbol{\tau},\quad 此处\boldsymbol{\tau} = \frac{\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}(\lambda)}{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda)}.$$

2. 考虑

$$\begin{aligned}\left|\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{X}}}{\partial\mu}\right|^2_{\mathbb{R}^3}(\lambda,\mu) &= \left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\cdot(|\boldsymbol{F}|\boldsymbol{F}^{-1})\cdot(|\boldsymbol{F}|\boldsymbol{F}^{-*})\cdot\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\\ &= \left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\cdot|\boldsymbol{F}|^2(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1}\cdot\left(\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right).\end{aligned}$$

由于

$$\begin{aligned}\overline{(\boldsymbol{F}^*\cdot\boldsymbol{F})}^{\cdot} &= 2\boldsymbol{F}^*\cdot\boldsymbol{D}\cdot\boldsymbol{F},\\ \overline{|\boldsymbol{F}|^2(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1}}^{\cdot} &= 2|\boldsymbol{F}|(\theta|\boldsymbol{F}|)(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1} + |\boldsymbol{F}|^2\overline{(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1}}^{\cdot},\end{aligned}$$

记 $\boldsymbol{F}^*\cdot\boldsymbol{F} = \overset{\circ}{\boldsymbol{C}}$, 则由 $\overset{\circ}{\boldsymbol{C}}\cdot\overset{\circ}{\boldsymbol{C}}^{-1} = \boldsymbol{I}$, 得

$$\overline{\overset{\circ}{\boldsymbol{C}}}^{\cdot}\cdot\overset{\circ}{\boldsymbol{C}}^{-1} + \overset{\circ}{\boldsymbol{C}}\cdot\overline{\overset{\circ}{\boldsymbol{C}}^{-1}}^{\cdot} = \boldsymbol{0}.$$

所以

$$\begin{aligned}\overline{\overset{\circ}{\boldsymbol{C}}^{-1}}^{\cdot} &= -\overset{\circ}{\boldsymbol{C}}^{-1}\cdot\overline{\overset{\circ}{\boldsymbol{C}}}^{\cdot}\cdot\overset{\circ}{\boldsymbol{C}}^{-1}\\ &= -\overset{\circ}{\boldsymbol{C}}^{-1}\cdot(2\boldsymbol{F}^*\cdot\boldsymbol{D}\cdot\boldsymbol{F})\cdot\overset{\circ}{\boldsymbol{C}}^{-1}\\ &= -2(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1}\cdot\boldsymbol{F}^*\cdot\boldsymbol{D}\cdot\boldsymbol{F}\cdot(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1}\\ &= -2\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-*}\cdot\boldsymbol{F}^*\cdot\boldsymbol{D}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-*}\\ &= -2\boldsymbol{F}^{-1}\cdot\boldsymbol{D}\cdot\boldsymbol{F}^{-*},\\ \overline{|\boldsymbol{F}|^2(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1}}^{\cdot} &= 2|\boldsymbol{F}|(\theta|\boldsymbol{F}|)(\boldsymbol{F}^*\cdot\boldsymbol{F})^{-1} - 2|\boldsymbol{F}|^2\boldsymbol{F}^{-1}\cdot\boldsymbol{D}\cdot\boldsymbol{F}^{-*}\\ &= 2|\boldsymbol{F}|^2(\theta\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-*} - \boldsymbol{F}^{-1}\cdot\boldsymbol{D}\cdot\boldsymbol{F}^{-*})\\ &= 2|\boldsymbol{F}|^2\boldsymbol{F}^{-1}\cdot(\theta\boldsymbol{I} - \boldsymbol{D})\cdot\boldsymbol{F}^{-*}.\end{aligned}$$

所以, 有

$$
\begin{aligned}
&2\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}\dot{\overline{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}}}\\
&=2\left[\left(\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\cdot|\boldsymbol{F}|\boldsymbol{F}^{-1}\right]\cdot(\theta\boldsymbol{I}-\boldsymbol{D})\cdot\left[|\boldsymbol{F}|\boldsymbol{F}^{-*}\cdot\left(\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\right]\\
&=2\left[|\boldsymbol{F}|\boldsymbol{F}^{-*}\cdot\left(\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\right]\cdot(\theta\boldsymbol{I}-\boldsymbol{D})\cdot\left[|\boldsymbol{F}|\boldsymbol{F}^{-*}\cdot\left(\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{\circ}{\boldsymbol{X}}}{\partial\mu}\right)\right]\\
&=2\left(\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right)\cdot(\theta\boldsymbol{I}-\boldsymbol{D})\cdot\left(\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right).
\end{aligned}
$$

故有

$$
\begin{aligned}
\frac{\dot{\overline{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}}}}{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}}
&=\frac{\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}}{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}}\cdot(\theta\boldsymbol{I}-\boldsymbol{D})\cdot\frac{\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}}{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}}\\
&=\boldsymbol{n}\cdot(\theta\boldsymbol{I}-\boldsymbol{D})\cdot\boldsymbol{n}=\theta-\boldsymbol{n}\cdot\boldsymbol{D}\cdot\boldsymbol{n},
\end{aligned}
$$

由此得

$$
\dot{\overline{\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}}}=(\theta-\boldsymbol{n}\cdot\boldsymbol{D}\cdot\boldsymbol{n})\left|\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\lambda}\times\frac{\partial \overset{t}{\boldsymbol{X}}}{\partial\mu}\right|_{\mathbb{R}^3}. \qquad \square
$$

由此性质的第一个等式可见, 当介质场中点点成立

$$
\dot{\overline{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}}}(\lambda)=0,
$$

亦即, 介质场中任意两相邻质点间的距离保持不变, 可将此类介质定义为**刚体**. 故有对刚体而言, 变形率张量为零张量.

2 应用事例

3 建立路径

- 不同于一般的文献, 本讲稿建立的变形刻画关系基于物质线, 物质面与物质体对参数的偏导数. 按向量值映照微分学, 这些结论是完全严格的而非差一个一阶无穷小量等.
- 变形刻画关系式的获得原则上仅需依赖于变形梯度的基本性质, 由此也表示变形梯度蕴含了变形的所有信息.