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On Two Kinds of Differential Operators on General Smooth Surfaces

XIE Xi-lin

(*Department of Mechanics & Engineering Science, Fudan University, Shanghai* 200433, *China*)

Abstract: Two kinds of differential operators that can be generally defined on an arbitrary smooth surface in a finite dimensional Euclid space are studied, one is termed as surface gradient and the other one as Levi-Civita gradient. The surface gradient operator is originated from the differentiability of a tensor field defined on the surface. Some integral and differential identities have been theoretically studied that play the important role in the studies on continuous mediums whose geometrical configurations can be taken as surfaces and on interactions between fluids and deformable boundaries. The definition of Levi-Civita gradient operator is based on Levi-Civita connections generally defined on Riemann manifolds. It can be used to set up some differential identities in the intrinsic/coordiantes-independent form that play the essential role in the theory of vorticity dynamics for two dimensional flows on general fixed smooth surfaces.

Keywords: surface gradient operator; Levi-Civita gradient operator; intrinsic generalized Stokes formulas; fluid-solid interactions with deformable boundaries; surface deformation theory; two dimensional flows on fixed smooth surface

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1 Introduction

Generally, an m-dimensional surface in $m+1$ Euclid space can be represented as

$$\boldsymbol{\Sigma}(x_\Sigma,\ t):\mathbb{R}^m \supset \mathscr{D}_x \ni \boldsymbol{x}_\Sigma \longmapsto \boldsymbol{\Sigma}(x_\Sigma,\ t) \in \mathbb{R}^{m+1}.$$

In the case that x_Σ is a nonsingular point, $\{\boldsymbol{g}_i(x_\Sigma,t):=\frac{\partial \boldsymbol{\Sigma}}{\partial x_\Sigma^i}(x_\Sigma,t)\}_{i=1}^m$ constitutes the so-called covariant basis of the tangent space $\boldsymbol{T}_x\boldsymbol{\Sigma}$ and there exists uniquely one direction $\boldsymbol{n}(x_\Sigma,t)$ that is particular to the tangent space, i. e. $(\boldsymbol{n},\boldsymbol{g}_i)(x_\Sigma,t)_{\mathbb{R}^{m+1}}=0(i=1,2,\cdots,m)$.

Two kinds of the fundamental affine tensor could be defined

$$\boldsymbol{G} \triangleq g_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j,\ g_{ij}:=(\boldsymbol{g}_i,\boldsymbol{g}_j)(x_\Sigma,t)_{\mathbb{R}_{m+1}},$$
$$\boldsymbol{K} \triangleq b_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j,\ b_{ij}:=\left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(x_\Sigma,t),\boldsymbol{n}\right)_{\mathbb{R}^{m+1}}$$

that are termed as the metric tensor and the curvature tensor respectively. Gaussian curvature is defined as $K_G:=\det[b_{ij}]/\det[g_{ij}]=\det[b_j^i]=:\det \boldsymbol{B}$ and mean curvature as $H:=b_s^s=:tr\,\boldsymbol{K}$. In the whole paper, Einstein summation convention is adopted accompanying with the indices are represented by lower, upper case letters or Greek alphabets in the related studies.

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Biography: XIE Xi-lin (1974—), male, Associate professor, E-mail: xiexilin@fudan.edu.cn.

Based on the differential calculus in $\mathbb{R}^{m+1}$, one has the following so-termed frame movement equations:

$$\begin{cases}\dfrac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(x_\Sigma,t)=\Gamma_{ji}^k\boldsymbol{g}_k+b_{ji}\boldsymbol{n}=\Gamma_{ji,k}\boldsymbol{g}^k+b_{ji}\boldsymbol{n}\\ \dfrac{\partial \boldsymbol{g}^i}{\partial x_\Sigma^j}(x_\Sigma,t)=-\Gamma_{jk}^i\boldsymbol{g}^k+b_j^i\boldsymbol{n}\end{cases};\quad \frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(x_\Sigma,t)=-b_{jk}\boldsymbol{g}^k=-b_j^k\boldsymbol{g}_k,$$

where $\Gamma_{ji,k}$ and Γ_{ji}^k are the Christoffel symbols of the first and second kinds respectively. In addition, one has the relation between metric tensor and Christoffel symbol

$$\Gamma_{ij,k}=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x_\Sigma^i}+\frac{\partial g_{ik}}{\partial x_\Sigma^j}-\frac{\partial g_{ij}}{\partial x_\Sigma^k}\right)(x_\Sigma,t).$$

An m dimensional smooth surface embedded in $m+1$ dimensional Euclid space is naturally a Riemann manifold with the metric represented by the metric tensor and the covariant derivative/differentiation denoted by ∇_l defined as, say $\boldsymbol{\Phi}:=\Phi^{ij}_{\cdot\cdot k}\boldsymbol{g}_i\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}^k\in\mathscr{T}^3(\boldsymbol{T\Sigma})$ is a tensor field with order 3 on the surface,

$$\nabla_l\Phi^{ij}_{\cdot\cdot k}\triangleq\frac{\partial\Phi^{ij}_{\cdot\cdot k}}{\partial x_\Sigma^l}(x_\Sigma,t)+\Gamma_{ls}^i\Phi^{sj}_{\cdot\cdot k}+\Gamma_{ls}^j\Phi^{is}_{\cdot\cdot k}-\Gamma_{lk}^s\Phi^{ij}_{\cdot\cdot s}.$$

The fundamentals of differential calculus on a surface can be referred to the monographs by Dubrovin et al[1] and Guo[2].

Two kinds of differential operators on the surface are to be studied that are termed as surface gradient operator and Levi-Civita gradient operator respectively. The whole content of the present paper can be divided into two parts. The first part is on the surface gradient tensor that is originated from the differentiation of a tensor field defined on the surface. As applications, four related aspects in fluid and solid mechanics are referred that include §2.1 intrinsic generalized Stokes formulas in $\mathbb{R}^3$ with three kinds of applications, §2.2 primary properties of deformation gradient tensor for thin enough continuous mediums, §2.3 strain tensor on an arbitrary deformable surface. The second part is on the Levi-Civita gradient operator that is based on Levi-Civita connection possessed by any Riemann manifold. As to its applications refer to §3.1 some primary identities in vorticity dynamics of two dimensional flows on fixed smooth surfaces and §3.2 some identities of affine surface tensors.

Generally, the surface gradient operator is more familiar to mechanicians and Levi-Civita connection is to mathematicians. However, all of the applications as indicated in the present paper are closely linked to the mechanics of continuous mediums whose geometrical configurations are either bulks or surfaces. And all of the related results accompanying with deductions are independent to other studies.

2 Surface gradient operator

Generally, the surface gradient operator $\overset{\Sigma}{\boldsymbol{\nabla}}\equiv\boldsymbol{g}^l\dfrac{\partial}{\partial x_\Sigma^l}$ is defined as, say $\boldsymbol{\Phi}\in\mathscr{T}^2(\mathbb{R}^3)$,

$$\begin{aligned}\overset{\Sigma}{\boldsymbol{\nabla}}\circ-\boldsymbol{\Phi}\equiv&\left(\boldsymbol{g}^l\frac{\partial}{\partial x_\Sigma^l}\right)\circ-(\Phi^i_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\Phi^i_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^3_{\cdot j}\boldsymbol{n}\otimes\boldsymbol{g}^j+\Phi^3_{\cdot 3}\boldsymbol{n}\otimes\boldsymbol{n})\triangleq\\ &\boldsymbol{g}^l\circ-\frac{\partial}{\partial x_\Sigma^l}(\Phi^i_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\Phi^i_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^3_{\cdot j}\boldsymbol{n}\otimes\boldsymbol{g}^j+\Phi^3_{\cdot 3}\boldsymbol{n}\otimes\boldsymbol{n})=\\ &[\nabla_l\Phi^i_{\cdot j}(\boldsymbol{g}^l\circ-\boldsymbol{g}_i)\otimes\boldsymbol{g}^j+\Phi^i_{\cdot j}b_{li}(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{g}^j+\Phi^i_{\cdot j}b_l^j(\boldsymbol{g}^l\circ-\boldsymbol{g}_i)\otimes\boldsymbol{n}]+\\ &[\nabla_l\Phi^i_{\cdot 3}(\boldsymbol{g}^l\circ-\boldsymbol{g}_i)\otimes\boldsymbol{n}+\Phi^i_{\cdot 3}b_{li}(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{n}-\Phi^i_{\cdot 3}b_l^s(\boldsymbol{g}^l\circ-\boldsymbol{g}_i)\otimes\boldsymbol{g}_s]+\\ &[\nabla_l\Phi^3_{\cdot j}(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{g}^j-\Phi^3_{\cdot j}b_l^s(\boldsymbol{g}^l\circ-\boldsymbol{g}_s)\otimes\boldsymbol{g}^j+\Phi^3_{\cdot j}b_l^j(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{n}]+\\ &[\nabla_l\Phi^3_{\cdot 3}(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{n}-\Phi^3_{\cdot 3}b_l^s(\boldsymbol{g}^l\circ-\boldsymbol{g}_s)\otimes\boldsymbol{n}-\Phi^3_{\cdot 3}b_l^s(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{g}_s],\end{aligned}$$

where $\circ -$ represents any available algebra tensor operator, ∇_l denotes the covariant derivative/differentation of the tensor component that is just effective to the indices with respect to the tangent plane, i. e. i, j in the above representations,

$$\nabla_l \Phi^{i}{}_{\cdot j} \triangleq \frac{\partial \Phi^{i}{}_{\cdot j}}{\partial x_\Sigma^l}(x_\Sigma, t) + \Gamma^i_{ls}\Phi^{s}{}_{\cdot j} - \Gamma^s_{lj}\Phi^{i}{}_{\cdot s}, \quad \nabla_l \Phi^{i}{}_{\cdot 3} \triangleq \frac{\partial \Phi^{i}{}_{\cdot 3}}{\partial x_\Sigma^l}(x_\Sigma, t) + \Gamma^i_{ls}\Phi^{s}{}_{\cdot 3},$$

$$\nabla_l \Phi^{3}{}_{\cdot j} \triangleq \frac{\partial \Phi^{3}{}_{\cdot j}}{\partial x_\Sigma^l}(x_\Sigma, t) - \Gamma^s_{lj}\Phi^{3}{}_{\cdot s}, \quad \nabla_l \Phi^{3}{}_{\cdot 3} \triangleq \frac{\partial \Phi^{3}{}_{\cdot 3}}{\partial x_\Sigma^l}(x_\Sigma, t),$$

where Γ^i_{ls} denotes Christoffel symbol of the second kind. The contravariant derivative relates generally to the co-variant one through $\nabla^l \triangleq g^{lt}\nabla_t$. The change of the order of co- and contravariant derivatives must be related to Riemannian-Christoffel tensor, that is,

$$\nabla_p \nabla^q \Phi^{i}{}_{\cdot j} = \nabla^q \nabla_p \Phi^{i}{}_{\cdot j} + R^{i\cdot\cdot q}_{\cdot sp\cdot}\Phi^{s}{}_{\cdot j} + R_{j\cdot p\cdot}^{\cdot s\cdot q}\Phi^{i}{}_{\cdot s},$$

where $R^{i\cdot\cdot q}_{\cdot sp\cdot} \triangleq b^i_p b^q_s - b_{sp}b^{iq}$ denotes the component of Riemannian-Christoffel tensor[2]. In addition, in the case of two dimensional Riemannian manifolds, Riemannian-Christoffel tensor can be represented by Gaussian curvature and metric tensor as revealed by the relation $R^{i\cdot\cdot q}_{\cdot sp\cdot} = K_G(\delta^i_p\delta^q_s - g_{sp}g^{iq})$.

It should be noted that the definition of the surface gradient operator is based on the differential calculus in the normed linear tensor space, namely, one has

$$\boldsymbol{\Phi}(x_\Sigma + \Delta x_\Sigma, t) - \boldsymbol{\Phi}(x_\Sigma, t) = \begin{cases} (\Delta x_\Sigma^s \boldsymbol{g}_s)\cdot(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{\Phi}) \\ (\boldsymbol{\Phi}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})\cdot(\Delta x_\Sigma^s \boldsymbol{g}_s) \end{cases} + \boldsymbol{o}(\Delta x_\Sigma).$$

Say $\boldsymbol{\Phi} = \Phi^{i}{}_{\cdot j3}\boldsymbol{g}_i\otimes\boldsymbol{g}^j\otimes\boldsymbol{n} \in \mathscr{T}^3(\mathbb{R}^3)$, one has

$$\boldsymbol{\Phi}(x_\Sigma + \Delta x_\Sigma, t) = \Phi^{i}{}_{\cdot j3}(x_\Sigma + \Delta x_\Sigma, t)(\boldsymbol{g}_i\otimes\boldsymbol{g}^j\otimes\boldsymbol{n})(x_\Sigma + \Delta x_\Sigma, t) \in \mathscr{T}^3(\mathbb{R}^3)$$

with the differentiations of the tensor component and basis vectors

$$\Phi^{i}{}_{\cdot j3}(x_\Sigma + \Delta x_\Sigma, t) = \Phi^{i}{}_{\cdot j3}(x_\Sigma, t) + \frac{\partial \Phi^{i}{}_{\cdot j3}}{\partial x_\Sigma^s}(x_\Sigma, t)\Delta x_\Sigma^s + o^{i}{}_{\cdot j3}(\Delta x_\Sigma) \in \mathbb{R},$$

$$\boldsymbol{g}_i(x_\Sigma + \Delta x_\Sigma, t) = \boldsymbol{g}_i(x_\Sigma, t) + \frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^s}(x_\Sigma, t)\Delta x_\Sigma^s + \boldsymbol{o}_i(\Delta x_\Sigma) \in \mathbb{R}^3,$$

$$\boldsymbol{g}^j(x_\Sigma + \Delta x_\Sigma, t) = \boldsymbol{g}^j(x_\Sigma, t) + \frac{\partial \boldsymbol{g}^j}{\partial x_\Sigma^s}(x_\Sigma, t)\Delta x_\Sigma^s + \boldsymbol{o}^j(\Delta x_\Sigma) \in \mathbb{R}^3,$$

$$\boldsymbol{n}(x_\Sigma + \Delta x_\Sigma, t) = \boldsymbol{n}(x_\Sigma, t) + \frac{\partial \boldsymbol{n}}{\partial x_\Sigma^s}(x_\Sigma, t)\Delta x_\Sigma^s + \boldsymbol{o}^3(\Delta x_\Sigma) \in \mathbb{R}^3.$$

Accompanying the multi-linearity of the representation of any simple tensor with the frame movement equations, the above mentioned representation can be attained. In the view of differentiation, the full dimensional gradient of a tensor filed defined on a domain can be taken as its derivative[3]. Similarly, the surface gradient of a tensor field defined on a surface is its derivative also.

Consequently, the partial derivative of the tensor with respect to one of the component of the surface coordinates can be determined:

$$\frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l}(x_\Sigma, t) \triangleq \lim_{\lambda\to 0}\frac{\boldsymbol{\Phi}(x_\Sigma + \lambda \boldsymbol{i}_l, t) - \boldsymbol{\Phi}(x_\Sigma, t)}{\lambda} = \boldsymbol{g}_l\cdot(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{\Phi}) = (\boldsymbol{\Phi}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})\cdot\boldsymbol{g}_l,$$

due to

$$\boldsymbol{\Phi}(x_\Sigma + \lambda \boldsymbol{i}_l, t) - \boldsymbol{\Phi}(x_\Sigma, t) = \begin{cases} (\lambda \boldsymbol{g}_l)\cdot(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{\Phi}) + \boldsymbol{o}(\lambda), \\ (\boldsymbol{\Phi}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})\cdot(\lambda \boldsymbol{g}_l) + \boldsymbol{o}(\lambda), \end{cases}$$

where $\boldsymbol{i}_l$ denotes the canonical basis vector in the parametric space.

2.1 Intrinsic generalized Stokes formulas in $\mathbb{R}^3$

In the first instance, the so termed semi-orthogonal curvilinear coordinates with respect to a surface is constructed as shown in Fig. 1.

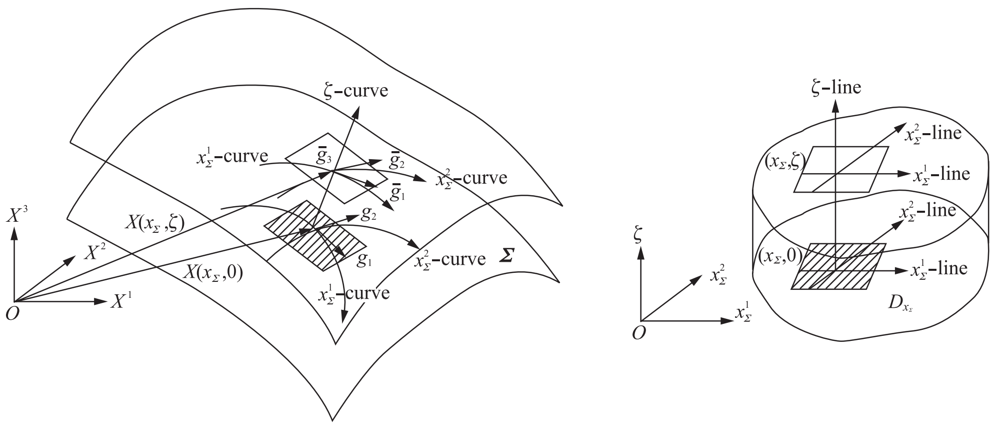

Fig. 1　Sketch of the semi-orthogonal curvilinear coordinates with respect to a surface Σ

Generally, the smooth surface in $\mathbb{R}^3$ takes the following form:

$$\boldsymbol{\Sigma}(x_\Sigma):\mathbb{R}^2\supset\mathscr{D}_{x_\Sigma}\ni x_\Sigma=\begin{bmatrix}x_\Sigma^1\\x_\Sigma^2\end{bmatrix}\longmapsto\boldsymbol{\Sigma}(x_\Sigma)=\begin{bmatrix}X^1\\X^2\\X^3\end{bmatrix}(x_\Sigma)\in\mathbb{R}^3,$$

where $x_\Sigma:=\{x_\Sigma^i\}_{i=1}^2$ are surface parameters, in other words Gaussian coordinates. Without lost of the generality, it is assumed that $\boldsymbol{\Sigma}(x_\Sigma)$ is an one-to-one/injective mapping on the definition domain $\mathscr{D}_{x_\Sigma}$. Subsequently, the mapping defined on the neighborhood of the surface can be constructed

$$\boldsymbol{X}(x_\Sigma,\zeta):\mathbb{R}^3\supset\mathscr{D}_x\ni\begin{bmatrix}x_\Sigma\\\zeta\end{bmatrix}\longmapsto\boldsymbol{X}(x_\Sigma,\zeta)\triangleq\boldsymbol{\Sigma}(x_\Sigma)+\zeta\boldsymbol{n}(x_\Sigma)\in\mathbb{R}^3,$$

where $\boldsymbol{n}(x_\Sigma)$ is the unit normal vector of the surface. And the definition domain is $\mathscr{D}_x=\mathscr{D}_{x_\Sigma}\times(-\lambda,\lambda)$ in which λ is a suitable positive number.

In order to calculate the determinant of the Jacobian matrix $D\boldsymbol{X}(x_\Sigma,\zeta)\in\mathbb{R}^{3\times3}$, the partial derivatives of $\boldsymbol{X}(x_\Sigma,\zeta)$ with respect to its all coordinates are firstly calculated

$$\begin{aligned}\bar{\boldsymbol{g}}_i(x_\Sigma,\zeta):=&\frac{\partial\boldsymbol{X}}{\partial x_\Sigma^i}(x_\Sigma,\zeta)=\frac{\partial\boldsymbol{\Sigma}}{\partial x_\Sigma^i}(x_\Sigma)+\zeta\frac{\partial\boldsymbol{n}}{\partial x_\Sigma^i}(x_\Sigma)=:\\&\boldsymbol{g}_i(x_\Sigma)+\zeta(-b_i^j)\boldsymbol{g}_j(x_\Sigma)=(\delta_i^j-\zeta b_i^j)\boldsymbol{g}_j(x_\Sigma)\qquad i=1,2,\\\bar{\boldsymbol{g}}_3(x_\Sigma,\zeta):=&\frac{\partial\boldsymbol{X}}{\partial\zeta}(x_\Sigma,\zeta)=\boldsymbol{n}(x_\Sigma)=\bar{\boldsymbol{g}}_3(x_\Sigma),\end{aligned}$$

where $\{\boldsymbol{g}_i(x_\Sigma)\}_{i=1}^2$ is the local covariant basis vectors of the surface. Secondly, the following calculation is carried out

$$\begin{aligned}\bar{\boldsymbol{g}}_1\times\bar{\boldsymbol{g}}_2=&(\delta_1^p-\zeta b_1^p)(\delta_2^q-\zeta b_2^q)\boldsymbol{g}_p\times\boldsymbol{g}_q=(\delta_1^p\delta_2^q-\zeta b_1^p\delta_2^q-\zeta\delta_1^p b_2^q+\zeta^2 b_1^p b_2^q)\boldsymbol{g}_p\times\boldsymbol{g}_q=\\&\boldsymbol{g}_1\times\boldsymbol{g}_2-\zeta b_s^s\boldsymbol{g}_1\times\boldsymbol{g}_2+\zeta^2(b_1^1b_2^2-b_1^2b_2^1)\boldsymbol{g}_1\times\boldsymbol{g}_2=\\&(1-\zeta b_s^s+\zeta^2\det[b_j^i])\boldsymbol{g}_1\times\boldsymbol{g}_2=(1-\zeta H+\zeta^2K_G)\boldsymbol{g}_1\times\boldsymbol{g}_2.\end{aligned}$$

Consequently, it is deduced that

$$\begin{aligned}\det D\boldsymbol{X}(x_\Sigma,\zeta)=&[\bar{\boldsymbol{g}}_1,\bar{\boldsymbol{g}}_2,\bar{\boldsymbol{g}}_3](x_\Sigma,\zeta)=\\&(1-\zeta H+\zeta^2K_G)[\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{n}](x_\Sigma)=:(1-\zeta H+\zeta^2K_G)\sqrt{g_\Sigma},\end{aligned}$$

where $\sqrt{g_\Sigma} := \det[g_{ij}](x_\Sigma) > 0$ is termed as the area element of the surface.

Finally, it can be concluded that the mapping $\boldsymbol{X}(x_\Sigma, \zeta)$ actualizes a smooth diffeomorphism between the definition domain $\mathscr{D}_{x_\Sigma} \times (-\lambda, \lambda)$ and the range of arrival, provided λ is small enough. Smooth diffeomorphism can also be termed as smooth curvilinear coordinates[4]. In the present case, it is evident that $\bar{\boldsymbol{g}}_3(x_\Sigma, \zeta)$ is perpendicular to $\{\bar{\boldsymbol{g}}_i(x_\Sigma, \zeta)\}_{i=1}^2$. Namely, the curvilinear coordinates $\{x_\Sigma, \zeta\}$ is semi-orthogonal. On the other hand, $\{x_\Sigma, \zeta\}$ is still a kind of full dimensional curvilinear coordinates so that the following identity is keeping valid

$$\boldsymbol{\nabla} = \boldsymbol{i}_\lambda \frac{\partial}{\partial X^\lambda}(X) = \bar{\boldsymbol{g}}^i(x_\Sigma, \zeta)\frac{\partial}{\partial x_\Sigma^i}(x_\Sigma, \zeta) + \bar{\boldsymbol{g}}^3(x_\Sigma, \zeta)\frac{\partial}{\partial \zeta}(x_\Sigma, \zeta) =$$
$$\bar{\boldsymbol{g}}^i(x_\Sigma, \zeta)\frac{\partial}{\partial x_\Sigma^i}(x_\Sigma, \zeta) + \boldsymbol{n}(x_\Sigma)\frac{\partial}{\partial \zeta}(x_\Sigma, \zeta) \qquad \forall (x_\Sigma, \zeta) \in \mathscr{D}_{x_\Sigma} \times (-\lambda, \lambda)$$

due to the chain-rule in differential calculus. On the surface, one has

$$\boldsymbol{\nabla} = \boldsymbol{i}_\lambda \frac{\partial}{\partial X^\lambda}(X) = \boldsymbol{g}^i(x_\Sigma)\frac{\partial}{\partial x_\Sigma^i}(x_\Sigma, 0) + \boldsymbol{n}(x_\Sigma)\frac{\partial}{\partial \zeta}(x_\Sigma, 0) = \overset{\Sigma}{\boldsymbol{\nabla}} + \boldsymbol{n}(x_\Sigma)\frac{\partial}{\partial \zeta}(x_\Sigma, 0)$$

through the continuously extension.

Proposition 1 (Intrinsic generalized Stokes formulas of the first kind)

$$\oint_{\partial\Sigma} \boldsymbol{\tau} \circ - \boldsymbol{\Phi} \mathrm{d}l = \int_\Sigma (\boldsymbol{n} \times \overset{\Sigma}{\boldsymbol{\nabla}}) \circ - \boldsymbol{\Phi} \mathrm{d}\sigma,$$
$$\oint_{\partial\Sigma} \boldsymbol{\Phi} \circ - \boldsymbol{\tau} \mathrm{d}l = \int_\Sigma \boldsymbol{\Phi} \circ - (\boldsymbol{n} \times \overset{\Sigma}{\boldsymbol{\nabla}}) \mathrm{d}\sigma.$$

Proof It is well known that Stokes formula in the fundamental calculus takes the following form

$$\oint_{\partial\Sigma} \boldsymbol{\tau} \cdot \boldsymbol{a} \mathrm{d}l = \int_\Sigma \boldsymbol{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) \mathrm{d}\sigma \quad \text{i. e} \quad \oint_{\partial\Sigma} \tau_\alpha a_\alpha \mathrm{d}l = \int_\Sigma n_\lambda e_{\lambda\mu\alpha} \frac{\partial a_\alpha}{\partial X^\mu} \mathrm{d}\sigma,$$

where all of the quantities are represented through the canonical basis, and X^μ denotes Cartesian coordinates in the full paper. Consequently, the vector field $\boldsymbol{a}$ should be extended differentially to a three dimensional open set in which the surface is embedded in order to fulfil the full dimensional curl operator. This kind of Stokes formula is termed as the prototype in the present paper.

In order to prove the second identity listed in Proposition 1, firstly the integrant of the curve integral is expanded through the canonical basis, that is

$$\boldsymbol{\Phi} \circ - \boldsymbol{\tau} = (\Phi_{\xi\eta} \boldsymbol{i}_\xi \otimes \boldsymbol{i}_\eta) \circ - (\tau_\alpha \boldsymbol{i}_\alpha) = \tau_\beta \delta_{\beta\alpha} \Phi_{\xi\eta} (\boldsymbol{i}_\xi \otimes \boldsymbol{i}_\eta \circ - \boldsymbol{i}_\alpha),$$

where $\boldsymbol{i}_\alpha$ and so on denote the canonical basis vectors.

Secondly, the Stokes formula in the prototype is adopted to attain the surface gradient

$$n_\theta e_{\theta\lambda\beta} \frac{\partial}{\partial X^\lambda}(\delta_{\beta\alpha} \Phi_{\xi\eta})(\boldsymbol{i}_\xi \otimes \boldsymbol{i}_\eta \circ - \boldsymbol{i}_\alpha) = n_\theta e_{\theta\lambda\alpha} \frac{\partial \Phi_{\xi\eta}}{\partial X^\lambda} \boldsymbol{i}_\xi \otimes \boldsymbol{i}_\eta \circ - \boldsymbol{i}_\alpha = n_\theta e_{\theta\lambda\alpha} \frac{\partial \boldsymbol{\Phi}}{\partial X^\lambda} \circ - \boldsymbol{i}_\alpha =$$
$$\frac{\partial \boldsymbol{\Phi}}{\partial X^\lambda} \circ - (\boldsymbol{n} \times \boldsymbol{i}_\lambda) =: \boldsymbol{\Phi} \circ - \left[\boldsymbol{n} \times \left(\boldsymbol{i}_\lambda \frac{\partial}{\partial X^\lambda}\right)\right] =: \boldsymbol{\Phi} \circ - (\boldsymbol{n} \times \boldsymbol{\nabla}).$$

Thirdly, the full dimensional gradient is represented through the surface gradient

$$\boldsymbol{\Phi} \circ - (\boldsymbol{n} \times \boldsymbol{\nabla}) = \boldsymbol{\Phi} \circ - \left[\boldsymbol{n} \times \left(\overset{\Sigma}{\boldsymbol{\nabla}} + \boldsymbol{n} \frac{\partial}{\partial X^3}\right)\right] = \boldsymbol{\Phi} \circ - (\boldsymbol{n} \times \overset{\Sigma}{\boldsymbol{\nabla}}),$$

in the process of the deduction the semi-orthogonal curvilinear coordinates with respect to the surface Σ is adopted. The proof of the first identity can be obtained in the same way[5]. Therefore, the proof is completed.

It should be pointed out that the surface gradients have nothing to do with the directional derivative with respect to the normal direction, in other words the quantity originally defined on the surface does not need to be extended if the surface gradient rather than the full dimensional gradient is adopted.

Proposition 2 (Intrinsic generalized Stokes formulas of the second kind)

$$\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\circ\!-\boldsymbol{\Phi}\,\mathrm{d}l=\int_{\Sigma}(\overset{\Sigma}{\boldsymbol{\nabla}}\circ\!-\boldsymbol{\Phi}+H\boldsymbol{n}\circ\!-\boldsymbol{\Phi})\,\mathrm{d}\sigma,$$

$$\oint_{\partial\Sigma}\boldsymbol{\Phi}\circ\!-(\boldsymbol{\tau}\times\boldsymbol{n})\,\mathrm{d}l=\int_{\Sigma}(\boldsymbol{\Phi}\circ\!-\overset{\Sigma}{\boldsymbol{\nabla}}+H\boldsymbol{\Phi}\circ\!-\boldsymbol{n})\,\mathrm{d}\sigma,$$

where the interaction direction $\boldsymbol{\tau}\times\boldsymbol{n}$ is perpendicular to the tangent vector but lies on the tangent plane as shown in Fig. 2.

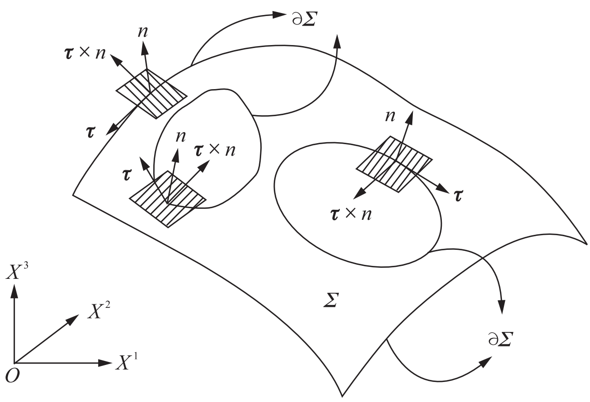

Fig. 2　Sketch of the generalized Stokes formulas of the second kind

Proof　The proof of the second identity is carried out as follows. And the first one can be verified in the same way[5].

Firstly, the integrant of the curve integral is expanded through the canonical basis:

$$\boldsymbol{\Phi}\circ\!-(\boldsymbol{\tau}\times\boldsymbol{n})=(\boldsymbol{\Phi}_{\xi\eta}\boldsymbol{i}_{\xi}\otimes\boldsymbol{i}_{\eta})\circ\!-(e_{\alpha\beta\gamma}\tau_{\beta}n_{\gamma}\boldsymbol{i}_{\alpha})=\tau_{\beta}e_{\beta\gamma\alpha}n_{\gamma}\Phi_{\xi\eta}(\boldsymbol{i}_{\xi}\otimes\boldsymbol{i}_{\eta}\circ\!-\boldsymbol{i}_{\alpha}).$$

Secondly, the curve integral is transferred to the surface integral according to the Stokes formula in the prototype. The deduction of the surface integrant is as follows:

$$\begin{aligned}&n_{\theta}e_{\theta\lambda\beta}\frac{\partial}{\partial X^{\lambda}}(e_{\beta\gamma\alpha}n_{\gamma}\Phi_{\xi\eta})(\boldsymbol{i}_{\xi}\otimes\boldsymbol{i}_{\eta}\circ\!-\boldsymbol{i}_{\alpha})=e_{\theta\lambda\beta}e_{\gamma\alpha\beta}n_{\theta}\frac{\partial}{\partial X^{\lambda}}(n_{\gamma}\Phi_{\xi\eta})(\boldsymbol{i}_{\xi}\otimes\boldsymbol{i}_{\eta}\circ\!-\boldsymbol{i}_{\alpha})=\\&(\delta_{\theta\gamma}\delta_{\lambda\alpha}-\delta_{\lambda\gamma}\delta_{\theta\alpha})n_{\theta}\frac{\partial}{\partial X^{\lambda}}(n_{\gamma}\Phi_{\xi\eta})(\boldsymbol{i}_{\xi}\otimes\boldsymbol{i}_{\eta}\circ\!-\boldsymbol{i}_{\alpha})=\\&\left[\frac{\partial\Phi_{\xi\eta}}{\partial X^{\alpha}}-n_{\alpha}\frac{\partial n_{\lambda}}{\partial X^{\lambda}}\Phi_{\xi\eta}-n_{\alpha}n_{\lambda}\frac{\partial\Phi_{\xi\eta}}{\partial X^{\lambda}}\right](\boldsymbol{i}_{\xi}\otimes\boldsymbol{i}_{\eta}\circ\!-\boldsymbol{i}_{\alpha})=\\&\boldsymbol{\Phi}\circ\!-\boldsymbol{\nabla}-(\boldsymbol{\nabla}\cdot\boldsymbol{n})(\boldsymbol{\Phi}\circ\!-\boldsymbol{n})-(\boldsymbol{n}\cdot(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi}))\circ\!-\boldsymbol{n}.\end{aligned}$$

Thirdly, the full dimensional gradient is represented by the surface gradient:

$$\begin{aligned}RHS=&\boldsymbol{\Phi}\circ\!-\left(\overset{\Sigma}{\boldsymbol{\nabla}}+\boldsymbol{n}\frac{\partial}{\partial x^{3}}\right)-\left[\left(\overset{\Sigma}{\boldsymbol{\nabla}}+\boldsymbol{n}\frac{\partial}{\partial x^{3}}\right)\cdot\boldsymbol{n}\right](\boldsymbol{\Phi}\circ\!-\boldsymbol{n})-\\&\boldsymbol{n}\cdot\left[\left(\overset{\Sigma}{\boldsymbol{\nabla}}+\boldsymbol{n}\frac{\partial}{\partial x^{3}}\right)\otimes\boldsymbol{\Phi}\right]\circ\!-\boldsymbol{n}=\boldsymbol{\Phi}\circ\!-\overset{\Sigma}{\boldsymbol{\nabla}}+H\boldsymbol{\Phi}\circ\!-\boldsymbol{n}.\end{aligned}$$

The proof is completed.

2.1.1　Some integral identities for soft matter studies

Yin[6] reported some kinds of novel integral identities that are taken as meaningful for soft matter studies. As a kind of applications, the intrinsic generalized Stokes formulas are utilized to deduce these identities as indicated in this subsection.

Firstly, the quantity termed as conjugate fundamental tensor[6] is introduced:

$$|\boldsymbol{K}|\boldsymbol{K}^{-1} =: \Delta_j^i \boldsymbol{g}_i \otimes \boldsymbol{g}^j,$$

where $\boldsymbol{K} \triangleq b_j^i \boldsymbol{g}_i \otimes \boldsymbol{g}^j$ is the curvature tensor, $[\Delta_j^i]$ denotes the adjugate matrix of $[b_j^i]$. Certainly, it should be pointed out that this quantity can only make sense in the case that $\boldsymbol{K}$ is nonsingular, i. e. $\det[b_j^i] \neq 0$.

On Δ_j^i, the following fundamental relations can be concluded:

$$\begin{aligned}
&\Delta_j^i = b_s^s \delta_j^i - b_j^i, \quad b_j^i = \Delta_s^s \delta_j^i - \Delta_j^i, \\
&e^{3ji} \Delta_i^l = -e^{3li} b_i^j, \quad \varepsilon^{3ji} \Delta_i^l = -\varepsilon^{3li} b_i^j, \\
&\nabla_i \Delta_j^i = 0.
\end{aligned}$$

The first two relationships can be directly verified. The last one is due to the Codazzi equation as indicated by Yin et al[7]. All of these relations play the essential role in the following deductions.

Proposition 3

$$\oint_{\partial\Sigma} \boldsymbol{\tau} \cdot \boldsymbol{K} \circ\!- \boldsymbol{\Phi} \mathrm{d}l = \int_{\Sigma} (\boldsymbol{n} \times \overset{\Sigma}{\hat{\boldsymbol{\nabla}}}) \circ\!- \boldsymbol{\Phi} \mathrm{d}\sigma,$$

where $\overset{\Sigma}{\hat{\boldsymbol{\nabla}}} := \hat{L}^{ij} \boldsymbol{g}_i \dfrac{\partial}{\partial x_\Sigma^j}, \hat{\boldsymbol{L}} := K_G \boldsymbol{K}^{-1}$.

Proof Firstly, it is worthy of mention that $K_G = \det[b_j^i] =: |\boldsymbol{K}|$ and $\overset{\Sigma}{\hat{\boldsymbol{\nabla}}} = |\boldsymbol{K}|\boldsymbol{K}^{-1} \cdot \overset{\Sigma}{\boldsymbol{\nabla}}$.

As the application of the intrinsic generalized Stokes formula of the first kind, one has

$$\oint_{\partial\Sigma} \boldsymbol{\tau} \cdot \boldsymbol{K} \circ\!- \boldsymbol{\Phi} \mathrm{d}l = \int_{\Sigma} (\boldsymbol{n} \times \overset{\Sigma}{\boldsymbol{\nabla}}) \cdot (\boldsymbol{K} \circ\!- \boldsymbol{\Phi}) \mathrm{d}\sigma$$

with

$$(n \times \overset{\Sigma}{\boldsymbol{\nabla}}) \cdot (\boldsymbol{K} \circ\!- \boldsymbol{\Phi}) = (\boldsymbol{n} \times \boldsymbol{g}^l) \cdot \left(\frac{\partial \boldsymbol{K}}{\partial x_\Sigma^l} \circ\!- \boldsymbol{\Phi} + \boldsymbol{K} \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l} \right) = (\boldsymbol{n} \times \boldsymbol{g}^l) \cdot \left(\boldsymbol{K} \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l} \right) = \varepsilon^{3lk} b_k^j \boldsymbol{g}_j \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l},$$

thanks to

$$\begin{aligned}
(\boldsymbol{n} \times \boldsymbol{g}^l) \cdot \frac{\partial \boldsymbol{K}}{\partial x_\Sigma^l} &= (\boldsymbol{n} \times \boldsymbol{g}^l) \cdot (\nabla_l b_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j + b_{ij} b_l^i \boldsymbol{n} \otimes \boldsymbol{g}^j + b_{ij} b_l^j \boldsymbol{g}^i \otimes \boldsymbol{n}) = \\
&\varepsilon^{3li} \nabla_l b_{ij} \boldsymbol{g}^j + \varepsilon^{3li} (b_{ij} b_l^j) \boldsymbol{n} = \boldsymbol{0}.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
(\boldsymbol{n} \times \overset{\Sigma}{\hat{\boldsymbol{\nabla}}}) \circ\!- \boldsymbol{\Phi} &= [\boldsymbol{n} \times (|\boldsymbol{K}|\boldsymbol{K}^{-1} \cdot \overset{\Sigma}{\boldsymbol{\nabla}})] \circ\!- \boldsymbol{\Phi} = \left[\boldsymbol{n} \times \left(\Delta_i^l \boldsymbol{g}^i \frac{\partial}{\partial x_\Sigma^l} \right) \right] \circ\!- \boldsymbol{\Phi} = \\
&[\boldsymbol{n} \times (\Delta_i^l \boldsymbol{g}^i)] \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l} = \varepsilon^{3ik} \Delta_i^l \boldsymbol{g}_k \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l} = \varepsilon^{3ij} \Delta_i^l \boldsymbol{g}_j \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l} = \\
&-\varepsilon^{3ji} \Delta_i^l \boldsymbol{g}_j \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l} = \varepsilon^{3li} b_i^j \boldsymbol{g}_j \circ\!- \frac{\partial \boldsymbol{\Phi}}{\partial x_\Sigma^l}.
\end{aligned}$$

It's the end of the proof.

Proposition 4

$$\oint_{\partial\Sigma} (\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \hat{\boldsymbol{L}} \circ\!- \boldsymbol{\Phi} \mathrm{d}l = \int_{\Sigma} \overset{\Sigma}{\hat{\boldsymbol{\nabla}}} \circ\!- \boldsymbol{\Phi} \mathrm{d}\sigma + \int_{\Sigma} 2K_G (\boldsymbol{n} \circ\!- \boldsymbol{\Phi}) \mathrm{d}\sigma,$$

where $\hat{\boldsymbol{L}} = |\boldsymbol{K}|\boldsymbol{K}^{-1}$.

Proof　On the left hand side, one has

$$\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot(|\boldsymbol{K}|\,\boldsymbol{K}^{-1})\circ\!-\boldsymbol{\Phi}\mathrm{d}l=\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot(|\boldsymbol{K}|\,\boldsymbol{K}^{-1}\circ\!-\boldsymbol{\Phi})\mathrm{d}l=$$
$$\int_{\Sigma}[\overset{\Sigma}{\nabla}\cdot(|\boldsymbol{K}|\,\boldsymbol{K}^{-1}\circ\!-\boldsymbol{\Phi})+H\boldsymbol{n}\cdot(|\boldsymbol{K}|\,\boldsymbol{K}^{-1}\circ\!-\boldsymbol{\Phi})]\mathrm{d}\sigma=$$
$$\int_{\Sigma}\overset{\Sigma}{\nabla}\cdot(|\boldsymbol{K}|\,\boldsymbol{K}^{-1}\circ\!-\boldsymbol{\Phi})\mathrm{d}\sigma.$$

To deal with

$$\overset{\Sigma}{\nabla}\cdot(|\boldsymbol{K}|\boldsymbol{K}^{-1}\circ\!-\boldsymbol{\Phi})=\boldsymbol{g}^l\cdot\frac{\partial}{\partial x_\Sigma^l}(|\boldsymbol{K}|\boldsymbol{K}^{-1}\circ\!-\boldsymbol{\Phi})=$$
$$\boldsymbol{g}^l\cdot\left[\frac{\partial}{\partial x_\Sigma^l}(|\boldsymbol{K}|\boldsymbol{K}^{-1})\circ\!-\boldsymbol{\Phi}+|\boldsymbol{K}|\boldsymbol{K}^{-1}\circ\!-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}\right],$$

one deduces the second term on the right hand side as

$$\boldsymbol{g}^l\cdot\left(|\boldsymbol{K}|\boldsymbol{K}^{-1}\circ\!-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}\right)=[\boldsymbol{g}^l\cdot(|\boldsymbol{K}|\boldsymbol{K}^{-1})]\circ\!-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}=(|\boldsymbol{K}|\boldsymbol{K}^{-1}\cdot\boldsymbol{g}^l)\circ\!-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}=$$
$$\left[|\boldsymbol{K}|\boldsymbol{K}^{-1}\cdot\left(\boldsymbol{g}^l\frac{\partial}{\partial x_\Sigma^l}\right)\right]\circ\!-\boldsymbol{\Phi}=\overset{\Sigma}{\tilde{\nabla}}\circ\!-\boldsymbol{\Phi},$$

and the first term on the right hand side as

$$\boldsymbol{g}^l\cdot\left[\frac{\partial}{\partial x_\Sigma^l}(|\boldsymbol{K}|\boldsymbol{K}^{-1})\circ\!-\boldsymbol{\Phi}\right]=\boldsymbol{g}^l\cdot\left[\frac{\partial}{\partial x_\Sigma^l}(\Delta_j^i\boldsymbol{g}_i\otimes\boldsymbol{g}^j)\circ\!-\boldsymbol{\Phi}\right]=$$
$$\boldsymbol{g}^l\cdot[(\nabla_l\Delta_j^i\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\Delta_j^ib_{li}\boldsymbol{n}\otimes\boldsymbol{g}^j+\Delta_j^ib_l^j\boldsymbol{g}_i\otimes\boldsymbol{n})\circ\!-\boldsymbol{\Phi}]=$$
$$(\nabla_l\Delta_j^l\boldsymbol{g}^j+\Delta_j^ib_i^j\boldsymbol{n})\circ\!-\boldsymbol{\Phi}=(\delta_j^ib_s^s-b_j^i)b_i^j\boldsymbol{n}\circ\!-\boldsymbol{\Phi}=(b_j^jb_s^s-b_j^ib_i^j)\boldsymbol{n}\circ\!-\boldsymbol{\Phi}=$$
$$2K_G\boldsymbol{n}\circ\!-\boldsymbol{\Phi}.$$

The last identity is due to the relationship

$$K_G=\det[b_j^i]=\frac{1}{2}\delta_{pq}^{ij}b_i^pb_j^q=\frac{1}{2}\begin{vmatrix}\delta_p^i & \delta_q^i\\ \delta_p^j & \delta_q^j\end{vmatrix}b_i^pb_j^q=\frac{1}{2}(b_j^jb_s^s-b_j^ib_i^j).$$

It's the end of the proof.

In studies by Yin with his collaborators[7,8] on some integral identities, the following one plays the essential role

$$\int_{\Sigma}\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})\mathrm{d}\sigma=\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})\mathrm{d}l\qquad\forall\boldsymbol{\Theta}\in\mathscr{T}^r(\boldsymbol{T\Sigma}),\forall\boldsymbol{\Phi}\in\mathscr{T}^p(\mathbb{R}^3).$$

Its validity can be confirmed as soon as the intrinsic generalized Stokes formula of the second kind is adopted, namely

$$\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})\mathrm{d}l=\int_{\Sigma}[\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})+H\boldsymbol{n}\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})]\mathrm{d}\sigma=\int_{\Sigma}\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})\mathrm{d}\sigma.$$

By other ways, one can do the following calculation, let $\boldsymbol{\Theta}=\Theta^{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j$ without lost of the generality,

$$\int_{\Sigma}\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})\mathrm{d}\sigma=\int_{\Sigma}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ\!-\boldsymbol{\Phi})(x_\Sigma,t)\mathrm{d}\sigma=$$
$$\int_{\mathscr{D}_x}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ\!-\boldsymbol{\Phi})(x_\Sigma,t)\mathrm{d}\sigma=\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot(\boldsymbol{\Theta}\circ\!-\boldsymbol{\Phi})\mathrm{d}l.$$

The first identity is due to

$$\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\Theta}\otimes\boldsymbol{\Phi})=\boldsymbol{g}^l\cdot\left[\frac{\partial}{\partial x_\Sigma^l}(\Theta^{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j)\circ-\boldsymbol{\Phi}+\boldsymbol{\Theta}\circ-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}\right]=$$

$$\boldsymbol{g}^l\cdot[\nabla_l\Theta^{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j+\Theta^{ij}(b_{li}\boldsymbol{n}\otimes\boldsymbol{g}_j+b_{lj}\boldsymbol{g}_i\otimes\boldsymbol{n})]\circ-\boldsymbol{\Phi}+\Theta^{ls}\boldsymbol{g}_s\circ-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}=$$

$$(\nabla_i\Theta^{ij}\boldsymbol{g}_j+\Theta^{ij}b_{ij}\boldsymbol{n})\circ-\boldsymbol{\Phi}+\Theta^{ls}\boldsymbol{g}_s\circ-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l},$$

where

$$\nabla_i\Theta^{ij}\boldsymbol{g}_j+\Theta^{ij}b_{ij}\boldsymbol{n}=\left(\frac{\partial\Theta^{ij}}{\partial x_\Sigma^i}+\Gamma_{is}^i\Theta^{sj}+\Gamma_{is}^j\Theta^{is}\right)\boldsymbol{g}_j+\Theta^{ij}b_{ij}\boldsymbol{n}=$$

$$\left(\frac{\partial\Theta^{sj}}{\partial x_\Sigma^s}+\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^s}\Theta^{sj}\right)\boldsymbol{g}_j+\Theta^{sj}(\Gamma_{sj}^k\boldsymbol{g}_k+b_{sj}\boldsymbol{n})=$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj})\boldsymbol{g}_j+\Theta^{sj}\frac{\partial\boldsymbol{g}_j}{\partial x_\Sigma^s}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j),$$

then

$$RHS=\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j)\circ-\boldsymbol{\Phi}+\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ-\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^s}\right]=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ-\boldsymbol{\Phi}).$$

The last identity is essentially due to the Green formula. In detail, firstly one has the relation as the prototype

$$\int_{\mathscr{D}_x}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}a^s)\mathrm{d}\sigma=\int_\Sigma(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{a}\mathrm{d}l\qquad\forall\boldsymbol{a}\in\boldsymbol{T\Sigma}.$$

As the left hand side is considered, it can be calculated as follows:

$$\int_{\mathscr{D}_x}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}a^s)\mathrm{d}\sigma=\int_{\mathscr{D}_x}\left[\frac{\partial}{\partial x_\Sigma^1}(\sqrt{g}a^1)+\frac{\partial}{\partial x_\Sigma^2}(\sqrt{g}a^2)\right]\mathrm{d}\sigma=\int_{\partial\mathscr{D}_x}[-\sqrt{g}a^2\boldsymbol{i}_1+\sqrt{g}a^1\boldsymbol{i}_2]\cdot\boldsymbol{\tau}\mathrm{d}l=$$

$$\int_a^b[-\sqrt{g}a^2\dot{x}^1(t)+\sqrt{g}a^1\dot{x}^2(t)]\mathrm{d}t=\int_a^b\sqrt{g}\,e_{3ij}a^i\dot{x}^j(t)\mathrm{d}t=\int_a^b\varepsilon_{3ij}a^i\dot{x}_j(t)\mathrm{d}t=$$

$$\int_a^b[\boldsymbol{n},a^i\boldsymbol{g}_i,\dot{x}^j(t)\boldsymbol{g}_j]\mathrm{d}t=\oint_{\partial\Sigma}[\boldsymbol{n},\boldsymbol{a},\boldsymbol{\tau}]\mathrm{d}t=\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{a}\mathrm{d}t.$$

Subsequently, the relation as the general type

$$\int_{\mathscr{D}_x}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ-\boldsymbol{\Phi})(x_\Sigma,t)\mathrm{d}\sigma=\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot(\boldsymbol{\Theta}\circ-\boldsymbol{\Phi})\mathrm{d}l$$

can be verified. The essentials of the deduction is to transfer some indices of the tensor with respect to the local bases to the ones with respect to the canonical bases, namely

$$\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ-\boldsymbol{\Phi}=\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ-(\boldsymbol{\Phi}^{\beta}{}_{\cdot\gamma}\boldsymbol{g}_\beta\otimes\boldsymbol{g}^\gamma)=\sqrt{g}\Theta^{s\alpha}\boldsymbol{i}_\alpha\circ-(\Phi^{\lambda}{}_{\cdot\mu}\boldsymbol{i}_\lambda\otimes\boldsymbol{i}^\mu),$$

where $\boldsymbol{\Phi}\in\mathscr{T}^2(\boldsymbol{T\Sigma})$ can be extended to $\mathscr{T}^2(\mathbb{R}^3)$ with the constrain $\boldsymbol{\Phi}(\cdot,\boldsymbol{n})=\boldsymbol{\Phi}(\boldsymbol{n},\cdot)=0\in\mathbb{R}$, then the transformation can be carried out

$$\Theta^{sj}\boldsymbol{g}_j=\Theta^{s\beta}\boldsymbol{g}_\beta:=\Theta(\boldsymbol{g}^s,\boldsymbol{g}^\beta)\boldsymbol{g}_\beta=\boldsymbol{\Theta}(\boldsymbol{g}^s,(\boldsymbol{g}^\beta,\boldsymbol{i}_\alpha)_{\mathbb{R}^3}\boldsymbol{i}^\alpha)\boldsymbol{g}_\beta=\boldsymbol{\Theta}(\boldsymbol{g}^s,\boldsymbol{i}^\alpha)[(\boldsymbol{i}_\alpha,\boldsymbol{g}^\beta)_{\mathbb{R}^3}\boldsymbol{g}_\beta]=:\Theta^{s\alpha}\boldsymbol{i}_\alpha.$$

Consequently, one can do the following deduction:

$$\int_{\mathscr{D}_x}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{sj}\boldsymbol{g}_j\circ-\boldsymbol{\Phi})\mathrm{d}\sigma=\left[\int_{\mathscr{D}x}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}\Theta^{s\alpha}\Phi^{\lambda}{}_{\cdot\mu})\mathrm{d}\sigma\right]\boldsymbol{i}_\alpha\circ-\boldsymbol{i}_\lambda\otimes\boldsymbol{i}^\mu=$$

$$\left[\oint_{\partial\Sigma}(\boldsymbol{n}\times\boldsymbol{\tau})\cdot(\Theta^{s\alpha}\Phi^{\lambda}{}_{\cdot\mu}\boldsymbol{g}_s)\right]\boldsymbol{i}_\alpha\circ-\boldsymbol{i}_\lambda\otimes\boldsymbol{i}^\mu=$$

$$\oint_{\partial\Sigma}(\boldsymbol{n}\times\boldsymbol{\tau})\cdot[\Theta^{s\alpha}\boldsymbol{g}_s\otimes\boldsymbol{i}_\alpha\circ-(\Phi^{\lambda}{}_{\cdot\mu}\boldsymbol{i}_\lambda\otimes\boldsymbol{i}^\mu)]\mathrm{d}l,$$

where the relation as the prototype is adopted.

The relations as the prototype and general type have been adopted directly by Yin et al[7,8] respectively.

2.1.2 A kind of ways to deduce governing equations for thin enough continuous mediums

To study the representation of the natural law of momentum conservation for the continuous medium whose geometrical configuration can be taken as a surface, the so termed surface stress can be introduced

$$\boldsymbol{t}=t^{i}{}_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+t^{i}{}_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}.$$

Subsequently, the momentum conservation can be set up in the integral form

$$\int_{\overset{t}{\Sigma}}\rho\boldsymbol{a}\,\mathrm{d}\sigma=\oint_{\partial\overset{t}{\Sigma}}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{t}\mathrm{d}l+\int_{\overset{t}{\Sigma}}\boldsymbol{f}_\Sigma\mathrm{d}\sigma,$$

where $\boldsymbol{f}_\Sigma$ denotes the distribution of the action imposed directly on the surface such as the weight, fraction and electromagnetic force. The differential equation of momentum conservation can be directly attained through the intrinsic generalized Stokes formula of the second kind

$$\rho\boldsymbol{a}=\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{t}+\boldsymbol{f}_\Sigma$$

with the component forms:

$$\rho a_l=\nabla_s t^{s}{}_{\cdot l}-b_{sl}t^{s}{}_{\cdot 3}+f_l,\ \rho a_n=\nabla_s t^{s}{}_{\cdot 3}+b^i_j t^{j}{}_{\cdot i}+f_n.$$

On the other hand, the moment of momentum conservation can be represented as

$$\int_{\overset{t}{\Sigma}}\rho\boldsymbol{a}\times\boldsymbol{\Sigma}\mathrm{d}\sigma=\oint_{\partial\overset{t}{\Sigma}}[(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{t}]\times\boldsymbol{\Sigma}\mathrm{d}l+\int_{\overset{t}{\Sigma}}\boldsymbol{f}_\Sigma\times\boldsymbol{\Sigma}\mathrm{d}\sigma+\int_{\overset{t}{\Sigma}}\boldsymbol{m}_\Sigma\mathrm{d}\sigma$$

with the differential form

$$\rho\boldsymbol{a}\times\boldsymbol{\Sigma}=\overset{\Sigma}{\boldsymbol{\nabla}}\cdot(\boldsymbol{t}\times\boldsymbol{\Sigma})+\boldsymbol{f}_\Sigma\times\boldsymbol{\Sigma}+\boldsymbol{m}_\Sigma=[(\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{t})\times\boldsymbol{\Sigma}+\boldsymbol{g}^l\cdot(\boldsymbol{t}\times\boldsymbol{g}_l)]+\boldsymbol{f}_\Sigma\times\boldsymbol{\Sigma}+\boldsymbol{m}_\Sigma,$$

where $\boldsymbol{m}_\Sigma$ denotes the surface force couple.

Substituting the governing equation of momentum conservation, one arrives at the governing equation of moment of momentum conservation

$$\boldsymbol{0}=\boldsymbol{g}^l\cdot(\boldsymbol{t}\times\boldsymbol{g}_l)+\boldsymbol{m}_\Sigma=-t^{ij}\varepsilon_{ij3}\boldsymbol{n}+\sqrt{g}(-t^{2}{}_{\cdot 3}\boldsymbol{g}^1+t^{1}{}_{\cdot 3}\boldsymbol{g}^2)+\boldsymbol{m}_\Sigma,\qquad g:=\det[g_{ij}].$$

Consequently, it can be concluded that the symmetry of the components of surface stress tensor on the tangent space, i.e. $t_{ij}=t_{ji}$, corresponds to the vanishing of the component of surface force couple in the surface normal direction. And the appearance of surface stress tensor in the surface normal direction, i.e. $t^{i}{}_{\cdot 3}\neq 0$, corresponds to the existence of components of surface force couple on the tangent space.

The governing equations of the statical force equilibrium of elastic plates and shells put forward by Chien[9] are included in the above mentioned equation of momentum conservation. Comparatively, the deduction based on the intrinsic generalized Stokes formula of the second kind seems more compactly. Both Chien[9] and Aris[10] have introduced the concept of membrane or surface stress tensor in their studies on solids or fluids whose geometrical configurations can be taken as surfaces. Subsequently, the stress force can be represented as the surface divergence of the stress tensor and the differential equations of nature laws can be readily deduced from the integral representations through the intrinsic generalized Stokes formula of the second kind.

2.1.3 A differential identity for vorticity dynamics

Proposition 5 On any deformable smooth surface, the following identity is keeping valid:

$$\boldsymbol{n}\cdot[\overset{\Sigma}{\boldsymbol{\nabla}}\times(\boldsymbol{n}\times\boldsymbol{\Phi})]=(\boldsymbol{n}\times\overset{\Sigma}{\boldsymbol{\nabla}})\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})=\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{\Phi}+H\boldsymbol{n}\cdot\boldsymbol{\Phi}\qquad\forall\boldsymbol{\Phi}\in\mathscr{T}^p(\mathbb{R}^3).$$

Proof through intrinsic generalized Stokes formulas This identity can be readily proved through the utilizations of the intrinsic generalized Stokes formula of the first kind

$$\oint_{\partial\Sigma}\boldsymbol{\tau}\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})\mathrm{d}l=\int_{\Sigma}(\boldsymbol{n}\times\overset{\Sigma}{\boldsymbol{\nabla}})\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})\mathrm{d}\sigma$$

and the one of the scecond kind

$$\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{\Phi}\mathrm{d}l=\int_{\Sigma}(\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{\Phi}+H\boldsymbol{n}\cdot\boldsymbol{\Phi})\mathrm{d}\sigma$$

accompanying with the fundamental relationship

$$\boldsymbol{\tau}\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})=(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{\Phi}\qquad\forall\boldsymbol{\Phi}\in\mathscr{T}^p(\mathbb{R}^3).$$

Therefore ,the proof is completed.

Proof through directly calculations On the other hand, one can prove this identity through direct calculations as follows.

Firstly, to calculate

$$(\boldsymbol{n}\times\overset{\Sigma}{\boldsymbol{\nabla}})\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})=(\boldsymbol{g}^3\times\boldsymbol{g}^l)\cdot\frac{\partial}{\partial x_\Sigma^l}(\boldsymbol{n}\times\boldsymbol{\Phi})(x_\Sigma,t)=$$
$$\varepsilon^{3lt}\boldsymbol{g}_t\cdot\left[\frac{\partial\boldsymbol{n}}{\partial x_\Sigma^l}(x_\Sigma,t)\times\boldsymbol{\Phi}+\boldsymbol{n}\times\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}(x_\Sigma,t)\right],$$

where the first term on the right hand side is

$$\varepsilon^{3lt}\boldsymbol{g}_t\cdot\left[\frac{\partial\boldsymbol{n}}{\partial x_\Sigma^l}(x_\Sigma,t)\times\boldsymbol{\Phi}\right]=\varepsilon^{3lt}\boldsymbol{g}_t\cdot[-b_l^i\boldsymbol{g}_i\times(\Phi^{3\alpha}\boldsymbol{g}_3\otimes\boldsymbol{g}_\alpha)]=$$
$$-\varepsilon^{3lt}\varepsilon_{ti3}b_l^i\Phi^{3\alpha}\boldsymbol{g}_\alpha=-(\delta_l^l\delta_i^t-\delta_t^l\delta_i^l)b_l^i\Phi^{3\alpha}\boldsymbol{g}_\alpha=$$
$$-(\delta_i^l-2\delta_i^l)b_l^i\Phi^{3\alpha}\boldsymbol{g}_\alpha=H\Phi^{3\alpha}\boldsymbol{g}_\alpha=H\boldsymbol{n}\cdot\boldsymbol{\Phi}$$

and the second term is

$$\varepsilon^{3lt}\boldsymbol{g}_t\cdot\left[\boldsymbol{n}\times\frac{\partial\boldsymbol{\Phi}}{\partial x_\Sigma^l}(x_\Sigma,t)\right]=\varepsilon^{3lt}\boldsymbol{g}_t\cdot\left[\boldsymbol{n}\times\frac{\partial}{\partial x_\Sigma^l}(\Phi^{i\alpha}\boldsymbol{g}_i\otimes\boldsymbol{g}_\alpha+\Phi^{3\alpha}\boldsymbol{n}\otimes\boldsymbol{g}_\alpha)(x_\Sigma,t)\right]=$$
$$\varepsilon^{3lt}\boldsymbol{g}_t\cdot\left[\boldsymbol{n}\times\left(\frac{\partial\Phi^{i\alpha}}{\partial x_\Sigma^l}(x_\Sigma,t)\boldsymbol{g}_i\otimes\boldsymbol{g}_\alpha+\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_i}{\partial x_\Sigma^l}(x_\Sigma,t)\otimes\boldsymbol{g}_\alpha+\Phi^{i\alpha}\boldsymbol{g}_i\otimes\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^l}(x_\Sigma,t)\right)+\right.$$
$$\left.\boldsymbol{n}\times\left(\Phi^{3\alpha}\frac{\partial\boldsymbol{n}}{\partial x_\Sigma^l}(x_\Sigma,t)\otimes\boldsymbol{g}_\alpha\right)\right]=$$
$$\varepsilon^{3lt}\left[\frac{\partial\Phi^{i\alpha}}{\partial x_\Sigma^l}(x_\Sigma,t)\varepsilon_{t3i}\boldsymbol{g}_\alpha+\Phi^{i\alpha}\varepsilon_{t3k}\Gamma_{li}^k\boldsymbol{g}_\alpha+\Phi^{3\alpha}\varepsilon_{t3k}(-b_l^k)\boldsymbol{g}_\alpha+\varepsilon_{t3i}\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^l}(x_\Sigma,t)\right]=$$
$$\varepsilon^{3lt}\varepsilon_{3it}\frac{\partial\Phi^{i\alpha}}{\partial x_\Sigma^l}(x_\Sigma,t)\boldsymbol{g}_\alpha+\varepsilon^{3lt}\varepsilon_{3kt}\Phi^{i\alpha}\Gamma_{li}^k\boldsymbol{g}_\alpha+\varepsilon^{3lt}\varepsilon_{3it}\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^l}(x_\Sigma,t)-\varepsilon^{3lt}\varepsilon_{3kt}\Phi^{3\alpha}b_l^k\boldsymbol{g}_\alpha=$$
$$(\delta_i^l\delta_t^t-\delta_i^t\delta_t^l)\frac{\partial\Phi^{i\alpha}}{\partial x_\Sigma^l}(x_\Sigma,t)\boldsymbol{g}_\alpha+(\delta_k^l\delta_t^t-\delta_k^t\delta_t^l)\Phi^{i\alpha}\Gamma_{li}^k\boldsymbol{g}_\alpha+(\delta_i^l\delta_t^t-\delta_i^t\delta_t^l)\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^l}(x_\Sigma,t)-(\delta_k^l\delta_t^t-\delta_k^t\delta_t^l)\Phi^{3\alpha}b_l^k\boldsymbol{g}_\alpha=$$
$$\frac{\partial\Phi^{k\alpha}}{\partial x_\Sigma^k}(x_\Sigma,t)\boldsymbol{g}_\alpha+\Gamma_{ki}^k\Phi^{i\alpha}\boldsymbol{g}_\alpha+\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^i}(x_\Sigma,t)-H\Phi^{3\alpha}\boldsymbol{g}_\alpha=$$
$$\left(\frac{\partial\Phi^{k\alpha}}{\partial x_\Sigma^k}(x_\Sigma,t)+\Gamma_{ki}^k\Phi^{i\alpha}\right)\boldsymbol{g}_\alpha+\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^i}(x_\Sigma,t)-H\boldsymbol{n}\cdot\boldsymbol{\Phi}.$$

Subsequently, one has

$$(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})=\left(\frac{\partial\Phi^{k\alpha}}{\partial x_\Sigma^k}(x_\Sigma,t)+\Gamma_{ki}^k\Phi^{i\alpha}\right)\boldsymbol{g}_\alpha+\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^i}(x_\Sigma,t).$$

Secondly, to calculate

$$\overset{\Sigma}{\nabla}\cdot\boldsymbol{\Phi}=\boldsymbol{g}^l\cdot\frac{\partial}{\partial x_\Sigma^l}(\Phi^{i\alpha}\boldsymbol{g}_i\otimes\boldsymbol{g}_\alpha+\Phi^{3\alpha}\boldsymbol{n}\otimes\boldsymbol{g}_\alpha)(x_\Sigma,t)=$$

$$\boldsymbol{g}^l\cdot\left[\frac{\partial\Phi^{i\alpha}}{\partial x_\Sigma^l}(x_\Sigma,t)\boldsymbol{g}_i\otimes\boldsymbol{g}_\alpha+\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_i}{\partial x_\Sigma^l}(x_\Sigma,t)\otimes\boldsymbol{g}_\alpha+\Phi^{i\alpha}\boldsymbol{g}_i\otimes\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^l}(x_\Sigma,t)+\Phi^{3\alpha}\frac{\partial\boldsymbol{n}}{\partial x_\Sigma^l}(x_\Sigma,t)\otimes\boldsymbol{g}_\alpha\right]=$$

$$\left(\frac{\partial\Phi^{k\alpha}}{\partial x_\Sigma^k}(x_\Sigma,t)+\Gamma_{ki}^k\Phi^{i\alpha}\right)\boldsymbol{g}_\alpha+\Phi^{i\alpha}\frac{\partial\boldsymbol{g}_\alpha}{\partial x_\Sigma^i}(x_\Sigma,t)-H\Phi^{3\alpha}\boldsymbol{g}_\alpha.$$

Summary, one arrives at

$$(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\cdot(\boldsymbol{n}\times\boldsymbol{\Phi})=\overset{\Sigma}{\nabla}\cdot\boldsymbol{\Phi}+H\boldsymbol{n}\cdot\boldsymbol{\Phi}.$$

It is the end of the proof.

In fluid mechanics, the nature law of momentum conservation is represented by so called Navier-Stokes equation (NSE):

$$\rho\boldsymbol{a}=\nabla\Pi-\mu\nabla\times\boldsymbol{\omega}+\rho\boldsymbol{f}_m,$$

where $\Pi:=-p+(\lambda+2\mu)\theta$, λ and μ denote different viscous coefficients, $\boldsymbol{\omega}:=\nabla\times\boldsymbol{V}$ is the vorticity. To deduce the representation of the flux of the dilation quantity on any smooth deformable boundary, $\cdot\,\boldsymbol{n}$ is taken on both side of NSE:

$$\frac{\partial\Pi}{\partial\boldsymbol{n}}:=\boldsymbol{n}\cdot\nabla\Pi=\rho\boldsymbol{n}\cdot\boldsymbol{a}+\mu\boldsymbol{n}\cdot(\nabla\times\boldsymbol{\omega})-\rho\boldsymbol{n}\cdot\boldsymbol{f}_m=$$

$$\rho\boldsymbol{n}\cdot\boldsymbol{a}+\mu(\boldsymbol{n}\times\nabla)\cdot\boldsymbol{\omega}-\rho\boldsymbol{n}\cdot\boldsymbol{f}_m=\rho\boldsymbol{n}\cdot\boldsymbol{a}+\mu(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\cdot\boldsymbol{\omega}-\rho\boldsymbol{n}\cdot\boldsymbol{f}_m.$$

On the other hand, $\times\boldsymbol{n}$ is taken on both sides of NSE to deduce the representation of the flux of the vorticity on the boundary:

$$\mu\frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{n}}=\rho\boldsymbol{n}\times\boldsymbol{a}-\boldsymbol{n}\times(\nabla\Pi)+\mu(\nabla\otimes\boldsymbol{\omega})\cdot\boldsymbol{n}+\rho\boldsymbol{f}_m\times\boldsymbol{n}=$$

$$\rho\boldsymbol{n}\times\boldsymbol{a}-(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\Pi+\mu(\nabla\otimes\boldsymbol{\omega})\cdot\boldsymbol{n}+\rho\boldsymbol{f}_m\times\boldsymbol{n},$$

where the following identity is adopted:

$$(\nabla\times\boldsymbol{\omega})\times\boldsymbol{n}=\boldsymbol{n}\cdot(\nabla\otimes\boldsymbol{\omega})-(\nabla\otimes\boldsymbol{\omega})\cdot\boldsymbol{n}.$$

Furthermore, one has the following relationships:

$$(\nabla\otimes\boldsymbol{\omega})\cdot\boldsymbol{n}=\overset{\Sigma}{\nabla}(\boldsymbol{\omega}\cdot\boldsymbol{n})-[\boldsymbol{\omega}\cdot(\overset{\Sigma}{\nabla}\otimes\boldsymbol{n})+(\overset{\Sigma}{\nabla}\cdot\boldsymbol{\omega})\boldsymbol{n}]=\overset{\Sigma}{\nabla}(\boldsymbol{\omega}\cdot\boldsymbol{n})-\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\omega}\otimes\boldsymbol{n}),$$

thanks to $\nabla\cdot\boldsymbol{\omega}=0$ and

$$\boldsymbol{\xi}\cdot(\overset{\Sigma}{\nabla}\otimes\boldsymbol{\eta})+(\overset{\Sigma}{\nabla}\cdot\boldsymbol{\xi})\boldsymbol{\eta}=\overset{\Sigma}{\nabla}\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\eta})\qquad\forall\,\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^3.$$

Finally, one arrives at the representation

$$\mu\frac{\partial\omega}{\partial n}=\rho\boldsymbol{n}\times\boldsymbol{a}-(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\Pi+\mu(\nabla\otimes\boldsymbol{\omega})\cdot\boldsymbol{n}+\rho\boldsymbol{f}_m\times\boldsymbol{n}=$$

$$\rho\boldsymbol{n}\times\boldsymbol{a}-(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\Pi+\mu(\boldsymbol{n}\times\overset{\Sigma}{\nabla})\cdot[(\boldsymbol{\omega}\times\boldsymbol{n})\otimes\boldsymbol{n}]+\rho\boldsymbol{f}_m\times\boldsymbol{n}+\mu[\overset{\Sigma}{\nabla}(\boldsymbol{\omega}\cdot\boldsymbol{n})+H(\boldsymbol{\omega}\cdot\boldsymbol{n})\boldsymbol{n}],$$

where $\mu[\overset{\Sigma}{\boldsymbol{\nabla}}(\boldsymbol{\omega}\cdot\boldsymbol{n})+H(\boldsymbol{\omega}\cdot\boldsymbol{n})\boldsymbol{n}]$ is the additional term contributed purely by the deformation of the boundary and equals to zero for any flow on the plane. Its mechanical meaning can be revealed by the following integral relation:

$$\int_{\Sigma}\mu\frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{n}}\mathrm{d}\sigma\sim\int_{\Sigma}\mu[\overset{\Sigma}{\boldsymbol{\nabla}}(\boldsymbol{\omega}\cdot\boldsymbol{n})+H(\boldsymbol{\omega}\cdot\boldsymbol{n})\boldsymbol{n}]\mathrm{d}\sigma=\oint_{\partial\Sigma}\mu(\boldsymbol{\tau}\times\boldsymbol{n})(\boldsymbol{\omega}\cdot\boldsymbol{n})\mathrm{d}l.$$

The representation of the vorticity flux on the fixed solid boundary can be referred to the monograph of Wu et al[11].

2.2 Primary properties of deformation gradient tensor for thin enough continuous mediums

The deformation gradient tensor plays the essential role in the whole theory of continuous mediums without regard to those geometrical configurations that are bulks corresponding to Euclid manifolds or surfaces to Riemann manifolds[5]. The latter theory is termed briefly as the surface deformation theory hereinafter. The primary properties of the deformation gradient tensor for continuous mediums whose geometrical configurations are surfaces can be concluded as follows.

Proposition 6 (properties of deformation gradient for surface deformation theory)

$$\dot{\boldsymbol{F}}=\boldsymbol{L}\cdot\boldsymbol{F},\quad \boldsymbol{L}:=\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}},$$
$$\overset{\cdot}{\overline{|\boldsymbol{F}|}}=\theta|\boldsymbol{F}|,\quad \theta:=\boldsymbol{V}\cdot\overset{\Sigma}{\boldsymbol{\nabla}}.$$

Denoted by $\boldsymbol{F}$, the deformation gradient with its determinant are defined as follows:

$$\boldsymbol{F}:=\frac{\partial x_{\Sigma}^{i}}{\partial\xi_{\Sigma}^{A}}(\xi_{\Sigma},t)\boldsymbol{g}_{i}\otimes\boldsymbol{G}^{A},\quad |\boldsymbol{F}|=\frac{\sqrt{g}}{\sqrt{G}}\det\left[\frac{\partial x_{\Sigma}^{i}}{\partial\xi_{\Sigma}^{A}}(\xi_{\Sigma},t)\right],$$

where $\{\xi_{\Sigma}^{A}\}_{A=1}^{m}$ and $\{x_{\Sigma}^{i}\}_{i=1}^{m}$ represent the Lagrangian and Eulerian coordinates in the parametric space respectively. The physical and parametric configurations for surface deformation is sketched in Fig. 3.

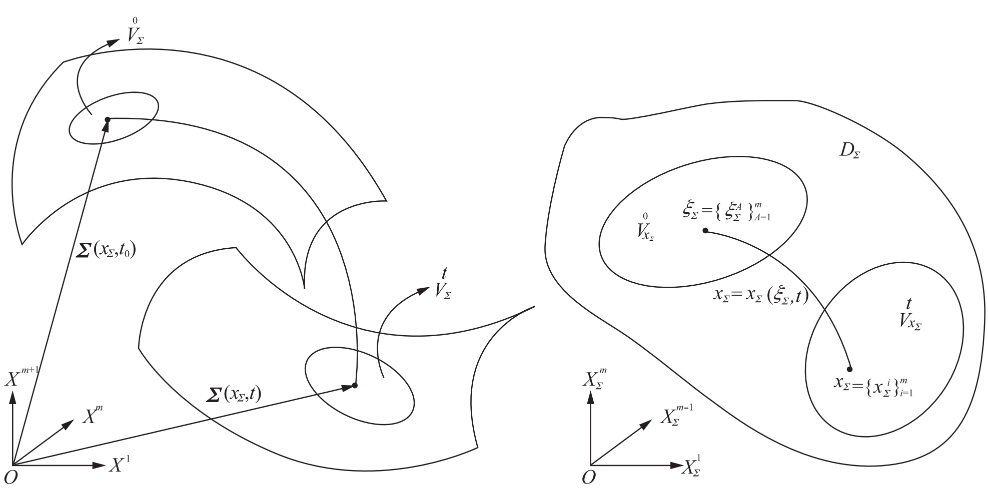

Fig. 3 Sketch of the physical and parametric configurations for a continuous mediums whose configuration is general smooth surface, $\overset{0}{V}_{\Sigma}$ and $\overset{t}{V}_{\Sigma}$ denote initial and current physical configurations respectively, $\overset{0}{V}_{\xi_{\Sigma}}$ and $\overset{t}{V}_{x_{\Sigma}}$ denote initial and current parametric configurations respectively, the mapping $D_{\Sigma}\times(t_0,t_0+T)\ni(x_{\Sigma},t)\longmapsto\boldsymbol{\Sigma}(x_{\Sigma},t)$ denotes the surface-self deformation.

Firstly, we put forward the following lemma for $g:=\det[g_{ij}]$.

Lemma 1 (Some identities on the determinant of the metric tensor)

$$\frac{1}{g}\frac{\partial g}{\partial x_{\Sigma}^{l}}(x_{\Sigma},t)=g^{ij}\frac{\partial g_{ij}}{\partial x_{\Sigma}^{l}}(x_{\Sigma},t);\quad \frac{1}{g}\frac{\partial g}{\partial t}(x_{\Sigma},t)=g^{ij}\frac{\partial g_{ij}}{\partial t}(x_{\Sigma},t).$$

Proof of the lemma　To consider

$$g := \det[g_{ij}] = \sum_{s=1}^{m} \Delta^{is} g_{is} \qquad \forall\, i=1,2,\cdots,m.$$

where Δ^{is} denotes the element in the ith row and sth column of the conjugate matrix of $[g_{pq}]$. Then, the determinant can be represented as

$$g = g(\{g_{ij}\,\text{included}\}) \quad \text{s. t.} \quad \frac{\partial g}{\partial g_{ij}}(\{g_{ij}\,\text{included}\}) = \Delta^{ij} \neq 0.$$

It means that if g_{ij} is included in the representation of g then $\Delta^{ij} \neq 0$, and if g does not include g_{ij} then $\Delta^{ij} = 0$.

Subsequently, one can do the deduction

$$\frac{\partial g}{\partial x_{\Sigma}^{l}} = \sum_{g_{ij}\,\text{included}} \frac{\partial g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial x_{\Sigma}^{l}}(x_{\Sigma},t) = \sum_{g_{ij}\,\text{included}} \Delta^{ij} \frac{\partial g_{ij}}{\partial x_{\Sigma}^{l}}(x_{\Sigma},t) = \sum_{p,q=1}^{m} \Delta^{pq} \frac{\partial g_{pq}}{\partial x_{\Sigma}^{l}}(x_{\Sigma},t) = g g^{pq} \frac{\partial g_{pq}}{\partial x_{\Sigma}^{l}}(x_{\Sigma},t).$$

The second identity can be proved similarly. As a corollary, one has the identity

$$\Gamma_{ij}^{i} \triangleq g^{ik}\Gamma_{ij,k} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_{\Sigma}^{j}}(x_{\Sigma},t), \quad \Gamma_{ij,k} \triangleq \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_{\Sigma}^{j}} + \frac{\partial g_{jk}}{\partial x_{\Sigma}^{i}} - \frac{\partial g_{ij}}{\partial x_{\Sigma}^{k}}\right)(x_{\Sigma},t).$$

Proof of the proposition　Firstly, to calculate the material derivative of the deformation gradient tensor

$$\dot{\boldsymbol{F}} = \overline{\frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}(\xi_{\Sigma},t)\boldsymbol{g}_i(x_{\Sigma},t) \otimes \boldsymbol{G}^A(\xi)}^{\boldsymbol{\cdot}} = \overline{\frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}}^{\boldsymbol{\cdot}}(\xi_{\Sigma},t)\boldsymbol{g}_i \otimes \boldsymbol{G}^A + \frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}(\xi_{\Sigma},t)\overline{\boldsymbol{g}_i(x_{\Sigma},t)}^{\boldsymbol{\cdot}} \otimes \boldsymbol{G}^A,$$

where

$$\overline{\frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}(\xi,t)}^{\boldsymbol{\cdot}} = \frac{\partial^2 x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A} \partial t}(\xi_{\Sigma},t) =: \frac{\partial \dot{x}_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}(\xi,t) = \frac{\partial x_{\Sigma}^{s}}{\partial \xi_{\Sigma}^{A}}(\xi,t)\frac{\partial \dot{x}_{\Sigma}^{i}}{\partial x_{\Sigma}^{s}}(x,t),$$

$$\overline{\boldsymbol{g}_i(x,t)}^{\boldsymbol{\cdot}} = \frac{\partial \boldsymbol{g}_i}{\partial t}(x,t) + \dot{x}_{\Sigma}^{s}\frac{\partial \boldsymbol{g}_i}{\partial x_{\Sigma}^{s}}(x,t) = \frac{\partial}{\partial x_{\Sigma}^{i}}\left(\frac{\partial \boldsymbol{\Sigma}}{\partial t}\right)(x,t) + \dot{x}_{\Sigma}^{s}\frac{\partial \boldsymbol{g}_s}{\partial x_{\Sigma}^{i}}(x,t).$$

Subsequently, one has

$$\dot{\boldsymbol{F}} = \frac{\partial x_{\Sigma}^{s}}{\partial \xi^{A}}\left[\frac{\partial \dot{x}_{\Sigma}^{i}}{\partial x_{\Sigma}^{s}}(x_{\Sigma},t)\boldsymbol{g}_i \otimes \boldsymbol{G}^A + \frac{\partial}{\partial x_{\Sigma}^{s}}\left(\frac{\partial \boldsymbol{\Sigma}}{\partial t}\right)(x_{\Sigma},t) \otimes \boldsymbol{G}^A + \dot{x}_{\Sigma}^{i}\frac{\partial \boldsymbol{g}_i}{\partial x_{\Sigma}^{s}}(x_{\Sigma},t) \otimes \boldsymbol{G}^A\right] =$$

$$\left[\frac{\partial}{\partial x_{\Sigma}^{s}}\left(\frac{\partial \boldsymbol{\Sigma}}{\partial t}\right)(x_{\Sigma},t) + \frac{\partial \dot{x}_{\Sigma}^{i}}{\partial x_{\Sigma}^{s}}(x_{\Sigma},t)\boldsymbol{g}_i + \dot{x}_{\Sigma}^{i}\frac{\partial \boldsymbol{g}^i}{\partial x_{\Sigma}^{s}}(x_{\Sigma},t)\right] \otimes \left[\frac{\partial x_{\Sigma}^{s}}{\partial \xi^{A}}(\xi,t)\boldsymbol{G}^A\right] =$$

$$\left[\frac{\partial}{\partial x_{\Sigma}^{s}}\left(\frac{\partial \boldsymbol{\Sigma}}{\partial t} + \dot{x}_{\Sigma}^{i}\boldsymbol{g}_i\right)(x_{\Sigma},t) \otimes \boldsymbol{g}^s\right] \cdot \left[\frac{\partial x_{\Sigma}^{t}}{\partial \xi^{A}}(\xi,t)\boldsymbol{g}_t \otimes \boldsymbol{G}^A\right] = (\boldsymbol{V} \otimes \overset{\Sigma}{\boldsymbol{\nabla}}) \cdot \boldsymbol{F}.$$

Then calculate the material derivative of the determinant of the deformation gradient tensor

$$\overline{|\boldsymbol{F}|}^{\boldsymbol{\cdot}} = \overline{\frac{\sqrt{g}(x_{\Sigma},t)}{\sqrt{G}(\xi_{\Sigma},t)i}\det\left[\frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}\right](\xi_{\Sigma},t)}^{\boldsymbol{\cdot}} = \frac{1}{\sqrt{G}}\overline{\sqrt{g}(x_{\Sigma},t)}^{\boldsymbol{\cdot}}\det\left[\frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}\right](x_{\Sigma},t) + \frac{\sqrt{g}}{\sqrt{G}}\overline{\det\left[\frac{\partial x_{\Sigma}^{i}}{\partial \xi_{\Sigma}^{A}}\right]}^{\boldsymbol{\cdot}}(x_{\Sigma},t),$$

where

$$\overline{\sqrt{g}(x_{\Sigma},t)}^{\boldsymbol{\cdot}} = \frac{\partial \sqrt{g}}{\partial t}(x_{\Sigma},t) + \dot{x}_{\Sigma}^{s}\frac{\partial \sqrt{g}}{\partial \dot{x}_{\Sigma}^{s}}(x_{\Sigma},t) =$$

$$\sqrt{g}\left[\frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial t}(x_{\Sigma},t) + \dot{x}_{\Sigma}^{s}\frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x_{\Sigma}^{s}}(x_{\Sigma},t)\right] = \sqrt{g}\left[\frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial t}(x_{\Sigma},t) + \Gamma_{st}^{t}\dot{x}_{\Sigma}^{s}\right]$$

and

$$\overline{\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t)}^{\bullet}=\frac{\partial \dot{x}_\Sigma^s}{\partial x_\Sigma^s}(x_\Sigma,t)\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t).$$

Subsequently, one has

$$\dot{\overline{|\boldsymbol{F}|}}=|\boldsymbol{F}|\cdot\left[\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}(x_\Sigma,t)+\frac{\partial \dot{x}_\Sigma^s}{\partial x_\Sigma^s}(x_\Sigma,t)+\Gamma_{sl}^s\dot{x}_\Sigma^l\right]=|\boldsymbol{F}|\cdot\left[\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}(x_\Sigma,t)+\nabla_s\dot{x}_\Sigma^s\right].$$

On the other hand, the divergence of the velocity can be calculated as follows:

$$\begin{aligned}\boldsymbol{V}\cdot\overset{\Sigma}{\boldsymbol{\nabla}}&=\frac{\partial\boldsymbol{V}}{\partial x_\Sigma^l}(x_\Sigma,t)\cdot g^l=\frac{\partial}{\partial x_\Sigma^l}\left(\frac{\partial\boldsymbol{\Sigma}}{\partial t}+\dot{x}_\Sigma^s\boldsymbol{g}_s\right)(x_\Sigma,t)\cdot\boldsymbol{g}^l=\\&\boldsymbol{g}^l\cdot\frac{\partial\boldsymbol{g}_l}{\partial t}(x_\Sigma,t)+\nabla_s\dot{x}_\Sigma^s=g^{lk}\boldsymbol{g}_k\cdot\frac{\partial\boldsymbol{g}_l}{\partial t}(x_\Sigma,t)+\nabla_s\dot{x}_\Sigma^s=\\&\frac{1}{2}g^{lk}\frac{\partial g_{lk}}{\partial t}(x_\Sigma,t)+\nabla_s\dot{x}_\Sigma^s=\frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial t}(x_\Sigma,t)+\nabla_s\dot{x}_\Sigma^s=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}(x_\Sigma,t)+\nabla_s\dot{x}_\Sigma^s.\end{aligned}$$

This ends the proof.

In addition, we give a proof of the identity that has been adopted previously.

Lemma 2

$$\overline{\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t)}^{\bullet}=\frac{\partial \dot{x}_\Sigma^s}{\partial x_\Sigma^s}(x_\Sigma,t)\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t).$$

Proof The determinant of a matrix can be represented through the permutation operator

$$\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t)=\sum_{\sigma\in P_m}\operatorname{sgn}\sigma\left[\frac{\partial x_\Sigma^1}{\partial \xi_\Sigma^{\sigma(1)}}\frac{\partial x_\Sigma^2}{\partial \xi_\Sigma^{\sigma(2)}}\cdots\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}\right](\xi_\Sigma,t).$$

Subsequently, one has

$$\overline{\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t)}^{\bullet}=\sum_{\sigma\in P_m}\operatorname{sgn}\sigma\left[\overline{\frac{\partial x_\Sigma^1}{\partial \xi_\Sigma^{\sigma(1)}}}^{\bullet}\frac{\partial x_\Sigma^2}{\partial \xi_\Sigma^{\sigma(2)}}\cdots\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}+\cdots+\frac{\partial x_\Sigma^1}{\partial \xi_\Sigma^{\sigma(1)}}\cdots\frac{\partial x_\Sigma^{m-1}}{\partial \xi_\Sigma^{\sigma(m-1)}}\overline{\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}}^{\bullet}\right](\xi_\Sigma,t).$$

To deal with the first term on the right hand side:

$$\begin{aligned}&\sum_{\sigma\in P_m}\operatorname{sgn}\sigma\left[\overline{\frac{\partial x_\Sigma^1}{\partial \xi_\Sigma^{\sigma(1)}}}^{\bullet}\frac{\partial x_\Sigma^2}{\partial \xi_\Sigma^{\sigma(2)}}\cdots\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}\right](\xi_\Sigma,t)=\sum_{\sigma\in P_m}\operatorname{sgn}\sigma\left[\frac{\partial \dot{x}_\Sigma^1}{\partial \xi_\Sigma^{\sigma(1)}}\frac{\partial x_\Sigma^2}{\partial \xi_\Sigma^{\sigma(2)}}\cdots\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}\right](\xi_\Sigma,t)=\\&\sum_{\sigma\in P_m}\operatorname{sgn}\sigma\left[\left(\frac{\partial \dot{x}_\Sigma^1}{\partial x_\Sigma^s}(x_\Sigma,t)\frac{\partial x_\Sigma^s}{\partial \xi_\Sigma^{\sigma(1)}}(\xi_\Sigma,t)\right)\frac{\partial x_\Sigma^2}{\partial \xi_\Sigma^{\sigma(2)}}(\xi_\Sigma,t)\cdots\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}(\xi_\Sigma,t)\right]=\\&\frac{\partial \dot{x}_\Sigma^1}{\partial x_\Sigma^s}(x_\Sigma,t)\sum_{\sigma\in P_m}\operatorname{sgn}\sigma\left[\frac{\partial x_\Sigma^s}{\partial \xi_\Sigma^{\sigma(1)}}\frac{\partial x_\Sigma^2}{\partial \xi_\Sigma^{\sigma(2)}}\cdots\frac{\partial x_\Sigma^m}{\partial \xi_\Sigma^{\sigma(m)}}\right](\xi_\Sigma,t)=\frac{\partial \dot{x}_\Sigma^1}{\partial x_\Sigma^1}(x_\Sigma,t)\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t).\end{aligned}$$

It is evident that the other terms can be similarly processed. Then the proof is completed.

It should be pointed out that the concluded properties of the deformation gradient tensor with the related lemmas in this subsection are keeping valid for continuous mediums whose geometrical configurations are surfaces with arbitrary finite dimensions. Therefore, the related results can be taken as the extension of the surface deformation theory with respect to the two dimensional surfaces as put forward by Xie et al[5].

In the monograph *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Aris[10] expatiated on the equations governing two dimensional flows on an arbitrary fixed surface. It is worthy of mention that in Aris' monograph, the following relation was adopted:

$$\overline{\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t)}^{\,\cdot}=\left[\frac{\partial \dot{x}_\Sigma^s}{\partial x_\Sigma^s}(x_\Sigma,t)+\Gamma_{sj}^s\dot{x}_\Sigma^j\right]\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t)=(\nabla_s\dot{x}_\Sigma^s)\det\left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}\right](\xi_\Sigma,t).$$

However, it is not true. Consequently, the transport equation on the fixed surface attained by Aris takes the following form, see (10.12.9) in Aris' monograph,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{\Sigma}}\Phi\mathrm{d}\sigma=\int_{\overset{t}{\Sigma}}\left[\dot{\Phi}+\Phi\left(\nabla_s V^s+\frac{\dot{g}}{2g}\right)\right]\mathrm{d}\sigma,\qquad V^s:=\dot{x}_\Sigma^s.$$

In fact, the true one should be

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{\Sigma}}\Phi\mathrm{d}\sigma=\int_{\overset{t}{\Sigma}}\left[\dot{\Phi}+\Phi\left(\frac{\partial V^s}{\partial x_\Sigma^s}(x_\Sigma,t)+\frac{\dot{g}}{2g}\right)\right]\mathrm{d}\sigma=\int_{\overset{t}{\Sigma}}[\dot{\Phi}+\Phi\,\nabla_s V^s]\mathrm{d}\sigma$$

accompanying with the relation

$$\frac{\dot{g}}{2g}=\frac{1}{2g}\frac{\partial g}{\partial x_\Sigma^i}(x_\Sigma)\dot{x}_\Sigma^i=\frac{1}{2g}\frac{\partial g}{\partial x_\Sigma^i}(x_\Sigma)V^i=\Gamma_{si}^s V^i.$$

The case studied by Aris[10] is the two dimensional flow on an arbitrary fixed surface. The surface deformation theory put forward by Xie et al[5] is available to two dimensional flows either on fixed surfaces or on deformable ones. Furthermore, a theoretical framework of vorticity dynamics for two dimensional flows on fixed surfaces has been provided recently by Xie[12].

2.3 Strain tensor on an arbitrary deformable surface

For any motion/deformation of continuous mediums, the strain tensor on an arbitrary deformable smooth surface/boundary can take the following representation.

Proposition 7 (Strain tensor on an arbitrary deformable surface)

$$\boldsymbol{D}\triangleq\frac{1}{2}(\boldsymbol{V}\otimes\boldsymbol{\nabla}+\boldsymbol{\nabla}\otimes\boldsymbol{V})=(\theta-\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V})\boldsymbol{n}\otimes\boldsymbol{n}+\frac{1}{2}[(\boldsymbol{\omega}+\boldsymbol{W})\times\boldsymbol{n}]\otimes\boldsymbol{n}+\frac{1}{2}\boldsymbol{n}\otimes[(\boldsymbol{\omega}+\boldsymbol{W})\times\boldsymbol{n}]+\overset{\Sigma}{\boldsymbol{D}},$$

where $\overset{\Sigma}{\boldsymbol{D}}\triangleq(\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}+\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V})/2$ is the strain of the boundary, $\boldsymbol{W}:=-(\overset{\Sigma}{\boldsymbol{\nabla}}V^3+\boldsymbol{V}\cdot\boldsymbol{K})\times\boldsymbol{n}$ is purely determined by the boundary.

Proof　Firstly, we introduce the intrinsic decomposition of any tensor field with respect to any direction.

Lemma 3

$$\boldsymbol{\Phi}=\begin{cases}\boldsymbol{e}\otimes(\boldsymbol{e},\boldsymbol{\Phi})_{\mathbb{R}^3}-[\boldsymbol{e},[\boldsymbol{e},\boldsymbol{\Phi}]]\\(\boldsymbol{\Phi},\boldsymbol{e})_{\mathbb{R}^3}\otimes\boldsymbol{e}-[[\boldsymbol{\Phi},\boldsymbol{e}],\boldsymbol{e}]\end{cases}\qquad\forall\,|\boldsymbol{e}|_{\mathbb{R}^3}=1,\quad\forall\,\boldsymbol{\Phi}\in\mathscr{T}^p(\mathbb{R}^3).$$

It can be directly verified.

As an application, one has

$$\boldsymbol{V}\otimes\boldsymbol{\nabla}=(\boldsymbol{V}\otimes\boldsymbol{\nabla},\boldsymbol{n})\otimes\boldsymbol{n}-[[\boldsymbol{V}\otimes\boldsymbol{\nabla},\boldsymbol{n}],\boldsymbol{n}]$$

with the relations

$$(\boldsymbol{V}\otimes\boldsymbol{\nabla},\boldsymbol{n})_{\mathbb{R}^3}=(\boldsymbol{V}\otimes\boldsymbol{\nabla}-\boldsymbol{\nabla}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}+(\boldsymbol{\nabla}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}=\boldsymbol{\omega}\times\boldsymbol{n}+(\boldsymbol{\nabla}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3},$$

$$(\boldsymbol{\nabla}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}=\left(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V}+\boldsymbol{n}\otimes\frac{\partial\boldsymbol{V}}{\partial x^3}(x_\Sigma,t),\boldsymbol{n}\right)_{\mathbb{R}^3}=(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}+\frac{\partial\boldsymbol{V}^3}{\partial x^3}(x_\Sigma,t)\boldsymbol{n}.$$

In addition, one has

$$(\boldsymbol{\nabla}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}=(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}+(\theta-\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V})\boldsymbol{n},$$

due to

$$\theta:=\boldsymbol{\nabla}\cdot\boldsymbol{V}=\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V}+\boldsymbol{n}\cdot\frac{\partial\boldsymbol{V}}{\partial x^3}(x,t)=\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V}+\frac{\partial V^3}{\partial x^3}.$$

Furthermore, one has based on the intrinsic decomposition

$$(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}=\boldsymbol{n}\otimes(\boldsymbol{n},(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3})_{\mathbb{R}^3}-[\boldsymbol{n},[\boldsymbol{n},(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}]]=$$
$$-[\boldsymbol{n},[\boldsymbol{n},(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}]]=:\boldsymbol{W}\times\boldsymbol{n},$$

where

$$\boldsymbol{W}:=-(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}\times\boldsymbol{n}=-(\overset{\Sigma}{\boldsymbol{\nabla}}V^3+\boldsymbol{V}\cdot\boldsymbol{K})\times\boldsymbol{n}\in\boldsymbol{T\Sigma},$$

due to

$$\begin{aligned}(\overset{\Sigma}{\boldsymbol{\nabla}}\otimes\boldsymbol{V},\boldsymbol{n})_{\mathbb{R}^3}=&\left(\boldsymbol{g}^l\otimes\frac{\partial}{\partial x_\Sigma^l}(V_i\boldsymbol{g}^i+V^3\boldsymbol{n})(x_\Sigma,t),\boldsymbol{n}\right)_{\mathbb{R}^3}=\\&\left(\nabla_l V_i\boldsymbol{g}^l\otimes\boldsymbol{g}^i+V_ib_l^i\boldsymbol{g}^l\otimes\boldsymbol{n}+\frac{\partial V^3}{\partial x_\Sigma^l}(x_\Sigma,t)\boldsymbol{g}^l\otimes\boldsymbol{n}+V^3\boldsymbol{g}^l\otimes\frac{\partial\boldsymbol{n}}{\partial x_\Sigma^l}(x_\Sigma,t),\boldsymbol{n}\right)_{\mathbb{R}^3}=\\&\left(V_ib_l^i+\frac{\partial V^3}{\partial x_\Sigma^l}(x_\Sigma,t)\right)\boldsymbol{g}^l=\overset{\Sigma}{\boldsymbol{\nabla}}V^3+\boldsymbol{V}\cdot\boldsymbol{K}\in\boldsymbol{T\Sigma}.\end{aligned}$$

As a summary, it is attained that

$$\boldsymbol{V}\otimes\boldsymbol{\nabla}=(\theta-\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V})\boldsymbol{n}\otimes\boldsymbol{n}+(\boldsymbol{\omega}\times\boldsymbol{n})\otimes\boldsymbol{n}+(\boldsymbol{W}\times\boldsymbol{n})\otimes\boldsymbol{n}-[[\boldsymbol{V}\otimes\boldsymbol{\nabla},\boldsymbol{n}],\boldsymbol{n}].$$

To deal with the last term, one has

$$-[[\boldsymbol{V}\otimes\boldsymbol{\nabla},\boldsymbol{n}],\boldsymbol{n}]=-[[\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}},\boldsymbol{n}],\boldsymbol{n}]=\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}-(\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}},\boldsymbol{n})_{\mathbb{R}^3}\otimes\boldsymbol{n}=\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}.$$

Consequently, one arrives at the identity that bridges the relation between the full dimensional velocity gradient and the one with respect to the surface gradient

$$\boldsymbol{V}\otimes\boldsymbol{\nabla}=(\theta-\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V})\boldsymbol{n}\otimes\boldsymbol{n}+(\boldsymbol{\omega}\times\boldsymbol{n})\otimes\boldsymbol{n}+(\boldsymbol{W}\times\boldsymbol{n})\otimes\boldsymbol{n}+\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}.$$

Taking account of the conjugate identity, one ends the proof.

The proved identity bridges the relation between the whole strain and the one purely due to the deformation of the surface. Consequently, one has the relation

$$\begin{aligned}|\Delta\boldsymbol{p}|_{\mathbb{R}^3}\dot{\overline{|\Delta\boldsymbol{p}|_{\mathbb{R}^3}}}=&\Delta\boldsymbol{p}\cdot\boldsymbol{D}\cdot\Delta\boldsymbol{p}=\\&\left(\frac{\partial\boldsymbol{V}}{\partial x^3}(x_\Sigma,t)+\boldsymbol{W}\times\boldsymbol{n},\Delta\boldsymbol{p}\right)_{\mathbb{R}^3}|\Delta\boldsymbol{p}_\perp|_{\mathbb{R}^3}+\Delta\boldsymbol{p}_{\|}\cdot\overset{\Sigma}{\boldsymbol{D}}\cdot\Delta\boldsymbol{p}_{\|}=\\&\left(\frac{\partial\boldsymbol{V}}{\partial x^3}(x_\Sigma,t)+\boldsymbol{W}\times\boldsymbol{n},\Delta\boldsymbol{p}\right)_{\mathbb{R}^3}|\Delta\boldsymbol{p}_\perp|_{\mathbb{R}^3}+|\Delta p_{\|}|_{\mathbb{R}^3}\dot{\overline{|\Delta\boldsymbol{p}_{\|}|_{\mathbb{R}^3}}},\end{aligned}$$

where $\Delta\boldsymbol{p}=\Delta\boldsymbol{p}_\perp+\Delta\boldsymbol{p}_{\|}$ denotes the directed line element, $\Delta\boldsymbol{p}_\perp\perp\boldsymbol{T\Sigma}$ and $\Delta\boldsymbol{p}_{\|}\in\boldsymbol{T\Sigma}$ are directed line elements with respect to the normal direction and tangent space respectively. As indicated by Xie et al[5], $|\Delta\boldsymbol{p}_{\|}|_{\mathbb{R}^3}\dot{\overline{|\Delta\boldsymbol{p}_{\|}|_{\mathbb{R}^3}}}=\Delta\boldsymbol{p}_{\|}\cdot\overset{\Sigma}{D}\cdot\Delta\boldsymbol{p}_{\|}$ is a kind of representations of the deformation of the boundary. In detail, it can be deduced

$$\frac{\partial \boldsymbol{V}}{\partial x^3}(x_\Sigma,t)=\left(\frac{\partial V^j}{\partial x^3}(x_\Sigma,t)-V^s b_s^j\right)\boldsymbol{g}_j+\frac{\partial V^3}{\partial x^3}(x_\Sigma,t)\boldsymbol{n}=:\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{n}}(x_\Sigma,t),$$

$$\boldsymbol{W}\times\boldsymbol{n}=\overset{\Sigma}{\boldsymbol{\nabla}}V^3+V^s b_s^j\boldsymbol{g}_j=\overset{\Sigma}{\boldsymbol{\nabla}}V^3+\boldsymbol{V}\cdot\boldsymbol{K}\in \boldsymbol{T\Sigma}.$$

It is evident that $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{n}}(x_\Sigma, t)$ represents the interaction between the boundary and fluid, $\boldsymbol{W}\times\boldsymbol{n}$ is determined by the boundary.

Wu et al[13] deduced a kind of novel representation of the strain tensor on an arbitrary deformable boundary based on the triple decomposition of the velocity gradient and the intrinsic decomposition, that is

$$\boldsymbol{D}=(\theta-\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V})\boldsymbol{n}\otimes\boldsymbol{n}+\frac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{n})\otimes\boldsymbol{n}+\frac{1}{2}\boldsymbol{n}\otimes(\boldsymbol{\omega}\times\boldsymbol{n})+$$
$$[(\boldsymbol{W}\times\boldsymbol{n})\otimes\boldsymbol{n}+\boldsymbol{n}\otimes(\boldsymbol{W}\times\boldsymbol{n})]-\overline{[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}]]}^{Sym},$$

where Sym represents the symmetrization of the affine tensor $[\boldsymbol{n},(\boldsymbol{n},\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})]$. The above representation can be deduced when the surface gradient of the velocity is rewritten as

$$\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}=\boldsymbol{n}\otimes(\boldsymbol{n},\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})_{\mathbb{R}^3}-[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}]]=\boldsymbol{n}\otimes(\boldsymbol{W}\times\boldsymbol{n})-[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}}]].$$

It should be mentioned that as presented in this subsection the deduced representation of the strain tensor on an arbitrary deformable boundary with the adoption of the intrinsic decomposition is gained from the enlightenment of the work by Wu et al[13].

3 Levi-Civita gradient operator

One can define the so-termed Levi-Civita connection operator $\boldsymbol{\nabla}\equiv\boldsymbol{g}^l\,\nabla_{\frac{\partial}{\partial x_\Sigma^l}}$,

$$\begin{aligned}\boldsymbol{\nabla}\circ-\boldsymbol{\Phi}&\equiv(\boldsymbol{g}^l\,\nabla_{\frac{\partial}{\partial x_\Sigma^l}})\circ-(\Phi^i{}_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\Phi^i{}_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^3{}_{\cdot j}\boldsymbol{n}\otimes\boldsymbol{g}^j+\Phi^3{}_{\cdot 3}\boldsymbol{n}\otimes\boldsymbol{n})\triangleq\\&\boldsymbol{g}^l\circ-\nabla_{\frac{\partial}{\partial x_\Sigma^l}}(\Phi^i{}_{\cdot j}\boldsymbol{g}_i\otimes\boldsymbol{g}^j+\Phi^i{}_{\cdot 3}\boldsymbol{g}_i\otimes\boldsymbol{n}+\Phi^3{}_{\cdot j}\boldsymbol{n}\otimes\boldsymbol{g}^j+\Phi^3{}_{\cdot 3}\boldsymbol{n}\otimes\boldsymbol{n})=\\&\nabla_l\Phi^i{}_{\cdot j}(\boldsymbol{g}^l\circ-\boldsymbol{g}_i)\otimes\boldsymbol{g}^j+\nabla_l\Phi^i{}_{\cdot 3}(\boldsymbol{g}^l\circ-\boldsymbol{g}_i)\otimes\boldsymbol{n}+\nabla_l\Phi^3{}_{\cdot j}(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{g}^j+\nabla_l\Phi^3{}_{\cdot 3}(\boldsymbol{g}^l\circ-\boldsymbol{n})\otimes\boldsymbol{n}.\end{aligned}$$

As compared to the surface gradient tensor, Levi-Civita gradient tensor is just effective to the components/indices of the tensor corresponding to the tangent plane, i. e. i, j in the above representations. In the present paper, Levi-Civita gradient operator takes the same denotation $\boldsymbol{\nabla}$ as the full dimensional gradient operator since the concrete meaning of $\boldsymbol{\nabla}$ can be determined in certain relations.

3.1 Some primary identities in vorticity dynamics of two dimensional flows on fixed smooth surfaces

Proposition 8 The Levi-Civita gradient operator with respect to initial physical configuration, denoted by $\mathring{\boldsymbol{\nabla}}\triangleq\boldsymbol{G}^L\mathring{\nabla}_{\frac{\partial}{\partial \xi_\Sigma^L}}$, can be similarly defined. And the order between $\mathring{\boldsymbol{\nabla}}$ and material derivative can be changed, namely

$$\mathring{\boldsymbol{\nabla}}\circ-\dot{\boldsymbol{\Phi}}=\overset{\cdot}{\overline{\mathring{\boldsymbol{\nabla}}\circ-\boldsymbol{\Phi}}}\qquad\forall\,\boldsymbol{\Phi}\in\mathscr{T}^p(\mathbb{R}^{m+1}),$$

where $\cdot$ represents the material derivative.

Proof One has, say $\boldsymbol{\Phi}\in\mathscr{T}^2(\mathbb{R}^3)$,

$$\mathring{\boldsymbol{\nabla}}\circ-\boldsymbol{\Phi}\equiv(\boldsymbol{G}^L\mathring{\nabla}_{\frac{\partial}{\partial\xi_\Sigma^L}})\circ-(\Phi^A{}_{\cdot B}\boldsymbol{G}_A\otimes\boldsymbol{G}^B+\Phi^A{}_{\cdot 3}\boldsymbol{G}_A\otimes\boldsymbol{N}+\Phi^3{}_{\cdot B}\boldsymbol{N}\otimes\boldsymbol{G}^B+\Phi^3{}_{\cdot 3}\boldsymbol{N}\otimes\boldsymbol{N})=$$
$$\mathring{\nabla}_L\Phi^A{}_{\cdot B}(\boldsymbol{G}^L\circ-\boldsymbol{G}_A)\otimes\boldsymbol{G}^B+\mathring{\nabla}_L\Phi^A{}_{\cdot 3}(\boldsymbol{G}^L\circ-\boldsymbol{G}_A)\otimes\boldsymbol{N}+\mathring{\nabla}_L\Phi^3{}_{\cdot B}(\boldsymbol{G}^L\circ-\boldsymbol{N})\otimes\boldsymbol{G}^B+$$
$$\mathring{\nabla}_L\Phi^3{}_{\cdot 3}(\boldsymbol{G}^L\circ-\boldsymbol{N})\otimes\boldsymbol{N},$$

where $\{\boldsymbol{G}_A(\xi_\Sigma)\}_{A=1}^2$ and $\{\boldsymbol{G}^A(\xi_\Sigma)\}_{A=1}^2$ denote the covariant and contravariant bases with respect to the initial physical configuration, and $\boldsymbol{N}(\xi)$ is the corresponding normal vector. All of them are independent on the time. Thanks to the relationships

$$\frac{\partial}{\partial t}(\mathring{\nabla}_L\Phi^A{}_{\cdot B})(\xi_\Sigma,t)=\mathring{\nabla}_L\left(\frac{\partial\Phi^A{}_{\cdot B}}{\partial t}(\xi_\Sigma,t)\right),\ \frac{\partial}{\partial t}(\mathring{\nabla}_L\Phi^A{}_{\cdot 3})(\xi_\Sigma,t)=\mathring{\nabla}_L\left(\frac{\partial\Phi^A{}_{\cdot 3}}{\partial t}(\xi_\Sigma,t)\right),$$
$$\frac{\partial}{\partial t}(\mathring{\nabla}_L\Phi^3{}_{\cdot B})(\xi_\Sigma,t)=\mathring{\nabla}_L\left(\frac{\partial\Phi^3{}_{\cdot B}}{\partial t}(\xi_\Sigma,t)\right),\ \frac{\partial}{\partial t}(\mathring{\nabla}_L\Phi^3{}_{\cdot 3})(\xi_\Sigma,t)=\mathring{\nabla}_L\left(\frac{\partial\Phi^3{}_{\cdot 3}}{\partial t}(\xi_\Sigma,t)\right),$$

the proof is completed.

Firstly, the follwing identity has been derived.

Proposition 9

$$[\mathring{\boldsymbol{\nabla}}\times(\boldsymbol{b}\cdot\boldsymbol{F})]\cdot\boldsymbol{N}=|\boldsymbol{F}|(\boldsymbol{\nabla}\times\boldsymbol{b})\cdot\boldsymbol{n},\qquad\forall\,\boldsymbol{b}\in\mathbb{R}^3,\ |\boldsymbol{F}|:=\frac{\sqrt{g}}{\sqrt{G}}\det\left[\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}\right](\xi,t),$$

where $\mathring{\boldsymbol{\nabla}}\triangleq\boldsymbol{G}^L\mathring{\boldsymbol{\nabla}}_{\frac{\partial}{\partial\xi_\Sigma^L}}$ and $\boldsymbol{\nabla}\triangleq\boldsymbol{g}^l\nabla_{\frac{\partial}{\partial x_\Sigma^l}}$ are Levi-Civita connection operators, $\boldsymbol{N}$ and $\boldsymbol{n}$ are surface normal vectors corresponding to the initial and current physical configurations respectively, $\sqrt{G}:=[\boldsymbol{G}_1,\boldsymbol{G}_2,\boldsymbol{N}]$, $\sqrt{g}:=[\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{n}]$.

Proof

$$[\mathring{\boldsymbol{\nabla}}\times(\boldsymbol{b}\cdot\boldsymbol{F})]\cdot\boldsymbol{N}=\left[(\boldsymbol{G}^B\mathring{\nabla}_{\frac{\partial}{\partial\xi_\Sigma^B}})\times\left(b_i\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}(\xi_\Sigma,t)\boldsymbol{G}^A\right)\right]\cdot\boldsymbol{N}=\mathring{\nabla}_B\left(b_i\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}(\xi_\Sigma,t)\varepsilon^{BA3}\right)=$$
$$\varepsilon^{BA3}\mathring{\nabla}_B\left(b_i\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}(\xi_\Sigma,t)\right)=\varepsilon^{BA3}\left[\frac{\partial}{\partial\xi_\Sigma^B}\left(b_i\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}\right)(\xi_\Sigma,t)-\Gamma_{BA}^L\left(b_i\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^L}(\xi_\Sigma,t)\right)\right]=$$
$$\varepsilon^{BA3}\frac{\partial}{\partial\xi_\Sigma^B}\left(b_i\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}\right)(\xi_\Sigma,t)=\frac{\partial b_i}{\partial x_\Sigma^s}(x_\Sigma,t)\left[\varepsilon^{BA3}\frac{\partial x_\Sigma^s}{\partial\xi_\Sigma^B}(\xi_\Sigma,t)\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}(\xi_\Sigma,t)\right]=$$
$$\frac{1}{\sqrt{G}}\det\left[\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}\right](\xi_\Sigma,t)e^{si3}\frac{\partial b_i}{\partial x_\Sigma^s}(x_\Sigma,t)=\frac{\sqrt{g}}{\sqrt{G}}\det\left[\frac{\partial x_\Sigma^i}{\partial\xi_\Sigma^A}\right](\xi_\Sigma,t)(\varepsilon^{si3}\nabla_s b_i)=|\boldsymbol{F}|(\varepsilon^{si3}\nabla_s b_i).$$

On the other hand, one has

$$\boldsymbol{\nabla}\times\boldsymbol{b}=\boldsymbol{\nabla}\times(b^i\boldsymbol{g}_i+b^3\boldsymbol{n})=\boldsymbol{g}^l\times\nabla_{\frac{\partial}{\partial x_\Sigma^l}}(b^i\boldsymbol{g}_i+b^3\boldsymbol{n})=\boldsymbol{g}^l\times\left(\nabla_l b_i\boldsymbol{g}^i+\frac{\partial b^3}{\partial x_\Sigma^l}\boldsymbol{n}\right)=\varepsilon^{li3}\nabla_l b_i\boldsymbol{n}+\varepsilon^{l3k}\frac{\partial b^3}{\partial x_\Sigma^l}\boldsymbol{g}_k.$$

This ends the proof.

As an application, one has

$$\omega^3:=(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot n=\frac{1}{|\boldsymbol{F}|}[\mathring{\boldsymbol{\nabla}}\times(\boldsymbol{V}\cdot\boldsymbol{F})]\cdot\boldsymbol{N}.$$

Subsequently, the governing equation of vorticity can be deduced

$$\dot{\omega}^3=-\frac{\theta}{|\boldsymbol{F}|}[\mathring{\boldsymbol{\nabla}}\times(\boldsymbol{V}\cdot\boldsymbol{F})]\cdot\boldsymbol{N}+\frac{1}{|\boldsymbol{F}|}[\mathring{\boldsymbol{\nabla}}\times(\boldsymbol{a}\cdot\boldsymbol{F})+\mathring{\boldsymbol{\nabla}}\times(\boldsymbol{V}\cdot(\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})\cdot\boldsymbol{F})]\cdot\boldsymbol{N}=$$
$$-\theta(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}+(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\boldsymbol{n}+\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\left(\frac{|\boldsymbol{V}|^2}{2}\right)\right)\cdot\boldsymbol{n}=-\theta\omega^3+(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\boldsymbol{n},$$

where the identities $\dot{\boldsymbol{F}}=(\boldsymbol{V}\otimes\overset{\Sigma}{\boldsymbol{\nabla}})\cdot\boldsymbol{F}$ and $\mathrm{d}|\boldsymbol{F}|/\mathrm{d}t=\theta|\boldsymbol{F}|$ are utilized.

Proposition 10

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{b})-\boldsymbol{\Delta b}+K_G\boldsymbol{b}\qquad\forall\,\boldsymbol{b}\in\boldsymbol{T\Sigma},\boldsymbol{\Delta b}\triangleq\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\otimes\boldsymbol{b}).$$

Proof

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b})=\boldsymbol{\nabla}\times\left[\left(\boldsymbol{g}^p\,\nabla_{\frac{\partial}{\partial x^p_\Sigma}}\right)\times(b_i\boldsymbol{g}^i)\right]=\left(\boldsymbol{g}^q\,\nabla_{\frac{\partial}{\partial x^q_\Sigma}}\right)\times[(\nabla_p b_i)\varepsilon^{pi3}\boldsymbol{n}]=\\ \varepsilon^{3kq}\varepsilon_{3pi}\nabla_q(\nabla^p b^i)\boldsymbol{g}_k=(\delta^k_p\delta^q_i-\delta^q_p\delta^k_i)\nabla_q(\nabla^p b^i)\boldsymbol{g}_k=\\ \nabla_i(\nabla^k b^i)\boldsymbol{g}_k-\nabla_p(\nabla^p b^i)\boldsymbol{g}_i=\nabla_i(\nabla^k b^i)\boldsymbol{g}_k-\boldsymbol{\Delta b}.\end{aligned}$$

Furthermore, one has

$$\nabla_i(\nabla^k b^i)\boldsymbol{g}_k=[\nabla^k(\nabla_i b^i)+R^{i\cdot\cdot k}_{\cdot si\cdot}b^s]\boldsymbol{g}_k=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{b})+K_G(\delta^i_i\delta^k_s-g_{si}g^{ik})b^s\boldsymbol{g}_k=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{b})+K_G\boldsymbol{b}.$$

It is the end of the proof.

The well known Stokes-Helmholtz decomposition in the present case as shown below takes the different form as compared to the one for Euclid space.

Proposition 11 (Stokes-Helmholtz Decomposition) For any $\boldsymbol{b}\in\boldsymbol{T\Sigma}$, one has $\boldsymbol{b}=\boldsymbol{\nabla}\phi+\boldsymbol{\nabla}\times(\psi\boldsymbol{n})$, where ϕ and ψ can be termed as the tangent plane potential and the normal potential respectively. Both of them are determined by the Possion equations:

$$\Delta\phi:=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)=g^{ij}\left[\frac{\partial^2\phi}{\partial x^i_\Sigma\partial^j_\Sigma}(x_\Sigma,t)-\Gamma^k_{ij}\frac{\partial\phi}{\partial x^k_\Sigma}(x_\Sigma,t)\right]=\boldsymbol{\nabla}\cdot\boldsymbol{b},$$

$$\Delta\psi:=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi)=g^{ij}\left[\frac{\partial^2\psi}{\partial x^i_\Sigma\partial x^j_\Sigma}(x_\Sigma,t)-\Gamma^k_{ij}\frac{\partial\psi}{\partial x^k_\Sigma}(x_\Sigma,t)\right]=-(\boldsymbol{\nabla}\times\boldsymbol{b})\cdot\boldsymbol{n}.$$

As an application, let us focus on two dimensional flows on general fixed smooth surface. The velocity $\boldsymbol{V}\in\boldsymbol{T\Sigma}$ can be represented as

$$\boldsymbol{V}=\boldsymbol{\nabla}\phi+\boldsymbol{\nabla}\times(\psi\boldsymbol{n}),\ \text{with}\begin{cases}\Delta\phi=\boldsymbol{\nabla}\cdot\boldsymbol{V}=:\theta,\\ \Delta\psi=-(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}=-\omega^3,\end{cases}$$

where θ is termed as the dilation quantity, the vorticity is defined as $\boldsymbol{\omega}\triangleq\omega^3\boldsymbol{n}=\boldsymbol{\nabla}\times\boldsymbol{V}$. In the case of $\theta=0$ i. e. the flow is incompressible, the velocity takes the representation $\boldsymbol{V}=\boldsymbol{\nabla}\times(\psi\boldsymbol{n})$ namely $V^i=-\varepsilon^{3ji}\frac{\partial\psi}{\partial x^j_\Sigma}(x_\Sigma,t)$, in which ψ is generally termed as stream function. Based on the intrinsic generalized Stokes formula of the second kind, one has

$$\oint_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{V}\mathrm{d}l=\int_\Sigma\overset{\Sigma}{\boldsymbol{\nabla}}\cdot\boldsymbol{V}\mathrm{d}\sigma=\int_\Sigma\boldsymbol{\nabla}\cdot\boldsymbol{V}\mathrm{d}\sigma=0.$$

In other words, the integral of $\int_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{V}\mathrm{d}l$ is independent on the integral paths. Furthermore, the stream function can be determined through

$$\psi(\boldsymbol{r},t)=\psi(\boldsymbol{r}_0,t)+\int_{\partial\Sigma}(\boldsymbol{\tau}\times\boldsymbol{n})\cdot\boldsymbol{V}\mathrm{d}l,$$

where $\boldsymbol{r}_0$ represents the reference point that can be an arbitrary point on the surface.

We have attained the governing equation of mass conservation

$$\dot{\rho}+\rho\theta=\left[\frac{\partial\rho}{\partial t}(x_\Sigma,t)+\boldsymbol{V}\cdot(\boldsymbol{\nabla}\rho)\right]+\rho\theta=0,$$

where ρ is the surface density.

Generally, denoted by $\boldsymbol{t}$ the surface stress that satisfies $\rho\boldsymbol{a}=\boldsymbol{\nabla}\cdot t+\rho\boldsymbol{f}_\Sigma\in\mathbb{R}^3$ can be assumed that

$$\boldsymbol{t}=(\gamma-p)\boldsymbol{I}+\mu(\boldsymbol{V}\otimes\boldsymbol{\nabla}+\boldsymbol{\nabla}\otimes\boldsymbol{V}),\quad \boldsymbol{I}=\delta^i_j\boldsymbol{g}_i\otimes\boldsymbol{g}^j,$$

where γ and μ denote coefficients of surface tension and inner fraction/viscousity respectively, p is the inner pressure.

Subsequently, the governing equations of momentum conservation with respect to the tangent plane and normal direction can be deduced

$$\rho\left[\frac{\partial\boldsymbol{V}}{\partial t}(x_\Sigma,t)+\boldsymbol{V}\cdot(\boldsymbol{\nabla}\otimes\boldsymbol{V}\right]=-\boldsymbol{\nabla}p+\mu(\Delta\boldsymbol{V}+\boldsymbol{\nabla}\theta+K_G\boldsymbol{V})+\rho\boldsymbol{f}_{sur,\Sigma}\in\boldsymbol{T\Sigma},$$
$$\rho(\boldsymbol{V}\otimes\boldsymbol{V}):\boldsymbol{K}=H(\gamma-p)+2\mu(\boldsymbol{V}\otimes\boldsymbol{\nabla}):\boldsymbol{K}+\rho f_{sur,n}\in\mathbb{R},$$

where $\boldsymbol{f}_{sur,\Sigma}$ and $f_{sur,n}$ denote the densities of the surface forces on the tangent plane and normal direction respectively.

Taking $\boldsymbol{\nabla}\cdot$ on both sides of the equation with respect to the tangent plane, one arrives at the governing equation of the dilation

$$\begin{aligned}\dot{\theta}=&-[(\boldsymbol{V}\otimes\boldsymbol{\nabla}):(\boldsymbol{\nabla}\otimes\boldsymbol{V})+\boldsymbol{V}\cdot\boldsymbol{\nabla}\theta+K_G|\boldsymbol{V}|^2]+\left[\frac{1}{\rho^2}\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}p-\frac{1}{\rho}\Delta p\right]-\\&\frac{\mu}{\rho^2}\boldsymbol{\nabla}\rho\cdot(\Delta\boldsymbol{V}+\boldsymbol{\nabla}\theta+K_G\boldsymbol{V})+\frac{2\mu}{\rho}[\Delta\theta+\boldsymbol{\nabla}\cdot(K_G\boldsymbol{V})]+\boldsymbol{\nabla}\cdot\boldsymbol{f}_{sur}=\\&-\left[\boldsymbol{D}:\boldsymbol{D}-\frac{|\boldsymbol{\omega}|^2}{2}+K_G|\boldsymbol{V}|^2\right]+\left[\frac{1}{\rho^2}\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}p-\frac{1}{\rho}\Delta p\right]-\\&\frac{\mu}{\rho^2}\boldsymbol{\nabla}\rho\cdot[-\boldsymbol{\nabla}\times\boldsymbol{\omega}+2(\boldsymbol{\nabla}\theta+K_G\boldsymbol{V})]+\frac{2\mu}{\rho}[\Delta\theta+\boldsymbol{\nabla}\cdot(K_G\boldsymbol{V})]+\boldsymbol{\nabla}\cdot\boldsymbol{f}_{sur},\end{aligned}$$

where denoted by $\boldsymbol{D}$ the strain tensor for two dimensional flows on fixed surfaces is defined by $\frac{1}{2}(\boldsymbol{V}\otimes\boldsymbol{\nabla}+\boldsymbol{\nabla}\otimes\boldsymbol{V})$.

On the other hand, the governing equation of the vorticity takes the following form:

$$\begin{aligned}\dot{\omega}^3=&-\theta\omega^3-\frac{1}{\rho^2}[\boldsymbol{\nabla}\rho,-\boldsymbol{\nabla}p+\mu[-\boldsymbol{\nabla}\times\boldsymbol{\omega}+2(\boldsymbol{\nabla}\theta+K_G\boldsymbol{V})],\boldsymbol{n}]+\\&\frac{\mu}{\rho}[\Delta\boldsymbol{\omega}+2\,\boldsymbol{\nabla}\times(K_G\boldsymbol{V})]\cdot\boldsymbol{n}-\frac{1}{\rho^2}[\boldsymbol{\nabla}\rho,\boldsymbol{f}_{sur},\boldsymbol{n}]+\frac{1}{\rho}(\boldsymbol{\nabla}\times\boldsymbol{f}_{sur})\cdot\boldsymbol{n}.\end{aligned}$$

As a summary, one can simulate generally spatial-temporal evolutions of compressible two dimensional flows on general fixed smooth surfaces through the governing equations of dilation and vorticity accompanying with the Possion equations for the velocity potentials, in addition the distribution of inner pressure can be updated based on the equation of momentum conservation in the normal direction.

3.2 Some identities of affine surface tensors

Proposition 12

$$\boldsymbol{\nabla}\times\boldsymbol{\Phi}\times\boldsymbol{\nabla}=[\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}-\boldsymbol{\Delta}(tr\boldsymbol{\Phi})]\boldsymbol{n}\otimes\boldsymbol{n}\qquad\forall\,\boldsymbol{\Phi}\in\mathscr{T}^2(\boldsymbol{T\Sigma}).$$

Proof Firstly, one should show that

$$(\boldsymbol{\nabla}\times\boldsymbol{\Phi})\times\boldsymbol{\nabla}=\boldsymbol{\nabla}\times(\boldsymbol{\Phi}\times\boldsymbol{\nabla})=:\boldsymbol{\nabla}\times\boldsymbol{\Phi}\times\boldsymbol{\nabla}.$$

The left hand side can be calculated as follows:

$$(\boldsymbol{\nabla}\times\boldsymbol{\Phi})\times\boldsymbol{\nabla}=\left[\left(\boldsymbol{g}^p\ \nabla_{\frac{\partial}{\partial x_\Sigma^p}}\right)\times(\Phi_{ij}\boldsymbol{g}^i\otimes\boldsymbol{g}^j)\right]\times\boldsymbol{\nabla}=\left[(\nabla_p\Phi_{ij})\varepsilon^{pi3}\boldsymbol{n}\otimes\boldsymbol{g}^j\right]\times\left(\boldsymbol{g}^q\ \nabla_{\frac{\partial}{\partial x_\Sigma^q}}\right)=$$
$$(\nabla_q\nabla^p\Phi^i{}_{\cdot j})\varepsilon_{pi3}\varepsilon^{jq3}\boldsymbol{n}\otimes\boldsymbol{n}=(\delta_p^j\delta_i^q-\delta_i^j\delta_p^q)(\nabla_q\nabla^p\Phi^i{}_{\cdot j})\boldsymbol{n}\otimes\boldsymbol{n}=$$
$$[\nabla_i\nabla^j\Phi^i{}_{\cdot j}-\nabla_q\nabla^q\Phi^i{}_{\cdot i}]\boldsymbol{n}\otimes\boldsymbol{n}=[\nabla_i\nabla^j\Phi^i{}_{\cdot j}-\boldsymbol{\Delta}(tr\,\boldsymbol{\Phi})]\boldsymbol{n}\otimes\boldsymbol{n},$$

where

$$\nabla_i\nabla^j\Phi^i{}_{\cdot j}=\nabla^j\nabla_i\Phi^i{}_{\cdot j}+R^{i\cdot\cdot j}_{\cdot si\cdot}\Phi^s{}_{\cdot j}+R_{j\cdot i\cdot}^{\cdot s\cdot j}\Phi^i{}_{\cdot s}=$$
$$\nabla^j\nabla_i\Phi^i{}_{\cdot j}+K_G(\delta_i^i\delta_s^j-g_{si}g^{ij})\Phi^s{}_{\cdot j}+K_G(g_{ji}g^{sj}-\delta_i^s\delta_j^j)\Phi^i{}_{\cdot s}=$$
$$\nabla^j\nabla_i\Phi^i{}_{\cdot j}+K_G\Phi^j{}_{\cdot j}-K_G\Phi^i{}_{\cdot i}=\nabla^j\nabla_i\Phi^i{}_{\cdot j}=\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}.$$

As a summary, one arrives at

$$(\boldsymbol{\nabla}\times\boldsymbol{\Phi})\times\boldsymbol{\nabla}=[\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}-\boldsymbol{\Delta}(tr\,\boldsymbol{\Phi})]\boldsymbol{n}\otimes\boldsymbol{n}.$$

On the other hand, $\boldsymbol{\nabla}\times(\boldsymbol{\Phi}\times\boldsymbol{\nabla})$ can similarly calculated with the same representation. This ends the proof.

Subsequently, it is evident that

$$\boldsymbol{\nabla}\times\boldsymbol{\Phi}\times\boldsymbol{\nabla}=\boldsymbol{0}\Leftrightarrow\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}-\boldsymbol{\Delta}(tr\,\boldsymbol{\Phi})=\boldsymbol{0}\in\mathbb{R}.$$

Proposition 13

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\Phi})=\boldsymbol{\nabla}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{\Phi})-\Delta\boldsymbol{\Phi}+K_G(\boldsymbol{\Phi}+\boldsymbol{\Phi}^*)-K_G(tr\,\boldsymbol{\Phi})\boldsymbol{I}\qquad\forall\,\boldsymbol{\Phi}\in\mathscr{T}^2(\boldsymbol{T\Sigma}).$$

Proof　On the left hand side, one can do the following deduction:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Phi}=\boldsymbol{\nabla}\times\left[\left(\boldsymbol{g}^p\ \nabla_{\frac{\partial}{\partial x_\Sigma^p}}\right)\times(\Phi_{ij}\boldsymbol{g}^i\otimes\boldsymbol{g}^j)\right]=\left(\boldsymbol{g}^q\ \nabla_{\frac{\partial}{\partial x_\Sigma^q}}\right)\times\left[\nabla_p\Phi_{ij}\varepsilon^{pi3}\boldsymbol{n}\otimes\boldsymbol{g}^j\right]=$$
$$(\nabla_q\nabla_p\Phi_{ij})\varepsilon^{q3k}\varepsilon^{pi3}\boldsymbol{g}_k\otimes\boldsymbol{g}^j=(\nabla^q\nabla_p\Phi_{ij})\varepsilon_{3kq}\varepsilon^{pi3}\boldsymbol{g}^k\otimes\boldsymbol{g}^j=$$
$$(\nabla^q\nabla_p\Phi_{ij})(\delta_k^p\delta_q^i-\delta_q^p\delta_k^i)\boldsymbol{g}^k\otimes\boldsymbol{g}^j=\nabla^i\nabla_k\Phi_{ij}\boldsymbol{g}^k\otimes\boldsymbol{g}^j-\nabla^p\nabla_p\Phi_{ij}\boldsymbol{g}^i\otimes\boldsymbol{g}^j=$$
$$\nabla^i\nabla_k\Phi_{ij}\boldsymbol{g}^k\otimes\boldsymbol{g}^j-\boldsymbol{\Delta\Phi},$$

where the first term on the right hand side can be processed as follows:

$$\nabla^i\nabla_k\Phi_{ij}\boldsymbol{g}^k\otimes\boldsymbol{g}^j=[\nabla_k\nabla^i\Phi_{ij}+R_{i\cdot\cdot k}^{\cdot si\cdot}\Phi_{sj}+R_{j\cdot\cdot k}^{\cdot si}\Phi_{is}]\boldsymbol{g}^k\otimes\boldsymbol{g}^j=$$
$$(\nabla_k\nabla^i\Phi_{ij})\boldsymbol{g}^k\otimes\boldsymbol{g}^j+K_G(\delta_i^i\delta_k^s-g^{si}g_{ik})\Phi_{sj}\boldsymbol{g}^k\otimes\boldsymbol{g}^j+K_G(\delta_j^i\delta_k^s-g^{si}g_{jk})\Phi_{is}\boldsymbol{g}^k\otimes\boldsymbol{g}^j=$$
$$\boldsymbol{\nabla}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{\Phi})+K_G(\boldsymbol{\Phi}+\boldsymbol{\Phi}^*)-K_G(tr\,\boldsymbol{\Phi})\boldsymbol{I}.$$

It is the end of the proof.

As indicated in this subsection, the change of the order of covariant or contra-variant differential operators defined on surfaces must be related to Riemann-Christoffel tensor. Consequently, the action of Levi-Civita gradient operators more than twice will generally lead to the appearance of the curvatures, such as Laplacian operator $\boldsymbol{\Delta\Phi}\triangleq\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi})$, double curl operator $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\Phi})$ and so on.

4　Conclusion

Two kinds of differential operators on the surface have been studied including definitions, properties and applications.

The first kind of differential operators on the surface is termed as the surface gradient operator. It can be taken as the derivative of a tensor field defined on the surface. Consequently, the partial derivative of a tensor field with respect to a curvilinear coordinate of the surface can be determined. According to the differential calculus in normed linear tensor spaces, the order of partial derivatives of a tensor field can be exchanged as the regularity of the tensor filed is provided. Firstly, all kinds of the intrinsic generalized Stokes formulas have been derived in which the relation between the full dimensional gradient operator and the surface gradient one is adopted. Particularly, one form of the intrinsic generalized Stokes formulas of the second kind have been used to deduce some integral identities studied by Yin[7]; to deduce the governing differential equations of momentum and moment of momentum conservations for continuous mediums whose geometrical configurations can be taken as surfaces that cover the governing equations for thin enough plates and shells as deduced by Chien[9]; to deduce a differential identity that plays the essential role to attain the representation of the vorticity flux on an arbitrary deformable solid boundary. Secondly, the primary properties of the deformation gradient tensor with respect to the continuous mediums whose geometrical configurations are surfaces with any finite dimensionality have been deduced. The deduction mistake made by Aris[10] in his studies on two dimensional flows on an arbitrary fixed surface have been pointed out in detail. Thirdly, the representations of strain tensor on an arbitrary deformable surface have been studied. The original result was attained by Wu et al[13].

The second kind of differential operators on the surface is termed as Levi-Civita gradient operator. Its definition is based on general Levi-Civita connection on the surface that is a typical Riemann manifold. It has been indicated that Levi-Civita operator is just effective to the indices with respect to the tangent space. Figuratively, Levi-Civita gradient of a tensor field just represents the feeling of an observer standing on the surface who is insensitive to the change in the normal direction, however, the surface gradient reflects the whole change of a tensor field as viewed from the outer space of the surface. The essential property of Levi-Civita gradient operator is that the change of the order of covariant or contravariant derivatives/differentiations should be related to Riemann-Christoffel tensor. Some differential identities have been set up based on Levi-Civita gradient operator that constitute the foundation of the theoretical framework of vorticity dynamics for two dimensional flows on an arbitrary fixed surface as indicated by Xie[12]. In the present case, there are some additional terms in the governing equations of momentum and vorticity that include curvatures of the surface explicitly. It implies that the geometrical quantities accompanying with the mechanical ones may appear explicitly in the governing equations of nature laws as the physical configurations of continuous mediums can be taken as surfaces.

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一般光滑曲面上的二类微分算子

谢锡麟

（复旦大学 力学与工程科学系，上海 200433）

摘　要：Euclid空间中一般光滑曲面上可以定义二类微分算子，一类称为曲面梯度算子，另一类称为Levi-Civita算子. 曲面梯度算子的定义源于定义于曲面上的张量场的可微性. 理论研究了若干曲面梯度算子的积分及微分恒等式，这些恒等式在研究几何形态为曲面的连续介质力学以及流体与可变形边界的相互作用中具有重要意义. Levi-Civita梯度算子的定义基于一般Riemann流形上的Levi-Civita联络. 基于Levi-Civita梯度算子可以建立一些内蕴/坐标无关的微分恒等式，这些恒等式为建立固定光滑曲面上二维流动的涡量动力学理论奠定了基础.

关键词：曲面梯度算子；Levi-Civita梯度算子；内蕴形式广义Stokes公式；可变形边界上的流固耦合；曲面变形理论；固定光滑曲面上二维流动

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Mechanism Research of Mechanical and Temperature Sensitivity of Mast Cells in Acupuncture and Moxibustion

ZHU Hai-ning[1]，ZHANG Di[1,2]，HUANG Meng[1]，YANG Yu-hua[1]，SA Zhe-yan[1]，DING Guang-hong[1,2]

（1. *Department of Mechanics and Engineering Science*，*Fudan University*，*Shanghai* 200433，*China*；
2. *Shanghai Research Center for Acupuncture and Meridians*，*Shanghai* 201203，*China*）

Abstract：The aim of the research was to observe the response characteristics of the mechanical and temperature sensitive channels on the mast cells membrane involving in the effect of acupuncture and moxibustion stimuli. The effect of the related channel-specific blocker on cell membrane currents activated by the mechanical and temperature stimuli，the intracellular calcium concentration and the release rate of histamine secretion was observed by means of the western blot of membrane proteins，the electrophysiological patch clamp technique，the flow chamber technique，the intracellular calcium fluorescence image and the fluorescence spectrophotometry. The experiments confirmed that the expression of TRPV2 on human mast cells provides an effective signal transduction pathway for their direct activation by the external mechanical and temperature stimuli. While the increase of intracellular free calcium concentration plays an important role in the process of the promotion of the histamine secretion release of mast cells. Such an initial transmission of the biological signal responses of mast cells provides a reliable scientific basis for the study of acupuncture signal transduction mechanisms.

Keywords：mast cells；TRPV2；mechano-sensitive；temperature-sensitive；acupuncture