张量代数—反称化算子及反对称张量

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1　知识要素

1.1　对称化算子与反称化算子

定义 1.1 (置换算子). 设有置换 $\sigma \in P_r$, 置换算子 I_σ 定义为

$$I_\sigma:\ \mathscr{T}^r(\mathbb{R}^m) \ni \boldsymbol{\Phi} \mapsto I_\sigma \boldsymbol{\Phi} \in \mathscr{T}^r(\mathbb{R}^m).$$

此处, $(I_\sigma \boldsymbol{\Phi})(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) \triangleq \boldsymbol{\Phi}(\boldsymbol{u}_{\sigma(1)}, \cdots, \boldsymbol{u}_{\sigma(r)}) \in \mathbb{R}$.

性质 1.1 (置换算子的线性性). 对 $\forall \boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathscr{T}^r(\mathbb{R}^m)$ 和 $\forall \alpha, \beta \in \mathbb{R}$, 有

$$I_\sigma(\alpha\boldsymbol{\Phi} + \beta\boldsymbol{\Psi}) = \alpha I_\sigma \boldsymbol{\Phi} + \beta I_\sigma \boldsymbol{\Psi}.$$

证明 根据置换算子的定义, 以及张量的线性性, 有

$$\begin{aligned} I_\sigma(\alpha\boldsymbol{\Phi} + \beta\boldsymbol{\Psi})(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) &= (\alpha\boldsymbol{\Phi} + \beta\boldsymbol{\Psi})(\boldsymbol{u}_{\sigma(1)}, \cdots, \boldsymbol{u}_{\sigma(r)}) \\ &= \alpha\boldsymbol{\Phi}(\boldsymbol{u}_{\sigma(1)}, \cdots, \boldsymbol{u}_{\sigma(r)}) + \beta\boldsymbol{\Psi}(\boldsymbol{u}_{\sigma(1)}, \cdots, \boldsymbol{u}_{\sigma(r)}) \\ &= \alpha I_\sigma\boldsymbol{\Phi}(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) + \beta I_\sigma\boldsymbol{\Psi}(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) \\ &= (\alpha I_\sigma\boldsymbol{\Phi} + \beta I_\sigma\boldsymbol{\Psi})(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r). \end{aligned}$$

□

根据置换算子的定义, 任意张量经过置换算子作用之后可以表示为

$$\begin{aligned} (I_\sigma\boldsymbol{\Phi})(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) &= \boldsymbol{\Phi}(\boldsymbol{u}_{\sigma(1)}, \cdots, \boldsymbol{u}_{\sigma(r)}) \\ &= (\boldsymbol{u}_{\sigma(1)}, \boldsymbol{g}_{i_1})_{\mathbb{R}^m} \cdots (\boldsymbol{u}_{\sigma(r)}, \boldsymbol{g}_{i_r})_{\mathbb{R}^m} \boldsymbol{\Phi}(\boldsymbol{g}^{i_1}, \cdots, \boldsymbol{g}^{i_r}) \\ &= \Phi^{i_1 \cdots i_r} (\boldsymbol{u}_1, \boldsymbol{g}_{\sigma^{-1}(i_1)})_{\mathbb{R}^m} \cdots (\boldsymbol{u}_r, \boldsymbol{g}_{\sigma^{-1}(i_r)})_{\mathbb{R}^m} \\ &= \Phi^{i_1 \cdots i_r} \boldsymbol{g}_{\sigma^{-1}(i_1)} \otimes \cdots \otimes \boldsymbol{g}_{\sigma^{-1}(i_r)}(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r), \end{aligned}$$

即有

$$I_\sigma\boldsymbol{\Phi} = \Phi^{i_1 \cdots i_r} \boldsymbol{g}_{\sigma^{-1}(i_1)} \otimes \cdots \otimes \boldsymbol{g}_{\sigma^{-1}(i_r)}.$$

又可得

$$I_\sigma\boldsymbol{\Phi} = \Phi^{\sigma(i_1) \cdots \sigma(i_r)} \boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_r}.$$

定义 1.2 (对称张量与反对称张量)**.** 如果张量 $\boldsymbol{\Phi} \in \mathscr{T}^r(\mathbb{R}^m)$ 满足 $I_\sigma \boldsymbol{\Phi} = \boldsymbol{\Phi}, \forall \sigma \in P_r$ 或者 $\Phi^{\sigma(i_1)\cdots\sigma(i_r)} = \Phi^{i_1\cdots i_r}$, 则张量 $\boldsymbol{\Phi}$ 称为**对称张量**, 记作 $\boldsymbol{\Phi} \in \mathbf{Sym}$ 或者 $\boldsymbol{\Phi} \in \mathscr{S}^r(\mathbb{R}^m)$. 如果张量 $\boldsymbol{\Phi} \in \mathscr{T}^r(\mathbb{R}^m)$ 对 $\forall \sigma \in P_r$ 满足 $I_\sigma \boldsymbol{\Phi} = \operatorname{sgn}\sigma \boldsymbol{\Phi}$ 或者 $\Phi^{\sigma(i_1)\cdots\sigma(i_r)} = \operatorname{sgn}\sigma \Phi^{i_1\cdots i_r}$, 则张量 $\boldsymbol{\Phi}$ 称为**反对称张量**, 记作 $\boldsymbol{\Phi} \in \mathbf{Skw}$ 或者 $\boldsymbol{\Phi} \in \Lambda^r(\mathbb{R}^m)$.

定义 1.3 (对称化算子和反称化算子)**.** 对称化算子 $\mathscr{S}$ 和反称化算子 $\mathscr{A}$ 分别定义为

$$\mathscr{S}(\boldsymbol{\Phi}):\ \mathscr{T}^r(\mathbb{R}^n) \ni \boldsymbol{\Phi} \mapsto \mathscr{S}\boldsymbol{\Phi} \triangleq \frac{1}{r!}\sum_{\sigma\in P_r} I_\sigma \boldsymbol{\Phi} \in \mathbf{Sym};$$
$$\mathscr{A}(\boldsymbol{\Phi}):\ \mathscr{T}^r(\mathbb{R}^n) \ni \boldsymbol{\Phi} \mapsto \mathscr{A}\boldsymbol{\Phi} \triangleq \frac{1}{r!}\sum_{\sigma\in P_r} \operatorname{sgn}\sigma I_\sigma \boldsymbol{\Phi} \in \Lambda^r(\mathbb{R}^n).$$

对于对称化算子, 设 $\forall \tau \in P_r$, 则有

$$I_\tau \mathscr{S}\boldsymbol{\Phi} = \frac{1}{r!} I_\tau \left(\sum_{\sigma\in P_r} I_\sigma \boldsymbol{\Phi}\right) = \frac{1}{r!}\sum_{\sigma\in P_r} I_{\tau\circ\sigma}\boldsymbol{\Phi} = \frac{1}{r!}\sum_{\gamma\in P_r} I_\gamma \boldsymbol{\Phi} = \mathscr{S}\boldsymbol{\Phi}.$$

因此 $\mathscr{S}\boldsymbol{\Phi}$ 是对称张量.

对于反称化算子, 设 $\forall \tau \in P_r$, 则有

$$I_\tau \mathscr{A}\boldsymbol{\Phi} = \frac{1}{r!} I_\tau \left(\sum_{\sigma\in P_r} \operatorname{sgn}\sigma I_\sigma \boldsymbol{\Phi}\right) = \frac{1}{r!}\sum_{\sigma\in P_r} \operatorname{sgn}\sigma I_{\tau\circ\sigma}\boldsymbol{\Phi} = \frac{1}{r!}\operatorname{sgn}\tau \sum_{\gamma\in P_r} I_\gamma \boldsymbol{\Phi} = \operatorname{sgn}\tau \mathscr{A}\boldsymbol{\Phi}.$$

因此 $\mathscr{A}\boldsymbol{\Phi}$ 是反对称张量.

性质 1.2 (反称化算子基本性质)**.** 反称化算子具有如下基本性质:

1. 线性性: 对 $\forall \boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathscr{T}^r(\mathbb{R}^m)$ 和 $\forall \alpha, \beta \in \mathbb{R}$, 有
$$\mathscr{A}(\alpha\boldsymbol{\Phi} + \beta\boldsymbol{\Psi}) = \alpha\mathscr{A}\boldsymbol{\Phi} + \beta\mathscr{A}\boldsymbol{\Psi};$$
2. $\mathscr{A}^2 = \mathscr{A}$, 更一般地, 有 $\mathscr{A}^k = \mathscr{A}$, $k \in \mathbb{N}$;
3. 对 $\forall \boldsymbol{\Phi} \in \mathscr{T}^r(\mathbb{R}^m), \boldsymbol{\Psi} \in \mathscr{T}^s(\mathbb{R}^m)$, 有
$$\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}) = \mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}) = \mathscr{A}(\boldsymbol{\Phi}\otimes\mathscr{A}\boldsymbol{\Psi}) = \mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\mathscr{A}\boldsymbol{\Psi}).$$

证明 可按置换算子的基本性质, 证明反称化算子的基本性质.

1. 根据置换算子的线性性, 这是显然的.
2. 设 $\boldsymbol{\Phi} \in \mathscr{T}^r(\mathbb{R}^m)$, 则有
$$\begin{aligned}\mathscr{A}^2\boldsymbol{\Phi} = \mathscr{A}(\mathscr{A}\boldsymbol{\Phi}) = \mathscr{A}&\left(\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_\sigma\boldsymbol{\Phi}\right)\\ &= \frac{1}{r!}\sum_{\beta\in P_r}\operatorname{sgn}\beta I_\beta\left(\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_\sigma\boldsymbol{\Phi}\right) = \frac{1}{r!}\sum_{\beta\in P_r}\left(\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}(\beta\circ\sigma) I_{\beta\circ\sigma}\boldsymbol{\Phi}\right)\\ &= \frac{1}{r!}\sum_{\beta\in P_r}\mathscr{A}\boldsymbol{\Phi} = \mathscr{A}\boldsymbol{\Phi}.\end{aligned}$$
由此, 即有 $\mathscr{A}^2 = \mathscr{A}$. 再根据数学归纳法, 易于证明 $\mathscr{A}^k = \mathscr{A}$.

3. 证明 $\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})=\mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})$. 由

$$\mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})=\mathscr{A}\left[\left(\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_\sigma\boldsymbol{\Phi}\right)\otimes\boldsymbol{\Psi}\right]=\mathscr{A}\left[\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma\left(I_\sigma\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}\right)\right].$$

此处 $I_\sigma\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}=\Phi^{\sigma(i_1)\cdots\sigma(i_r)}\Psi^{j_1\cdots j_s}\boldsymbol{g}_{i_1}\otimes\cdots\otimes\boldsymbol{g}_{i_r}\otimes\boldsymbol{g}_{j_1}\otimes\cdots\otimes\boldsymbol{g}_{j_s}$, 作置换 $\sigma\in P_r$ 的延拓 $\hat{\sigma}\in P_{r+1}$, 如下所示:

$$\sigma=\begin{pmatrix} i_1 & \cdots & i_r\\ \sigma(i_1) & \cdots & \sigma(i_r)\end{pmatrix}\in P_r,\quad \hat{\sigma}=\begin{pmatrix} i_1 & \cdots & i_r & j_1 & \cdots & j_s\\ \sigma(i_1) & \cdots & \sigma(i_r) & j_1 & \cdots & j_s\end{pmatrix}\in P_{r+s},$$

因此有 $I_\sigma\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}=I_{\hat{\sigma}}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})$. 所以, 有

$$\begin{aligned}\mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})&=\mathscr{A}\left[\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_{\hat{\sigma}}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})\right]\\&=\frac{1}{(r+s)!}\sum_{\hat{\beta}\in P_{r+s}}\operatorname{sgn}\hat{\beta}I_{\hat{\beta}}\left[\frac{1}{r!}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_{\hat{\sigma}}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})\right]\\&=\frac{1}{r!}\sum_{\sigma\in P_r}\left[\frac{1}{(r+s)!}\sum_{\hat{\beta}\in P_{r+s}}\operatorname{sgn}(\hat{\beta}\circ\hat{\sigma})I_{\hat{\beta}\circ\hat{\sigma}}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})\right]=\frac{1}{r!}\sum_{\sigma\in P_r}\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})\\&=\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}).\end{aligned}$$

用类似的方法可以证明 $\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})=\mathscr{A}(\boldsymbol{\Phi}\otimes\mathscr{A}\boldsymbol{\Psi})$.

按以上结论, 可得

$$\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})=\mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})=\mathscr{A}(\mathscr{A}\boldsymbol{\Phi}\otimes\mathscr{A}\boldsymbol{\Psi}).$$ □

1.2 反对称张量的外积运算

定义 1.4 (外积运算). 反对称张量的外积运算定义如下:

$$\wedge:\ \Lambda^p(\mathbb{R}^m)\times\Lambda^q(\mathbb{R}^m)\ni\{\boldsymbol{\Phi},\boldsymbol{\Psi}\}\mapsto\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}\in\Lambda^{p+q}(\mathbb{R}^m),$$

其中

$$\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}\triangleq\frac{(p+q)!}{p!\,q!}\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi})=\frac{1}{p!\,q!}\sum_{\sigma\in P_{p+q}}\operatorname{sgn}\sigma I_\sigma(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}).$$

性质 1.3 (外积运算基本性质). 外积运算具有如下基本性质:

1. 线性性: 对 $\forall\boldsymbol{\Phi},\boldsymbol{\Psi}\in\Lambda^p(\mathbb{R}^m),\boldsymbol{\Theta}\in\Lambda^q(\mathbb{R}^m)$ 和 $\forall\alpha,\beta\in\mathbb{R}$, 有

$$(\alpha\boldsymbol{\Phi}+\beta\boldsymbol{\Psi})\wedge\boldsymbol{\Theta}=\alpha\boldsymbol{\Phi}\wedge\boldsymbol{\Theta}+\beta\boldsymbol{\Psi}\wedge\boldsymbol{\Theta};$$

2. 对 $\forall\boldsymbol{\Phi}\in\Lambda^p(\mathbb{R}^m),\boldsymbol{\Psi}\in\Lambda^q(\mathbb{R}^m),\boldsymbol{\Theta}\in\Lambda^r(\mathbb{R}^m)$, 有

$$(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi})\wedge\boldsymbol{\Theta}=\boldsymbol{\Phi}\wedge(\boldsymbol{\Psi}\wedge\boldsymbol{\Theta})=\frac{(p+q+r)!}{p!q!r!}\mathscr{A}(\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}\otimes\boldsymbol{\Theta})\in\Lambda^{p+q+r}(\mathbb{R}^m),$$

由此, 上述两种表达都统一记作 $\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}\wedge\boldsymbol{\Theta}$;

3. 对 $\forall \boldsymbol{\Phi} \in \Lambda^p(\mathbb{R}^m), \boldsymbol{\Psi} \in \Lambda^q(\mathbb{R}^m)$, 则有

$$\boldsymbol{\Phi} \wedge \boldsymbol{\Psi} = (-1)^{pq} \boldsymbol{\Psi} \wedge \boldsymbol{\Phi}.$$

证明 可基于反称化算子的基本性质, 证明外积运算的基本性质.

1. 根据反称化算子的线性性, 这是显然的.

2. 根据反称化算子的性质 (3), 有

$$\begin{aligned}(\boldsymbol{\Phi} \wedge \boldsymbol{\Psi}) \wedge \boldsymbol{\Theta} &= \frac{(p+q)!}{p!q!} \mathscr{A}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi}) \wedge \boldsymbol{\Theta} \\ &= \frac{(p+q)!}{p!q!} \frac{(p+q+r)!}{(p+q)!r!} \mathscr{A}[\mathscr{A}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi}) \otimes \boldsymbol{\Theta}] \\ &= \frac{(p+q+r)!}{p!q!r!} \mathscr{A}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi} \otimes \boldsymbol{\Theta}).\end{aligned}$$

同理可得

$$\boldsymbol{\Phi} \wedge (\boldsymbol{\Psi} \wedge \boldsymbol{\Theta}) = \frac{(p+q+r)!}{p!q!r!} \mathscr{A}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi} \otimes \boldsymbol{\Theta}).$$

因此有

$$(\boldsymbol{\Phi} \wedge \boldsymbol{\Psi}) \wedge \boldsymbol{\Theta} = \boldsymbol{\Phi} \wedge (\boldsymbol{\Psi} \wedge \boldsymbol{\Theta}) = \frac{(p+q+r)!}{p!q!r!} \mathscr{A}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi} \otimes \boldsymbol{\Theta}).$$

3. 根据外积运算的定义, 有

$$\begin{aligned}\boldsymbol{\Phi} \wedge \boldsymbol{\Psi} &= \frac{(p+q)!}{p!q!} \mathscr{A}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi}) = \frac{1}{p!q!} \sum_{\sigma \in P_{p+q}} \operatorname{sgn} \sigma I_\sigma(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi}) \\ &= \frac{1}{p!q!} \sum_{\sigma \in P_{p+q}} \operatorname{sgn} \sigma I_\sigma(\Phi^{i_1 \cdots i_p} \Psi^{j_1 \cdots j_q} \boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_p} \otimes \boldsymbol{g}_{j_1} \otimes \cdots \otimes \boldsymbol{g}_{j_q}) \\ &= \frac{1}{p!q!} \sum_{\sigma \in P_{p+q}} \operatorname{sgn} \sigma \Phi^{i_1 \cdots i_p} \Psi^{j_1 \cdots j_q} I_\sigma(\boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_p} \otimes \boldsymbol{g}_{j_1} \otimes \cdots \otimes \boldsymbol{g}_{j_q}).\end{aligned}$$

引入置换

$$\tau^{-1} = \begin{pmatrix} i_1 & \cdots & i_q & i_{q+1} & \cdots & i_p & j_1 & \cdots & j_q \\ j_1 & \cdots & j_q & i_1 & \cdots & i_{p-q} & i_{p-q+1} & \cdots & i_p \end{pmatrix} \in P_{p+q},$$

此置换将哑标 $i_1, \cdots, i_p$ 与哑标 $j_1, \cdots, j_q$ 整体换位 (此处假设了 $q < p$, 另一种情况的构造方法完全类似). 则有

$$\begin{aligned}\boldsymbol{\Phi} \wedge \boldsymbol{\Psi} &= \frac{1}{p!q!} \sum_{\sigma \in P_{p+q}} \operatorname{sgn}(\sigma \circ \tau) \Phi^{i_1 \cdots i_p} \Psi^{j_1 \cdots j_q} I_{\sigma \circ \tau}(\boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_p} \otimes \boldsymbol{g}_{j_1} \otimes \cdots \otimes \boldsymbol{g}_{j_q}) \\ &= \frac{1}{p!q!} \operatorname{sgn} \tau \sum_{\sigma \in P_{p+q}} \operatorname{sgn} \sigma \Phi^{i_1 \cdots i_p} \Psi^{j_1 \cdots j_q} I_\sigma(\boldsymbol{g}_{\tau^{-1}(i_1)} \otimes \cdots \otimes \boldsymbol{g}_{\tau^{-1}(i_p)} \otimes \boldsymbol{g}_{\tau^{-1}(j_1)} \otimes \cdots \\ &\quad \otimes \boldsymbol{g}_{\tau^{-1}(j_q)}) \\ &= \frac{1}{p!q!} \operatorname{sgn} \tau \sum_{\sigma \in P_{p+q}} \operatorname{sgn} \sigma \Phi^{i_1 \cdots i_p} \Psi^{j_1 \cdots j_q} I_\sigma(\boldsymbol{g}_{j_1} \otimes \cdots \otimes \boldsymbol{g}_{j_q} \otimes \boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_p}) \\ &= \operatorname{sgn} \tau \frac{1}{p!q!} \sum_{\sigma \in P_{p+q}} \operatorname{sgn} \sigma I_\sigma(\boldsymbol{\Psi} \otimes \boldsymbol{\Phi}) = \operatorname{sgn} \tau \boldsymbol{\Psi} \wedge \boldsymbol{\Phi}.\end{aligned}$$

以下考虑 $\operatorname{sgn}\tau$, 依次将 $i_1,\cdots,i_p$ 移动到 $j_1,\cdots,j_q$ 之前, 总共需要 pq 次操作, 因此

$$\operatorname{sgn}\tau=(-1)^{pq}.$$

综上, 有

$$\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}=(-1)^{pq}\boldsymbol{\Psi}\wedge\boldsymbol{\Phi}.$$ □

1.3 反对称张量的表示

反对称张量也常称为**外形式**, r 阶反对称张量则称为 r-**形式**. 显然 $\mathbb{R}^m$ 上全体 r-形式组成的空间是 r 阶张量空间 $\mathscr{T}^r(\mathbb{R}^m)$ 的一个子空间, 称为 $\mathbb{R}^m$ 上的 r-形式空间, 记作 $\Lambda^r(\mathbb{R}^m)$.

定义 1.5 (简单 r-形式)**.** 设 $\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r\in\mathbb{R}^m$, 则 $\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_r\in\Lambda^r(\mathbb{R}^m)$ 称为简单 r-形式.

根据外积运算的基本性质 (1), 有

$$\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_r=\frac{r!}{1!\cdots1!}\mathscr{A}(\boldsymbol{u}_1\otimes\cdots\otimes\boldsymbol{u}_r)=r!\mathscr{A}(\boldsymbol{u}_1\otimes\cdots\otimes\boldsymbol{u}_r)\in\Lambda^r(\mathbb{R}^m).$$

即简单 r-形式是 r 阶反对称张量.

定理 1.4. 设 $\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r\in\mathbb{R}^m$, $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_r\in\mathbb{R}^m$, 则有

$$\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_r(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_r)=\begin{vmatrix}(\boldsymbol{u}_1,\boldsymbol{v}_1)_{\mathbb{R}^m}&\cdots&(\boldsymbol{u}_1,\boldsymbol{v}_r)_{\mathbb{R}^m}\\ \vdots&&\vdots\\ (\boldsymbol{u}_r,\boldsymbol{v}_1)_{\mathbb{R}^m}&\cdots&(\boldsymbol{u}_r,\boldsymbol{v}_r)_{\mathbb{R}^m}\end{vmatrix}.$$

证明 根据外积的定义可有

$$\begin{aligned}&\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_r(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_r)=r!\mathscr{A}(\boldsymbol{u}_1\otimes\cdots\otimes\boldsymbol{u}_r)(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_r)\\
&\quad=\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_\sigma(\boldsymbol{u}_1\otimes\cdots\otimes\boldsymbol{u}_r)(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_r)\\
&\quad=\sum_{\sigma\in P_r}\operatorname{sgn}\sigma(\boldsymbol{u}_1\otimes\cdots\otimes\boldsymbol{u}_r)(\boldsymbol{v}_{\sigma(1)},\cdots,\boldsymbol{v}_{\sigma(r)})\quad(\text{置换算子的定义})\\
&\quad=\sum_{\sigma\in P_r}\operatorname{sgn}\sigma(\boldsymbol{u}_1,\boldsymbol{v}_{\sigma(1)})_{\mathbb{R}^m}\cdots(\boldsymbol{u}_r,\boldsymbol{v}_{\sigma(r)})_{\mathbb{R}^m}\quad(\text{简单张量的定义})\\
&\quad=\begin{vmatrix}(\boldsymbol{u}_1,\boldsymbol{v}_1)_{\mathbb{R}^m}&\cdots&(\boldsymbol{u}_1,\boldsymbol{v}_r)_{\mathbb{R}^m}\\ \vdots&&\vdots\\ (\boldsymbol{u}_r,\boldsymbol{v}_1)_{\mathbb{R}^m}&\cdots&(\boldsymbol{u}_r,\boldsymbol{v}_r)_{\mathbb{R}^m}\end{vmatrix}\quad(\text{行列式的定义}).\end{aligned}$$ □

推论 1.4.1. 对 $\forall\sigma\in P_r$, 有

$$\begin{aligned}I_\sigma(\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r})&=\boldsymbol{g}_{\sigma(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma(i_r)}=\boldsymbol{g}_{\sigma^{-1}(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma^{-1}(i_r)}\\
&=\operatorname{sgn}\sigma\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}.\end{aligned}$$

证明 根据置换算子的定义, 有

$$\begin{aligned}
&I_\sigma(\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r})(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r)=\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}(\boldsymbol{u}_{\sigma(1)},\cdots,\boldsymbol{u}_{\sigma(r)})\\
&\quad=\begin{vmatrix}(\boldsymbol{g}_{i_1},\boldsymbol{u}_{\sigma(1)})_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_1},\boldsymbol{u}_{\sigma(r)})_{\mathbb{R}^m}\\ \vdots & & \vdots\\ (\boldsymbol{g}_{i_r},\boldsymbol{u}_{\sigma(1)})_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_r},\boldsymbol{u}_{\sigma(r)})_{\mathbb{R}^m}\end{vmatrix}=\operatorname{sgn}\sigma\begin{vmatrix}(\boldsymbol{g}_{i_1},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_1},\boldsymbol{u}_r)_{\mathbb{R}^m}\\ \vdots & & \vdots\\ (\boldsymbol{g}_{i_r},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_r},\boldsymbol{u}_r)_{\mathbb{R}^m}\end{vmatrix}\\
&\quad=\operatorname{sgn}\sigma\,\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r).
\end{aligned}$$

同样有

$$\begin{aligned}
&I_\sigma(\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r})(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r)\\
&\quad=\operatorname{sgn}\sigma\begin{vmatrix}(\boldsymbol{g}_{i_1},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_1},\boldsymbol{u}_r)_{\mathbb{R}^m}\\ \vdots & & \vdots\\ (\boldsymbol{g}_{i_r},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_r},\boldsymbol{u}_r)_{\mathbb{R}^m}\end{vmatrix}=\begin{vmatrix}(\boldsymbol{g}_{\sigma(i_1)},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{\sigma(i_1)},\boldsymbol{u}_r)_{\mathbb{R}^m}\\ \vdots & & \vdots\\ (\boldsymbol{g}_{\sigma(i_r)},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{\sigma(i_r)},\boldsymbol{u}_r)_{\mathbb{R}^m}\end{vmatrix}\\
&\quad=\boldsymbol{g}_{\sigma(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma(i_r)}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r),\\
&I_\sigma(\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r})(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r)\\
&\quad=\begin{vmatrix}(\boldsymbol{g}_{i_1},\boldsymbol{u}_{\sigma(1)})_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_1},\boldsymbol{u}_{\sigma(r)})_{\mathbb{R}^m}\\ \vdots & & \vdots\\ (\boldsymbol{g}_{i_r},\boldsymbol{u}_{\sigma(1)})_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{i_r},\boldsymbol{u}_{\sigma(r)})_{\mathbb{R}^m}\end{vmatrix}=\begin{vmatrix}(\boldsymbol{g}_{\sigma^{-1}(i_1)},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{\sigma^{-1}(i_1)},\boldsymbol{u}_r)_{\mathbb{R}^m}\\ \vdots & & \vdots\\ (\boldsymbol{g}_{\sigma^{-1}(i_r)},\boldsymbol{u}_1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_{\sigma^{-1}(i_r)},\boldsymbol{u}_r)_{\mathbb{R}^m}\end{vmatrix}\\
&\quad=\boldsymbol{g}_{\sigma^{-1}(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma^{-1}(i_r)}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_r).
\end{aligned}$$

综上, 有

$$I_\sigma(\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r})=\operatorname{sgn}\sigma\,\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}=\boldsymbol{g}_{\sigma(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma(i_r)}.$$ □

引理 1.5. 如果反对称张量的分量有两个指标相同, 则该分量必为零.

证明 设 $\boldsymbol{\Phi}\in\Lambda^r(\mathbb{R}^m)$, 则其分量满足

$$\Phi^{\sigma(k_1)\cdots\sigma(k_r)}=\operatorname{sgn}\sigma\,\Phi^{k_1\cdots k_r},\quad\forall\,\sigma\in P_r.$$

设该分量的第 i 个和第 j 个指标是相同的, 即 $k_i=k_j$. 令

$$\sigma=\begin{bmatrix}1 & \cdots & i & \cdots & j & \cdots & r\\ 1 & \cdots & j & \cdots & i & \cdots & r\end{bmatrix},$$

则有 $\operatorname{sgn}\sigma=-1$. 所以有

$$\Phi^{k_1\cdots k_i\cdots k_j\cdots k_r}=-\Phi^{k_1\cdots k_j\cdots k_i\cdots k_r}=-\Phi^{k_1\cdots k_i\cdots k_j\cdots k_r},$$

即 $\Phi^{k_1\cdots k_i\cdots k_j\cdots k_r}=0\,(k_i=k_j)$. □

此引理表明, 如果某外形式空间 $r>m$, 则必然有 $\Lambda^r(\mathbb{R}^m)=\{\boldsymbol{0}\}$, 即仅含有 r 阶零张量. 根据引理1.5(第6页)), 可以得出如下的反对称张量表示定理.

定理 1.6 (反对称张量表示定理)**.** 设 $\boldsymbol{\Phi} \in \Lambda^r(\mathbb{R}^m)$, 则反对称张量 $\boldsymbol{\Phi}$ 可以表示为

$$\begin{aligned}\boldsymbol{\Phi} &= \Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\otimes\cdots\otimes\boldsymbol{g}_{i_r}\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r} = \frac{1}{r!}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}.\end{aligned}$$

证明 根据引理1.5(第6页), 除去 $\boldsymbol{\Phi}$ 中必为零的分量 (即有相同指标的分量), 即有

$$\begin{aligned}\boldsymbol{\Phi} &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\sum_{\sigma\in P_r}\Phi^{\sigma(i_1)\cdots\sigma(i_r)}\boldsymbol{g}_{\sigma(i_1)}\otimes\cdots\otimes\boldsymbol{g}_{\sigma(i_r)}\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma\Phi^{i_1\cdots i_r}\boldsymbol{g}_{\sigma(i_1)}\otimes\cdots\otimes\boldsymbol{g}_{\sigma(i_r)}\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\Phi^{i_1\cdots i_r}\sum_{\sigma\in P_r}\operatorname{sgn}\sigma I_{\sigma^{-1}}(\boldsymbol{g}_{i_1}\otimes\cdots\otimes\boldsymbol{g}_{i_r})\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\Phi^{i_1\cdots i_r}\left[r!\mathscr{A}(\boldsymbol{g}_{i_1}\otimes\cdots\otimes\boldsymbol{g}_{i_r})\right]\ (\text{反称化算子的定义})\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}.\end{aligned}$$

定理的第一个式子得证. 以下证明定理的第二种表示, 有

$$\begin{aligned}\boldsymbol{\Phi} &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\operatorname{sgn}\sigma\Phi^{\sigma(i_1)\cdots\sigma(i_r)}\operatorname{sgn}\sigma I_{\sigma^{-1}}(\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}),\ \forall\sigma\in P_r\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\Phi^{\sigma(i_1)\cdots\sigma(i_r)}\boldsymbol{g}_{\sigma(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma(i_r)},\ \forall\sigma\in P_r\\ &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant m}\sum_{\sigma\in P_r}\frac{1}{r!}\Phi^{\sigma(i_1)\cdots\sigma(i_r)}\boldsymbol{g}_{\sigma(i_1)}\wedge\cdots\wedge\boldsymbol{g}_{\sigma(i_r)}\\ &= \frac{1}{r!}\sum_{i_1,\cdots,i_r}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r} = \frac{1}{r!}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}.\end{aligned}$$

□

定理1.6(第7页) 表明, $\{\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}\}_{1\leqslant i_1<\cdots<i_r\leqslant m}$ 为 r-形式空间 $\Lambda^r(\mathbb{R}^m)$ 的基, 其维数为

$$\binom{r}{m} = m(m-1)\cdots(m-(r-1)) = \frac{m!}{(m-r)!}.$$

设有反对称张量 $\boldsymbol{\Phi}\in\Lambda^r(\mathbb{R}^m), \boldsymbol{\Psi}\in\Lambda^s(\mathbb{R}^m)$, 其表示为

$$\begin{aligned}\boldsymbol{\Phi} &= \frac{1}{r!}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r},\\ \boldsymbol{\Psi} &= \frac{1}{s!}\Psi^{j_1\cdots j_s}\boldsymbol{g}_{j_1}\wedge\cdots\wedge\boldsymbol{g}_{j_s},\end{aligned}$$

则 $\boldsymbol{\Phi}$ 和 $\boldsymbol{\Psi}$ 的外积 $\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}$ 的表示为

$$\begin{aligned}\boldsymbol{\Phi}\wedge\boldsymbol{\Psi} &= \left(\frac{1}{r!}\Phi^{i_1\cdots i_r}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}\right)\wedge\left(\frac{1}{s!}\Psi^{j_1\cdots j_s}\boldsymbol{g}_{j_1}\wedge\cdots\wedge\boldsymbol{g}_{j_s}\right)\\ &= \frac{1}{r!s!}\Phi^{i_1\cdots i_r}\Psi^{j_1\cdots j_s}\boldsymbol{g}_{i_1}\wedge\cdots\wedge\boldsymbol{g}_{i_r}\wedge\boldsymbol{g}_{j_1}\wedge\cdots\wedge\boldsymbol{g}_{j_s}\in\Lambda^{r+s}(\mathbb{R}^m).\end{aligned}$$

此结论基于外积运算基本性质1.3(第3页) 的 (1) 和 (2).

2 应用事例

2.1 有关向量组的应用

引理 2.1 (向量组线性相关性的外积表示)**.** 向量组 $\{\boldsymbol{g}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性相关的充分必要条件为

$$\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_r = \boldsymbol{0} \in \Lambda^r(\mathbb{R}^m).$$

证明 可直接基于运算, 证明充分必要性.

1. 证明必要性. 设有 $\{\boldsymbol{g}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性相关, 需证 $\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_r = \boldsymbol{0} \in \Lambda^r(\mathbb{R}^m)$.

   由于 $\{\boldsymbol{g}_i\}_{i=1}^r$ 线性相关, 不妨设有 $\boldsymbol{g}_r = c_1\boldsymbol{g}_1 + \cdots + c_{r-1}\boldsymbol{g}_{r-1} \in \mathbb{R}^m$, 故有

$$\begin{aligned}\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{r-1} \wedge \boldsymbol{g}_r &= \boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{r-1} \wedge \left(c_1\boldsymbol{g}_1 + \cdots + c_{r-1}\boldsymbol{g}_{r-1}\right)\\ &= \boldsymbol{0} + \cdots + \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0} \in \Lambda^r(\mathbb{R}^m).\end{aligned}$$

2. 证明充分性. 设有 $\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_r = \boldsymbol{0} \in \Lambda^r(\mathbb{R}^m)$, 需证 $\{\boldsymbol{g}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性相关.

   利用反证法, 设 $\{\boldsymbol{g}_i\}_{i=1}^r$ 线性无关, 可补充 $\{\boldsymbol{g}_j\}_{j=r+1}^m$ 使得 $\{\boldsymbol{g}_i\}_{i=1}^r \cup \{\boldsymbol{g}_j\}_{j=r+1}^m$ 为 $\mathbb{R}^m$ 的一组基, 由此存在对偶基 $\{\boldsymbol{g}^\alpha\}_{\alpha=1}^m$. 考虑到

$$\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_r(\boldsymbol{g}^1, \cdots, \boldsymbol{g}^r) = \begin{vmatrix} (\boldsymbol{g}_1, \boldsymbol{g}^1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_1, \boldsymbol{g}^r)_{\mathbb{R}^m} \\ \vdots & & \vdots \\ (\boldsymbol{g}_r, \boldsymbol{g}^1)_{\mathbb{R}^m} & \cdots & (\boldsymbol{g}_r, \boldsymbol{g}^r)_{\mathbb{R}^m} \end{vmatrix} = \det \boldsymbol{I}_r = 1,$$

   即有 $\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_r \neq \boldsymbol{0} \in \Lambda^r(\mathbb{R}^m)$, 即得矛盾. □

由引理2.1(第8页) 可得如下引理.

引理 2.2 (向量组线性无关性的外积表示)**.** 向量组 $\{\boldsymbol{g}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性无关的充分必要条件为

$$\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_r \neq \boldsymbol{0} \in \Lambda^r(\mathbb{R}^m).$$

按引理2.1(第8页) 和引理2.2(第8页), 可有两向量组之间的关系.

引理 2.3 (Cartan 引理)**.** 设有向量组 $\{\boldsymbol{u}_i\}_{i=1}^r, \{\boldsymbol{v}_i\}_{i=1}^r \subset \mathbb{R}^m$, 满足

$$\sum_{i=1}^r \boldsymbol{u}_i \wedge \boldsymbol{v}_i = \boldsymbol{0} \in \Lambda^2(\mathbb{R}^m).$$

如有 $\{\boldsymbol{v}_i\}_{i=1}^r$ 线性无关, 则有

$$\boldsymbol{u}_i = \sum_{s=1}^r P_{si}\boldsymbol{v}_s, \quad i = 1, \cdots, r,$$

且 $P_{ij} = P_{ji}, \quad i, j = 1, \cdots, r.$

证明 由于 $\{\boldsymbol{v}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性无关, 则可补充 $\{\boldsymbol{v}_{r+1}, \cdots, \boldsymbol{v}_m\} \subset \mathbb{R}^m$, 使得 $\{\boldsymbol{v}_\alpha\}_{\alpha=1}^m$ 为 $\mathbb{R}^m$ 的一组基. 由此有

$$\boldsymbol{u}_i = \sum_{\alpha=1}^m P_{\alpha i}\boldsymbol{v}_\alpha, \quad i = 1, \cdots, r.$$

由

$$\begin{aligned}
\mathbf{0} = \sum_{i=1}^r \boldsymbol{u}_i \wedge \boldsymbol{v}_i &= \sum_{i=1}^r \sum_{\alpha=1}^m P_{\alpha i}\boldsymbol{v}_\alpha \wedge \boldsymbol{v}_i \\
&= \sum_{i=1}^r \sum_{j=1}^r P_{ji}\boldsymbol{v}_j \wedge \boldsymbol{v}_i + \sum_{i=1}^r \sum_{j=r+1}^m P_{ji}\boldsymbol{v}_j \wedge \boldsymbol{v}_i \\
&= \sum_{1\leqslant j<i\leqslant r} (P_{ji} - P_{ij})\boldsymbol{v}_j \wedge \boldsymbol{v}_i + \sum_{i=1}^r \sum_{j=r+1}^m P_{ji}\boldsymbol{v}_j \wedge \boldsymbol{v}_i \in \Lambda^2(\mathbb{R}^m),
\end{aligned}$$

考虑到 $\{\boldsymbol{v}_\beta \wedge \boldsymbol{v}_\alpha\}_{1\leqslant\beta<\alpha\leqslant m}$ 为 $\Lambda^2(\mathbb{R}^m)$ 的一个基, 有

$$P_{ij} = P_{ji}, \quad i, j = 1, \cdots, r; \quad P_{ji} = 0, \quad i = 1, \cdots, r, \ j = r+1, \cdots, m.$$

故有

$$\boldsymbol{u}_i = \sum_{j=1}^r P_{ji}\boldsymbol{v}_j, \quad i = 1, \cdots, r. \qquad \square$$

引理 2.4 (两向量组之间的关系)**.** 设有 $\forall\{\boldsymbol{u}_i, \boldsymbol{v}_i\}_{i=1}^r, \{\boldsymbol{\xi}_i, \boldsymbol{\eta}_i\}_{i=1}^r \subset \mathbb{R}^m$ 为两个向量组, 如有

1. $\{\boldsymbol{u}_i, \boldsymbol{v}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性无关;
2. $\displaystyle\sum_{i=1}^r \boldsymbol{u}_i \wedge \boldsymbol{v}_i = \sum_{i=1}^r \boldsymbol{\xi}_i \wedge \boldsymbol{\eta}_i \in \Lambda^2(\mathbb{R}^m)$,

则有 $\{\boldsymbol{\xi}_i, \boldsymbol{\eta}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性无关, 且 $\{\boldsymbol{\xi}_i, \boldsymbol{\eta}_i\}_{i=1}^r$ 可由 $\{\boldsymbol{u}_i, \boldsymbol{v}_i\}_{i=1}^r$ 线性表示.

证明 计算

$$\begin{aligned}
\left(\sum_{i=1}^r \boldsymbol{u}_i \wedge \boldsymbol{v}_i\right)^r &\triangleq (\boldsymbol{u}_1 \wedge \boldsymbol{v}_1 + \cdots + \boldsymbol{u}_r \wedge \boldsymbol{v}_r) \wedge \cdots \wedge (\boldsymbol{u}_1 \wedge \boldsymbol{v}_1 + \cdots + \boldsymbol{u}_r \wedge \boldsymbol{v}_r) \\
&= \sum_{\sigma\in P_r} \left(\boldsymbol{u}_{\sigma(1)} \wedge \boldsymbol{v}_{\sigma(1)}\right) \wedge \cdots \wedge \left(\boldsymbol{u}_{\sigma(m)} \wedge \boldsymbol{v}_{\sigma(m)}\right) \\
&= \left(\sum_{\sigma\in P_r} \operatorname{sgn}^2\sigma\right) [(\boldsymbol{u}_1 \wedge \boldsymbol{v}_1) \wedge \cdots \wedge (\boldsymbol{u}_r \wedge \boldsymbol{v}_r)] \\
&= r!\, \boldsymbol{u}_1 \wedge \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{u}_r \wedge \boldsymbol{v}_r,
\end{aligned}$$

故有

$$\boldsymbol{u}_1 \wedge \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{u}_r \wedge \boldsymbol{v}_r = \boldsymbol{\xi}_1 \wedge \boldsymbol{\eta}_1 \wedge \cdots \wedge \boldsymbol{\xi}_r \wedge \boldsymbol{\eta}_r.$$

由 $\{\boldsymbol{u}_i, \boldsymbol{v}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性无关, 故上式为非零 $2r$-形式. 按向量组线性无关的外积表示, 可有 $\{\boldsymbol{\xi}_i, \boldsymbol{\eta}_i\}_{i=1}^r \subset \mathbb{R}^m$ 线性无关.

另考虑到

$$\boldsymbol{u}_1\wedge\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{u}_r\wedge\boldsymbol{v}_r\wedge\boldsymbol{\xi}_i=\boldsymbol{\xi}_1\wedge\boldsymbol{\eta}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\eta}_r\wedge\boldsymbol{\xi}_i=\mathbf{0}\in\Lambda^{2r+1}(\mathbb{R}^m),$$

其中 $i=1,\cdots,r$. 结合向量组线性相关性的外积表示, 以及 $\{\boldsymbol{u}_i,\boldsymbol{v}_i\}_{i=1}^r$ 的线性无关性, 有 $\boldsymbol{\xi}_i\in\mathbb{R}^m(i=1,\cdots,r)$ 可由 $\{\boldsymbol{u}_i,\boldsymbol{v}_i\}_{i=1}^r$ 线性表示. □

引理 2.5 (外形式的低一阶表示). 设 $\{\boldsymbol{\xi}_i\}_{i=1}^r$ 为线性无关向量组, 则对 $\forall\boldsymbol{\Phi}\in\Lambda^p(\mathbb{R}^m)$ 有表示

$$\boldsymbol{\Phi}=\boldsymbol{\xi}_1\wedge\boldsymbol{\psi}_1+\cdots+\boldsymbol{\xi}_r\wedge\boldsymbol{\psi}_r,\quad \boldsymbol{\psi}_1,\cdots,\boldsymbol{\psi}_r\in\Lambda^{p-1}(\mathbb{R}^m)$$

的充分必要条件为

$$\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\Phi}=\mathbf{0}\in\Lambda^{r+p}(\mathbb{R}^m).$$

证明 按线性代数中的结论, 可将线性无关组 $\{\boldsymbol{\xi}_i\}_{i=1}^r\subset\mathbb{R}^m$ 扩充为 $\mathbb{R}^m$ 中的一组基 $\{\boldsymbol{\xi}_\alpha\}_{\alpha=1}^m$. 由此可有 $\{\boldsymbol{\xi}_{i_1}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}\}_{1\leqslant i_1<\cdots<i_p\leqslant m}$ 为 $\Lambda^p(\mathbb{R}^m)$ 的一组基, 即有

$$\begin{aligned}\boldsymbol{\Phi}&=\sum_{1\leqslant i_1<\cdots<i_p\leqslant m}\Phi^{i_1\cdots i_p}\boldsymbol{\xi}_{i_1}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}\\&=\boldsymbol{\xi}_1\wedge\left(\sum_{2\leqslant i_2<\cdots<i_p\leqslant m}\Phi^{1i_2\cdots i_p}\boldsymbol{\xi}_{i_2}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}\right)+\cdots\\&\quad+\boldsymbol{\xi}_r\wedge\left(\sum_{r+1\leqslant i_2<\cdots<i_p\leqslant m}\Phi^{ri_2\cdots i_p}\boldsymbol{\xi}_{i_2}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}\right)\\&\quad+\sum_{r+1\leqslant i_1<\cdots<i_p\leqslant m}\Phi^{i_1\cdots i_p}\boldsymbol{\xi}_{i_1}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}\\&=:\boldsymbol{\xi}_1\wedge\boldsymbol{\psi}_1+\cdots+\boldsymbol{\xi}_r\wedge\boldsymbol{\psi}_r+\sum_{r+1\leqslant i_1<\cdots<i_p\leqslant m}\Phi^{i_1\cdots i_p}\boldsymbol{\xi}_{i_1}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}.\end{aligned}$$

当 $r+p>m$ 时, 上式最后一项自动消失. 在此情形下即有

$$\boldsymbol{\Phi}=\boldsymbol{\xi}_1\wedge\boldsymbol{\psi}_1+\cdots+\boldsymbol{\xi}_r\wedge\boldsymbol{\psi}_r,\quad \boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\Phi}=\mathbf{0}.$$

亦即结论自然成立.

当 $r+p\leqslant m$ 时, 设有 $\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\Phi}=\mathbf{0}\in\Lambda^{r+p}(\mathbb{R}^m)$, 则

$$\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\Phi}=\sum_{r+1\leqslant i_1<\cdots<i_p\leqslant m}\Phi^{i_1\cdots i_p}\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\xi}_{i_1}\wedge\cdots\wedge\boldsymbol{\xi}_{i_p}=\mathbf{0}\in\Lambda^{r+p}(\mathbb{R}^m).$$

由于 $\{\boldsymbol{\xi}_{j_1}\wedge\cdots\wedge\boldsymbol{\xi}_{j_{r+p}}\}_{1\leqslant j_1<\cdots<j_{r+p}\leqslant m}$ 为 $\Lambda^{r+p}(\mathbb{R}^m)$ 的基, 故按上式有

$$\Phi^{i_1\cdots i_p}=0,\quad r+1\leqslant i_1<\cdots<i_p\leqslant m,$$

即有 $\boldsymbol{\Phi}=\boldsymbol{\xi}_1\wedge\boldsymbol{\psi}_1+\cdots+\boldsymbol{\xi}_r\wedge\boldsymbol{\psi}_r$.

反之, 设有 $\boldsymbol{\Phi}=\boldsymbol{\xi}_1\wedge\tilde{\boldsymbol{\psi}}_1+\cdots+\boldsymbol{\xi}_r\wedge\tilde{\boldsymbol{\psi}}_r$, 其中 $\tilde{\boldsymbol{\psi}}_1,\cdots,\tilde{\boldsymbol{\psi}}_r\in\Lambda^{p-1}(\mathbb{R}^m)$, 考虑到 $\{\boldsymbol{\xi}_{j_1}\wedge\cdots\wedge\boldsymbol{\xi}_{j_{r+p}}\}_{1\leqslant j_1<\cdots<j_{r+p}\leqslant m}$ 为 $\Lambda^{r+p}(\mathbb{R}^m)$ 的基, 则有

$$\Phi^{i_1\cdots i_p}=0,\quad r+1\leqslant i_1<\cdots<i_p\leqslant m,$$

故有

$$\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_r\wedge\boldsymbol{\Phi}=\mathbf{0}\in\Lambda^{r+p}(\mathbb{R}^m).$$ □

2.2 广义 Kronecker 符号

定义 2.1 (广义 Kronecker 符号)**.** $\mathbb{R}^m$ 空间中的广义 Kronecker 符号定义为

$$\delta^{i_1\cdots i_r}_{j_1\cdots j_r} \triangleq \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_r} \\ \vdots & & \vdots \\ \delta^{i_r}_{j_1} & \cdots & \delta^{i_r}_{j_r} \end{vmatrix},$$

此处 $i_1,\cdots,i_r=1,\cdots,m;\ j_1,\cdots,j_r=1,\cdots,m.$

按定理1.4(第5页), 有

$$\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = \boldsymbol{g}^{i_1}\wedge\cdots\wedge\boldsymbol{g}^{i_r}(\boldsymbol{g}_{j_1},\cdots,\boldsymbol{g}_{j_r}) = \boldsymbol{g}_{j_1}\wedge\cdots\wedge\boldsymbol{g}_{j_r}(\boldsymbol{g}^{i_1},\cdots,\boldsymbol{g}^{i_r}).$$

性质 2.6 (广义 Kronecker 符号基本性质)**.** 广义 Kronecker 符号具有如下基本性质:

1. $\delta^{i_1\cdots i_r s}_{j_1\cdots j_r s} = (m-r)\delta^{i_1\cdots i_r}_{j_1\cdots j_r}$;
2. $\delta^{i_1\cdots i_r s_t\cdots s_1}_{j_1\cdots j_r s_t\cdots s_1} = \dfrac{(m-r)!}{(m-r-t)!}\delta^{i_1\cdots i_r}_{j_1\cdots j_r}$.

证明 可直接通过行列式计算, 证明广义 Kronecker 符号的基本性质.

1. 根据定义, 如果 $j_1,\cdots,j_r$ 互不相同, 则有

$$\begin{aligned}\delta^{i_1\cdots i_r s}_{j_1\cdots j_r s} &\triangleq \sum_{s=1}^{m}\begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_r} & \delta^{i_1}_{s} \\ \vdots & & \vdots & \vdots \\ \delta^{i_r}_{j_1} & \cdots & \delta^{i_r}_{j_r} & \delta^{i_r}_{s} \\ \delta^{s}_{j_1} & \cdots & \delta^{s}_{j_r} & \delta^{s}_{s} \end{vmatrix} = \sum_{s\neq\{j_1,\cdots,j_r\}}\begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_r} & \delta^{i_1}_{s} \\ \vdots & & \vdots & \vdots \\ \delta^{i_r}_{j_1} & \cdots & \delta^{i_r}_{j_r} & \delta^{i_r}_{s} \\ \delta^{s}_{j_1} & \cdots & \delta^{s}_{j_r} & \delta^{s}_{s} \end{vmatrix} \\ &= \sum_{s\neq\{j_1,\cdots,j_r\}}\begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_r} & \delta^{i_1}_{s} \\ \vdots & & \vdots & \vdots \\ \delta^{i_r}_{j_1} & \cdots & \delta^{i_r}_{j_r} & \delta^{i_r}_{s} \\ 0 & \cdots & 0 & \delta^{s}_{s} \end{vmatrix} = \sum_{s\neq\{j_1,\cdots,j_r\}}\begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_r} \\ \vdots & & \vdots \\ \delta^{i_r}_{j_1} & \cdots & \delta^{i_r}_{j_r} \end{vmatrix} \\ &= \sum_{s\neq\{j_1,\cdots,j_r\}}\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = (m-r)\delta^{i_1\cdots i_r}_{j_1\cdots j_r}.\end{aligned}$$

如果 $j_1,\cdots,j_r$ 中至少两个相同, 则有 $\delta^{i_1\cdots i_r s}_{j_1\cdots j_r s} = \delta^{i_1\cdots i_r}_{j_1\cdots j_r} = 0$. 综上有

$$\delta^{i_1\cdots i_r s}_{j_1\cdots j_r s} = (m-r)\delta^{i_1\cdots i_r}_{j_1\cdots j_r}.$$

2. 根据上一条性质, 有

$$\begin{aligned}\delta^{i_1\cdots i_r s_t\cdots s_1}_{j_1\cdots j_r s_t\cdots s_1} &= [m-(r+t-1)]\delta^{i_1\cdots i_r s_t\cdots s_2}_{j_1\cdots j_r s_t\cdots s_2} \\ &= [m-(r+t-1)]\,[m-(r+t-2)]\,\delta^{i_1\cdots i_r s_t\cdots s_3}_{j_1\cdots j_r s_t\cdots s_3} \\ &= \cdots = [m-(r+t-1)]\cdots(m-r)\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = \frac{(m-r)!}{(m-r-t)!}\delta^{i_1\cdots i_r}_{j_1\cdots j_r}.\end{aligned}$$

□

设矩阵 $\boldsymbol{A}=\begin{pmatrix}A_{i_1j_1} & \cdots & A_{i_1j_r}\\ \vdots & & \vdots\\ A_{i_rj_1} & \cdots & A_{i_rj_r}\end{pmatrix}\in\mathbb{R}^{r\times r}$, 则基于广义 Kronecker 符号, 可得行列式的另一表达式

$$\begin{aligned}\det\boldsymbol{A}&=\frac{1}{r!}\sum_{\alpha,\beta\in P_r}\operatorname{sgn}\alpha\operatorname{sgn}\beta A_{\alpha(i_1),\beta(j_1)}\cdots A_{\alpha(i_r),\beta(j_r)}\\&=\frac{1}{r!}\sum_{\alpha,\beta\in P_r}\operatorname{sgn}\alpha\operatorname{sgn}\beta\left(\delta^{p_1}_{\alpha(i_1)}\cdots\delta^{p_r}_{\alpha(i_r)}\right)\left(\delta^{q_1}_{\beta(j_1)}\cdots\delta^{q_r}_{\beta(j_r)}\right)A_{p_1q_1}\cdots A_{p_rq_r}\\&=\frac{1}{r!}\begin{vmatrix}\delta^{p_1}_{i_1} & \cdots & \delta^{p_1}_{i_r}\\ \vdots & & \vdots\\ \delta^{p_r}_{i_1} & \cdots & \delta^{p_r}_{i_r}\end{vmatrix}\begin{vmatrix}\delta^{q_1}_{j_1} & \cdots & \delta^{q_1}_{j_r}\\ \vdots & & \vdots\\ \delta^{q_r}_{j_1} & \cdots & \delta^{q_r}_{j_r}\end{vmatrix}A_{p_1q_1}\cdots A_{p_rq_r}\\&=\frac{1}{r!}\delta^{p_1\cdots p_r}_{i_1\cdots i_r}\delta^{q_1\cdots q_r}_{j_1\cdots j_r}A_{p_1q_1}\cdots A_{p_rq_r}.\end{aligned}$$

为讨论方便, 考虑 $\boldsymbol{A}=\left(A_{ij}\right)\in\mathbb{R}^{r\times r}$, 此处 $i,j=1,\cdots,r$, 则有

$$\det\boldsymbol{A}=\frac{1}{r!}\delta^{p_1\cdots p_r}_{1\cdots r}\delta^{q_1\cdots q_r}_{1\cdots r}A_{p_1q_1}\cdots A_{p_rq_r}.$$

考虑

$$\begin{aligned}\frac{\partial\det\boldsymbol{A}}{\partial A_{ij}}&=\frac{1}{r!}\delta^{p_1\cdots p_r}_{1\cdots r}\delta^{q_1\cdots q_r}_{1\cdots r}\left(\frac{\partial A_{p_1q_1}}{\partial A_{ij}}A_{p_2q_2}\cdots A_{p_rq_r}+\cdots+A_{p_1q_1}\cdots A_{p_{r-1}q_{r-1}}\frac{\partial A_{p_rq_r}}{\partial A_{ij}}\right)\\&=\frac{1}{r!}\delta^{p_1\cdots p_r}_{1\cdots r}\delta^{q_1\cdots q_r}_{1\cdots r}\left(\delta_{p_1i}\delta_{q_1j}A_{p_2q_2}\cdots A_{p_rq_r}+\cdots+A_{p_1q_1}\cdots A_{p_{r-1}q_{r-1}}\delta_{p_ri}\delta_{q_rj}\right)\\&=\frac{1}{r!}\Big(\delta^{i\,p_2\cdots p_r}_{1\,2\cdots r}\delta^{j\,q_2\cdots q_r}_{1\,2\cdots r}A_{p_2q_2}\cdots A_{p_rq_r}\\&\qquad+\cdots+\delta^{p_1\cdots p_{r-1}\,i}_{1\cdots(r-1)\,r}\delta^{q_1\cdots q_{r-1}\,j}_{1\cdots(r-1)\,r}A_{p_1q_1}\cdots A_{p_{r-1}q_{r-1}}\Big)\\&=\frac{1}{(r-1)!}\delta^{i\,p_2\cdots p_r}_{1\,2\cdots r}\delta^{j\,q_2\cdots q_r}_{1\,2\cdots r}A_{p_2q_2}\cdots A_{p_rq_r}=:(\operatorname{adj}\boldsymbol{A})_{ji},\end{aligned}$$

式中 $\operatorname{adj}\boldsymbol{A}$ 表示矩阵 $\boldsymbol{A}$ 的**伴随矩阵**, 即

$$\operatorname{adj}\boldsymbol{A}=\begin{pmatrix}\varDelta_{11} & \cdots & \varDelta_{1m}\\ \vdots & & \vdots\\ \varDelta_{m1} & \cdots & \varDelta_{mm}\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}\varDelta_{11} & \cdots & \varDelta_{m1}\\ \vdots & & \vdots\\ \varDelta_{1m} & \cdots & \varDelta_{mm}\end{pmatrix},$$

此处 $\varDelta_{ij}$ 表示矩阵 $\boldsymbol{A}$ 元素 A_{ij} 的代数余子式. 计算

$$
\begin{aligned}
(\mathrm{adj}\boldsymbol{A})_{ji}A_{ik} &= \frac{1}{(r-1)!}\delta^{ip_2\cdots p_r}_{12\cdots r}\delta^{jq_2\cdots q_r}_{12\cdots r}A_{ik}A_{p_2q_2}\cdots A_{p_rq_r} \\
&= \frac{1}{(r-1)!}\left(\sum_{\sigma\in P_r}\mathrm{sgn}\,\sigma\delta^{i}_{\sigma(1)}\delta^{p_2}_{\sigma(2)}\cdots\delta^{p_r}_{\sigma(r)}\right)A_{ik}A_{p_2q_2}\cdots A_{p_rq_r}\delta^{jq_2\cdots q_r}_{12\cdots r} \\
&= \frac{1}{(r-1)!}\left(\sum_{\sigma\in P_r}\mathrm{sgn}\,\sigma A_{\sigma(1)k}A_{\sigma(2)q_2}\cdots A_{\sigma(r)q_r}\right)\delta^{jq_2\cdots q_r}_{12\cdots r} \\
&= \frac{1}{(r-1)!}\begin{vmatrix} A_{1k} & A_{1q_2} & \cdots & A_{1q_r} \\ \vdots & \vdots & & \vdots \\ A_{rk} & A_{rq_2} & \cdots & A_{rq_r}\end{vmatrix}\delta^{jq_2\cdots q_r}_{12\cdots r} \\
&= \frac{(-1)^j}{(r-1)!}\begin{vmatrix} A_{1k} & A_{1q_2} & \cdots & A_{1q_r} \\ \vdots & \vdots & & \vdots \\ A_{rk} & A_{rq_2} & \cdots & A_{rq_r}\end{vmatrix}\delta^{q_2\cdots j\cdots q_r}_{1\cdots j\cdots r}
\end{aligned}
$$

式中 $\{q_2,\cdots,q_r\}$ 只能为 $\{\sigma(1),\cdots,\overset{\circ}{j},\cdots,\sigma(r)\}$, $\forall\sigma\in P_{r-1}$, 结合行列式的基本性质, 可见当 $j\neq k$ 时, 有 $(\mathrm{adj}\boldsymbol{A})_{ji}A_{ik}=0$. 当 $j=k$ 时, 有

$$
\begin{aligned}
(\mathrm{adj}\boldsymbol{A})_{ji}A_{ij} &= \frac{1}{(r-1)!}\begin{vmatrix} A_{1q_2} & \cdots & A_{1j} & \cdots & A_{1q_r} \\ \vdots & & \vdots & & \vdots \\ A_{rq_2} & \cdots & A_{rj} & \cdots & A_{rq_r}\end{vmatrix}\delta^{q_2\cdots j\cdots q_r}_{1\cdots j\cdots r} \quad (\text{对 } j \text{ 不求和}) \\
&= \frac{1}{(r-1)!}\sum_{\sigma\in P_{r-1}}\begin{vmatrix} A_{1\sigma(1)} & \cdots & A_{1j} & \cdots & A_{1\sigma(r)} \\ \vdots & & \vdots & & \vdots \\ A_{r\sigma(1)} & \cdots & A_{rj} & \cdots & A_{r\sigma(r)}\end{vmatrix}\delta^{\sigma(1)\cdots j\cdots\sigma(r)}_{1\ \cdots j\cdots\ r} \\
&= \frac{1}{(r-1)!}\left(\sum_{\sigma\in P_{r-1}}\mathrm{sgn}^2\sigma\right)\begin{vmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{r1} & \cdots & A_{rr}\end{vmatrix} = \det\boldsymbol{A},
\end{aligned}
$$

此处 $\sigma=\begin{pmatrix} 1 & \cdots & \overset{\circ}{j} & \cdots & r \\ \sigma(1) & \cdots & \sigma(\overset{\circ}{j}) & \cdots & \sigma(r)\end{pmatrix}\in P_{r-1}$. 综上, 有

$$(\mathrm{adj}\boldsymbol{A})_{ji}A_{ik}=\delta_{jk}\det\boldsymbol{A}$$

或者当 $\det\boldsymbol{A}\neq 0$ 时, 有

$$\frac{\mathrm{adj}\boldsymbol{A}}{\det\boldsymbol{A}}\boldsymbol{A}=\boldsymbol{I}_r,$$

即有

$$\boldsymbol{A}^{-1}=\frac{\mathrm{adj}\boldsymbol{A}}{\det\boldsymbol{A}}.$$

2.3 Eddington 张量

Eddington 张量的定义可基于外积运算, 其几何意义可理解为 Euclid 空间中的体积单元.

定义 2.2 (Eddington 张量)**.** 设 $\{\boldsymbol{e}_i\}_{i=1}^m$ 为 $\mathbb{R}^m$ 空间中的一组单位正交基, 定义

$$\boldsymbol{\varepsilon} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m \in \Lambda^m(\mathbb{R}^m),$$

称为 Eddington 张量.

考虑 $\forall \{\boldsymbol{e}_{(i)}\}_{i=1}^m$ 为 $\mathbb{R}^m$ 的另一组单位正交基, 设有

$$\begin{pmatrix} \boldsymbol{e}_{(1)} & \cdots & \boldsymbol{e}_{(m)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_m \end{pmatrix} \boldsymbol{P},$$

其中 $\boldsymbol{P}$ 为正交矩阵, 而且设 $\det \boldsymbol{P} = 1$. 由此即有 $\boldsymbol{e}_{(i)} = P_{si}\boldsymbol{e}_s$. 计算

$$\boldsymbol{e}_{(1)} \wedge \cdots \wedge \boldsymbol{e}_{(m)} = (P_{s_1,1}\boldsymbol{e}_{s_1}) \wedge \cdots \wedge (P_{s_m,m}\boldsymbol{e}_{s_m}) = P_{s_1,1} \cdots P_{s_m,m}\boldsymbol{e}_{s_1} \wedge \cdots \wedge \boldsymbol{e}_{s_m}.$$

根据简单外形式的性质, $s_1, \cdots, s_m$ 中只要有任意两数相等, 结果即为零. 因此可有

$$\begin{aligned}
\boldsymbol{e}_{(1)} \wedge \cdots \wedge \boldsymbol{e}_{(m)} &= \sum_{\sigma \in P_m} P_{\sigma(1),1} \cdots P_{\sigma(m),m}\boldsymbol{e}_{\sigma(1)} \wedge \cdots \wedge \boldsymbol{e}_{\sigma(m)} \\
&= \sum_{\sigma \in P_m} \operatorname{sgn} \sigma P_{\sigma(1),1} \cdots P_{\sigma(m),m}\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m \\
&= \det (P_{ij})\, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m,
\end{aligned}$$

故 Eddington 张量的定义不依赖于单位正交基的选取.

对于一般基 $\{\boldsymbol{g}_i\}_{i=1}^m$, 设有

$$\begin{pmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{i}_1 & \cdots & \boldsymbol{i}_m \end{pmatrix} \boldsymbol{P},$$

即有 $\boldsymbol{g}_i = P_{si}\boldsymbol{i}_s$. 所以

$$\begin{aligned}
\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_m &= (P_{s_1,1}\boldsymbol{i}_{s_1}) \wedge \cdots \wedge (P_{s_m,m}\boldsymbol{i}_{s_m}) = \det (P_{ij})\, \boldsymbol{i}_1 \wedge \cdots \wedge \boldsymbol{i}_m \\
&= \det \begin{pmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m \end{pmatrix} \boldsymbol{i}_1 \wedge \cdots \wedge \boldsymbol{i}_m,
\end{aligned}$$

故有

$$\boldsymbol{\varepsilon} = \frac{1}{\sqrt{g}}\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_m.$$

此处 $\sqrt{g} = \det \begin{pmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m \end{pmatrix} > 0$ 称为体积单元. 同理可有 $\boldsymbol{\varepsilon} = \sqrt{g}\boldsymbol{g}^1 \wedge \cdots \wedge \boldsymbol{g}^m$.

性质 2.7.

$$\varepsilon^{i_1 \cdots i_m}\varepsilon_{j_1 \cdots j_m} = \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_m} \\ \vdots & & \vdots \\ \delta^{i_m}_{j_1} & \cdots & \delta^{i_m}_{j_m} \end{vmatrix}$$

证明 根据张量分量和 Eddington 张量的定义, 可有

$$\begin{aligned}
\varepsilon^{i_1\cdots i_m} &= \boldsymbol{\varepsilon}(\boldsymbol{g}^{i_1},\cdots,\boldsymbol{g}^{i_m}) = \sqrt{g}\boldsymbol{g}_1\wedge\cdots\boldsymbol{g}_m(\boldsymbol{g}^{i_1},\cdots,\boldsymbol{g}^{i_m})\\
&= \sqrt{g}\begin{vmatrix}\delta_1^{i_1} & \cdots & \delta_1^{i_m}\\ \vdots & & \vdots\\ \delta_m^{i_1} & \cdots & \delta_m^{i_m}\end{vmatrix},\\
\varepsilon_{j_1\cdots j_m} &= \boldsymbol{\varepsilon}(\boldsymbol{g}_{j_1},\cdots,\boldsymbol{g}_{j_m}) = \frac{1}{\sqrt{g}}\boldsymbol{g}^1\wedge\cdots\boldsymbol{g}^m(\boldsymbol{g}_{j_1},\cdots,\boldsymbol{g}_{j_m})\\
&= \frac{1}{\sqrt{g}}\begin{vmatrix}\delta_{j_1}^1 & \cdots & \delta_{j_m}^1\\ \vdots & & \vdots\\ \delta_{j_1}^m & \cdots & \delta_{j_m}^m\end{vmatrix},
\end{aligned}$$

所以

$$\varepsilon^{i_1\cdots i_m}\varepsilon_{j_1\cdots j_m} = \begin{vmatrix}\delta_1^{i_1} & \cdots & \delta_1^{i_m}\\ \vdots & & \vdots\\ \delta_m^{i_1} & \cdots & \delta_m^{i_m}\end{vmatrix}\begin{vmatrix}\delta_{j_1}^1 & \cdots & \delta_{j_m}^1\\ \vdots & & \vdots\\ \delta_{j_1}^m & \cdots & \delta_{j_m}^m\end{vmatrix} = \begin{vmatrix}\delta_{j_1}^{i_1} & \cdots & \delta_{j_m}^{i_1}\\ \vdots & & \vdots\\ \delta_{j_1}^{i_m} & \cdots & \delta_{j_m}^{i_m}\end{vmatrix}.$$

上式最后一步利用了行列式的性质**??**(第**??**页), 即关系式 $\det(\boldsymbol{AB}) = \det\boldsymbol{A}\det\boldsymbol{B}$. □

2.4 Hodge 星算子

Hodge 星算子的定义基于外积运算.

定义 2.3 (Hodge 星算子). Hodge 星算子用映照形式可以表示为

$$*: \Lambda^r(\mathbb{R}^m)\ni\boldsymbol{\Phi}\mapsto *\boldsymbol{\Phi}\in\Lambda^s(\mathbb{R}^m),\quad s=m-r,$$

满足

$$\boldsymbol{\varepsilon}\diamondsuit(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}) = (*\boldsymbol{\Phi})\diamondsuit\boldsymbol{\Psi} \triangleq \frac{1}{s!}(*\boldsymbol{\Phi})\odot\boldsymbol{\Psi},\quad \forall\boldsymbol{\Psi}\in\Lambda^s(\mathbb{R}^m).$$

此处 $\boldsymbol{\varepsilon}\in\Lambda^m(\mathbb{R}^m)$ 为 m 阶 Eddington 张量, 运算 $\diamondsuit$ 称为**外全点积**.

性质 2.8 (Hodge 星算子基本性质). Hodge 星算子具有如下基本性质:

1. 对 $\forall\boldsymbol{\Phi}\in\Lambda^r(\mathbb{R}^m)$, 有
$$*\boldsymbol{\Phi} = \frac{(-1)^{rs}}{r!}\boldsymbol{\varepsilon}\begin{pmatrix}r\\ \bullet\end{pmatrix}\boldsymbol{\Phi},\quad s=m-r;$$

2. 对 $\forall\boldsymbol{\Phi}\in\Lambda^r(\mathbb{R}^m)$, 有
$$*(*\boldsymbol{\Phi}) = (-1)^{rs}\boldsymbol{\Phi},\quad s=m-r.$$

证明 可直接通过张量整体形式的计算, 证明 Hodge 星算子的基本性质.

1. 计算

$$
\begin{aligned}
\boldsymbol{\varepsilon} \lozenge (\boldsymbol{\Phi} \wedge \boldsymbol{\Psi}) &= \boldsymbol{\varepsilon} \lozenge \left(\frac{1}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\boldsymbol{g}^{i_1} \wedge \cdots \wedge \boldsymbol{g}^{i_r} \wedge \boldsymbol{g}^{j_1} \wedge \cdots \wedge \boldsymbol{g}^{j_s}\right) \\
&= \boldsymbol{\varepsilon} \lozenge \left[\frac{m!}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\mathscr{A}\left(\boldsymbol{g}^{i_1} \otimes \cdots \otimes \boldsymbol{g}^{i_r} \otimes \boldsymbol{g}^{j_1} \otimes \cdots \otimes \boldsymbol{g}^{j_s}\right)\right] \\
&= \boldsymbol{\varepsilon} \lozenge \left[\frac{1}{r!s!}\sum_{\sigma\in P_m} \operatorname{sgn}\sigma \Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\boldsymbol{g}^{\sigma^{-1}(i_1)} \otimes \cdots \otimes \boldsymbol{g}^{\sigma^{-1}(i_r)} \otimes \boldsymbol{g}^{\sigma^{-1}(j_1)} \otimes \cdots \otimes \boldsymbol{g}^{\sigma^{-1}(j_s)}\right] \\
&= \boldsymbol{\varepsilon} \lozenge \left[\frac{1}{r!s!}\sum_{\sigma\in P_m} \operatorname{sgn}\sigma \Phi_{\sigma(i_1)\cdots\sigma(i_r)}\Psi_{\sigma(j_1)\cdots\sigma(j_s)}\boldsymbol{g}^{i_1} \otimes \cdots \otimes \boldsymbol{g}^{i_r} \otimes \boldsymbol{g}^{j_1} \otimes \cdots \otimes \boldsymbol{g}^{j_s}\right] \\
&= \frac{1}{r!s!m!}\sum_{\sigma\in P_m} \operatorname{sgn}\sigma \Phi_{\sigma(i_1)\cdots\sigma(i_r)}\Psi_{\sigma(j_1)\cdots\sigma(j_s)}\varepsilon^{i_1\cdots i_r j_1\cdots j_s} \\
&= \frac{1}{r!s!m!}\sum_{\sigma\in P_m} \operatorname{sgn}\sigma \Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\varepsilon^{\sigma^{-1}(i_1)\cdots\sigma^{-1}(i_r)\sigma^{-1}(j_1)\cdots\sigma^{-1}(j_s)} \\
&= \frac{1}{r!s!m!}\sum_{\sigma\in P_m} \operatorname{sgn}\sigma \Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\operatorname{sgn}\sigma^{-1}\varepsilon^{i_1\cdots i_r j_1\cdots j_s} \\
&= \frac{1}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\varepsilon^{i_1\cdots i_r j_1\cdots j_s}.
\end{aligned}
$$

另有

$$
(*\boldsymbol{\Phi}) \lozenge \boldsymbol{\Psi} = \frac{1}{s!}(*\Phi)^{j_1\cdots j_s}\Psi_{j_1\cdots j_s},
$$

所以, 有

$$
(*\Phi)^{j_1\cdots j_s} = \frac{1}{r!}\Phi_{i_1\cdots i_r}\varepsilon^{i_1\cdots i_r j_1\cdots j_s} = \frac{(-1)^{rs}}{r!}\varepsilon^{j_1\cdots j_s i_1\cdots i_r}\Phi_{i_1\cdots i_r},
$$

即有

$$
*\boldsymbol{\Phi} = \frac{(-1)^{rs}}{r!}\boldsymbol{\varepsilon}\binom{r}{\cdot}\boldsymbol{\Phi}.
$$

2. 计算

$$
\begin{aligned}
(\boldsymbol{\Phi}) &= *\left[\frac{(-1)^{rs}}{r!}\boldsymbol{\varepsilon}\binom{r}{\cdot}\boldsymbol{\Phi}\right] = \frac{(-1)^{sr}}{s!}\boldsymbol{\varepsilon}\binom{s}{\cdot}\left[\frac{(-1)^{rs}}{r!}\boldsymbol{\varepsilon}\binom{r}{\cdot}\boldsymbol{\Phi}\right] \\
&= \frac{1}{r!s!}\boldsymbol{\varepsilon}\binom{s}{\cdot}\left[\boldsymbol{\varepsilon}\binom{r}{\cdot}\boldsymbol{\Phi}\right] = \frac{1}{r!s!}\boldsymbol{\varepsilon}\binom{s}{\cdot}\left(\varepsilon^{j_1\cdots j_s i_1\cdots i_r}\Phi_{i_1\cdots i_r}\boldsymbol{g}_{j_1} \otimes \cdots \otimes \boldsymbol{g}_{j_s}\right) \\
&= \frac{1}{r!s!}\varepsilon_{k_1\cdots k_r j_1\cdots j_s}\varepsilon^{j_1\cdots j_s i_1\cdots i_r}\Phi_{i_1\cdots i_r}\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} \\
&= \frac{(-1)^{rs}}{r!s!}\varepsilon_{k_1\cdots k_r j_1\cdots j_s}\varepsilon^{i_1\cdots i_r j_1\cdots j_s}\Phi_{i_1\cdots i_r}\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} \\
&= \frac{(-1)^{rs}}{r!s!}\delta^{i_1\cdots i_r j_1\cdots j_s}_{k_1\cdots k_r j_1\cdots j_s}\Phi_{i_1\cdots i_r}\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} \\
&= \frac{(-1)^{rs}}{r!}\delta^{i_1\cdots i_r}_{k_1\cdots k_r}\Phi_{i_1\cdots i_r}\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} \\
&= \frac{(-1)^{rs}}{r!}\left(\sum_{\sigma\in P_r}\operatorname{sgn}\sigma\delta^{i_1}_{\sigma(k_1)}\cdots\delta^{i_r}_{\sigma(k_r)}\Phi_{i_1\cdots i_r}\right)\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} \\
&= \frac{(-1)^{rs}}{r!}\left(\sum_{\sigma\in P_r}\operatorname{sgn}\sigma\Phi_{\sigma(i_1)\cdots\sigma(i_r)}\right)\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} \\
&= (-1)^{rs}\Phi_{k_1\cdots k_r}\boldsymbol{g}^{k_1} \otimes \cdots \otimes \boldsymbol{g}^{k_r} = (-1)^{rs}\boldsymbol{\Phi}.
\end{aligned}
$$

□

作为例子, 考虑 $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, 则 $\boldsymbol{a} \wedge \boldsymbol{b} \in \Lambda^2(\mathbb{R}^3)$, 且有

$$\begin{aligned}
*(\boldsymbol{a} \wedge \boldsymbol{b}) &= \frac{(-1)^{2\times 1}}{2!}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}(\boldsymbol{a} \wedge \boldsymbol{b}) = \frac{1}{2}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}[2\mathscr{A}(\boldsymbol{a}\otimes\boldsymbol{b})] \\
&= \frac{1}{2}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}\left[\sum_{\sigma\in P_2}\operatorname{sgn}\sigma I_\sigma\left(a_i b_j \boldsymbol{g}^i \otimes \boldsymbol{g}^j\right)\right] = \frac{1}{2}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}\left[a_i b_j\left(\boldsymbol{g}^i\otimes\boldsymbol{g}^j - \boldsymbol{g}^j\otimes\boldsymbol{g}^i\right)\right] \\
&= \frac{1}{2}\left(\varepsilon^{ijk}a_i b_j \boldsymbol{g}_k - \varepsilon^{jik}a_i b_j \boldsymbol{g}_k\right) = \varepsilon^{ijk}a_i b_j \boldsymbol{g}_k = \boldsymbol{a}\times\boldsymbol{b} \in \mathbb{R}^3.
\end{aligned}$$

亦即, 对 $\mathbb{R}^3$ 中任意 2 个向量的叉乘, 可通过两者的外积再取 Hodeg 算子来表示.

再考虑 $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^4$, 则 $\boldsymbol{a} \wedge \boldsymbol{b} \in \Lambda^2(\mathbb{R}^4)$, 且有

$$\begin{aligned}
*(\boldsymbol{a} \wedge \boldsymbol{b}) &= \frac{(-1)^{2\times 2}}{2!}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}(\boldsymbol{a} \wedge \boldsymbol{b}) = \frac{1}{2}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}[2\mathscr{A}(\boldsymbol{a}\otimes\boldsymbol{b})] \\
&= \frac{1}{2}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}\left[\sum_{\sigma\in P_2}\operatorname{sgn}\sigma I_\sigma\left(a_i b_j \boldsymbol{g}^i \otimes \boldsymbol{g}^j\right)\right] = \frac{1}{2}\boldsymbol{\varepsilon}\begin{pmatrix}2\\ \cdot\end{pmatrix}\left[a_i b_j\left(\boldsymbol{g}^i\otimes\boldsymbol{g}^j - \boldsymbol{g}^j\otimes\boldsymbol{g}^i\right)\right] \\
&= \frac{1}{2}\left(\varepsilon^{stij}a_i b_j \boldsymbol{g}_s\otimes\boldsymbol{g}_t - \varepsilon^{stji}a_i b_j \boldsymbol{g}_s\otimes\boldsymbol{g}_t\right) = \varepsilon^{stij}a_i b_j \boldsymbol{g}_s\otimes\boldsymbol{g}_t \in \mathscr{T}^2(\mathbb{R}^4) \\
&= \varepsilon^{\sigma^{-1}(s)\sigma^{-1}(t)ij}a_i b_j \boldsymbol{g}_{\sigma(s)}\otimes\boldsymbol{g}_{\sigma(t)}, \quad \forall \sigma \in P_2 \\
&= \varepsilon^{stij}a_i b_j\left[\operatorname{sgn}\sigma I_\sigma\left(\boldsymbol{g}_s\otimes\boldsymbol{g}_t\right)\right], \quad \forall \sigma \in P_2 \\
&= \frac{1}{2!}\varepsilon^{stij}a_i b_j\left[\sum_{\sigma\in P_2}\operatorname{sgn}\sigma I_\sigma\left(\boldsymbol{g}_s\otimes\boldsymbol{g}_t\right)\right] = \varepsilon^{stij}a_i b_j \mathscr{A}(\boldsymbol{g}_s\otimes\boldsymbol{g}_t) \\
&= \frac{1}{2}\varepsilon^{stij}a_i b_j \boldsymbol{g}_s\wedge\boldsymbol{g}_t.
\end{aligned}$$

另考虑 $\{\boldsymbol{g}_i\}_{i=1}^{p+1} \subset \mathbb{R}^{p+1}$ 为一个基, 而 $\{\boldsymbol{g}^i\}_{i=1}^{p+1} \subset \mathbb{R}^{p+1}$ 为其对偶基, 可计算

$$\begin{aligned}
&*(\boldsymbol{g}_1\wedge\cdots\wedge\boldsymbol{g}_p) = \frac{(-1)^{p\times 1}}{p!}\boldsymbol{\varepsilon}\begin{pmatrix}p\\ \cdot\end{pmatrix}(\boldsymbol{g}_1\wedge\cdots\wedge\boldsymbol{g}_p) = \frac{(-1)^p}{p!}\boldsymbol{\varepsilon}\begin{pmatrix}p\\ \cdot\end{pmatrix}\left[p!\mathscr{A}(\boldsymbol{g}_1\otimes\cdots\otimes\boldsymbol{g}_p)\right] \\
&= \frac{(-1)^p}{p!}\boldsymbol{\varepsilon}\begin{pmatrix}p\\ \cdot\end{pmatrix}\left[\sum_{\sigma\in P_p}\operatorname{sgn}\sigma I_\sigma(\boldsymbol{g}_1\otimes\cdots\otimes\boldsymbol{g}_p)\right] \\
&= \frac{(-1)^p}{p!}\sum_{\sigma\in P_p}\operatorname{sgn}\sigma\boldsymbol{\varepsilon}\begin{pmatrix}p\\ \cdot\end{pmatrix}\boldsymbol{g}_{\sigma(1)}\otimes\cdots\otimes\boldsymbol{g}_{\sigma(p)} = \frac{(-1)^p}{p!}\sum_{\sigma\in P_p}\operatorname{sgn}\sigma\varepsilon_{s\sigma(1)\cdots\sigma(p)}\boldsymbol{g}^s \\
&= \frac{(-1)^p}{p!}\left(\sum_{\sigma\in P_p}\operatorname{sgn}^2\sigma\right)\varepsilon_{s1\cdots p}\boldsymbol{g}^s = \varepsilon_{1\cdots p(p+1)}\boldsymbol{g}^{p+1} = \frac{1}{\sqrt{g}}\boldsymbol{g}^{p+1}.
\end{aligned}$$

此处 $\sqrt{g} = \det\begin{pmatrix}\boldsymbol{g}_1 & \cdots & \boldsymbol{g}_p & \boldsymbol{g}_{p+1}\end{pmatrix}$. 上述分析中利用了

$$\varepsilon_{1\cdots p(p+1)} \triangleq \boldsymbol{\varepsilon}(\boldsymbol{g}_1,\cdots,\boldsymbol{g}_{p+1}) = \frac{1}{\sqrt{g}}\boldsymbol{g}^1\wedge\cdots\wedge\boldsymbol{g}^{p+1}(\boldsymbol{g}_1,\cdots,\boldsymbol{g}_{p+1}) = \frac{1}{\sqrt{g}}.$$

3 建立路径

- 置换运算的基本性质决定了置换算子的基本性质; 而置换算子的基本性质又决定了外积运算的基本性质. 亦即, 置换运算及其基本性质是本质.

- 外积运算可谓 “一种神奇的运算”, 藉此可以解释诸多反对称张量的内在性质.