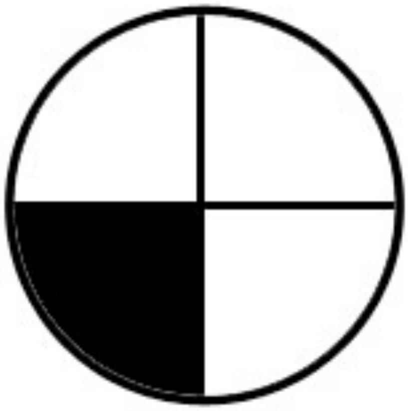
Calculus related puzzles

Buffon's needle problem

A needle 2 inches long is dropped at random onto a floor made with wooden boards 2 inches in width, placed side by side. What is the probability that the needle falls across one of the cracks?

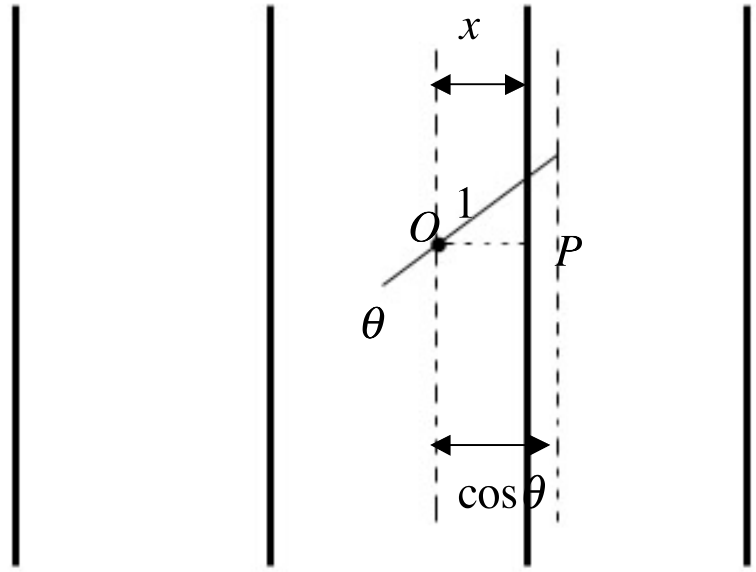
This puzzle certainly deals with the aspect of probability, something many of you should be familiar with. Let's have a small discussion about it. Say you have to close your eyes and point to one of the sections of the wheel below.



Your chances of pointing to a black one is $\frac{1}{4}$. Why? Simply because there are 4 different sections, or 4 different outcomes and out of those 4, only 1 of them is black. Now, what are your chances on pointing to a black one with the wheel below.

It is now $\frac{3}{8}$. 3 black sections out of 8 available outcomes. Simply enough to understand.

Now, how about this needle problem? Obviously, it seems hard to visualize what is the total possible amount of outcomes. To help us, we create a coordinate system. Define x and θ as shown below where x is the distance *OP* from the midpoint of the needle to the nearest crack, and θ is the smallest angle between *OP* and the needle.



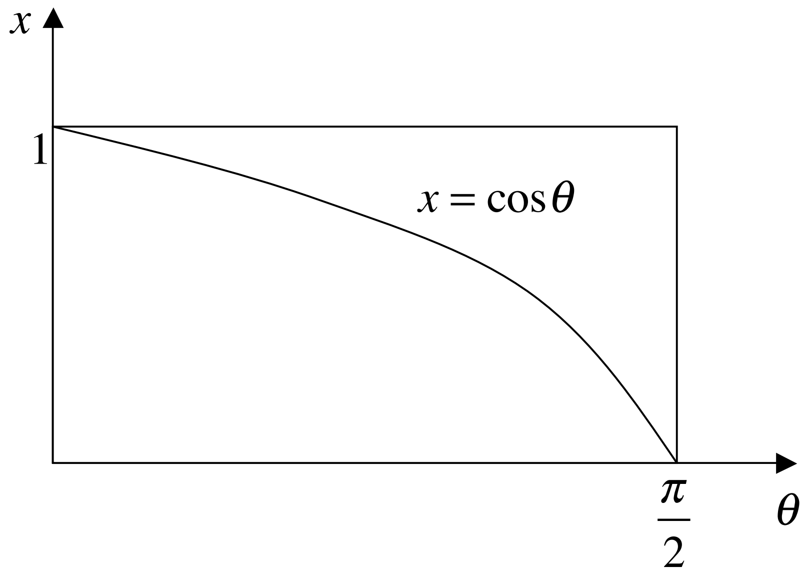
Notice that a random toss of the needle can be 'marked' out by this new coordinate system with the variables in the interval

$$0 \le x \le 1 \text{ and } 0 \le \theta \le \frac{\pi}{2}$$

which covers all the random positions the needle can fall to. Further, notice that the outcome with are interest in can be written as the inequality

$$x < \cos\theta$$

By plotting the graph $x = \cos\theta$ as follows,



we see that this inequality describes the shaded region under the graph. hence, we conclude that the probability of the needle falling across a crack equals to the following ratio of areas:

$$\frac{\text{area under curve}}{\text{area of rectange}} = \frac{\int_0^{\pi/2} \cos\theta \, d\theta}{\pi/2} = \frac{1}{\pi/2} = \frac{\pi}{2}$$

which is slightly less than $2/3$.

Who would have thought that a question starting out on probability could have used the integral calculus to solve it.

Here is an experimental procedures to think about. Say that I decided to carry out the experiment of tossing and recording the amount of times it falls across the crack. I toss the needle n times and record when it falls k times. I happen to be very free to the point where I performed this experiment an infinite amount of times. Mathematically then, we have

$$\lim_{n\to\infty} \frac{k}{n} = \frac{2}{\pi}$$

which reduces to

$$\frac{k}{n} \cong \frac{2}{\pi}$$

assuming I carry out the experiment infinite amount of times. And by doing so, I have just found the value of π, or at least a close approximation by theory, without any use of geometry let alone circles. Now you can tell your friends how to get its value without wrapping a string about a tennis ball, which was how I did it in middle school.