复旦大学力学与工程科学系

2011 ～ 2012 学年第一学期期末考试试卷

□A 卷　　　□B 卷

课程名称：Calculus on Differential Manifolds　课程代码：MATH120008.09

开课院系：力学与工程科学系　　　　考试形式：开卷/闭卷/课程论文

姓名：________　学号：________　专业：________

题 号	1/(1)	1/(2)	1/(3)	1/(4)	2/(1)	2/(2)	2/(3)	2/(4)	2/(5)	2/(6)
得 分										
题 号	3/(1)	3/(2)	3/(3)	4/(1)	4/(2)	5/(1)	5/(2)	5/(3)	5/(4)	5/(5)
得 分										
题 号	5/(6)	5/(7)	6/(1)	6/(2)						总分
得 分										

Problem 1 (The Application of the Diffeomorphism) *The concept of diffeomorphism with its related results play the essential role to recognize the so-called differential manifolds.*

1. *To narrate the content of rank theorem.*
2. *To narrate the definition of differential manifolds.*
3. *To proof the following proposition. If a chart $\varphi : I^k \to U \subset S$ belongs to class $C^{(1)}(I^k, \mathbb{R}^n)$ and has maximal rank at each point of the cube I^k,there exists a number $\varepsilon > 0$ and a diffeomorphism $\varphi_\varepsilon : I^n_\varepsilon \to \mathbb{R}^n$ of the cube $I^n_\varepsilon := \{t \in \mathbb{R}^n || t^i| \le \varepsilon_i, i = 1, \cdots, n\}$ of dimension n in $\mathbb{R}^n$ such that $\varphi|_{I^k \cap I^n_\varepsilon} = \varphi_\varepsilon|_{I^k \cap I^n_\varepsilon}$.*
4. *To prove that the unite sphere in $\mathbb{R}^3$ is the differential manifold.*

Problem 2 (Tangent and cotangent mapping) *Let ϕ be the differential mapping between the differential manifolds M and N with $\dim M = m$ and $\dim N = n$ respectively. Its corresponding tangent mapping is defined as follows*

$$(\phi^*\omega)_{i_1,\cdots,i_r}(x) \triangleq \left[\frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_r}}{\partial x^{i_r}}\right](x)\omega_{\alpha_1,\cdots,\alpha_r}(y(x))$$

1. *To prove that*

$$\{(\phi^*\omega)_{i_1,\cdots,i_r}(x)\} \in \wedge^r(M)$$

2. *To prove the general identity*

$$\begin{aligned}(\phi^*\omega)(x) &\equiv \phi^*(y(x))\omega(y(x)) = \phi^*(y(x))\left[\frac{1}{r!}\omega_{\alpha_1,\cdots,\alpha_r}(y(x))dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_r}\right] \\ &= \frac{1}{r!}\sum_{1\le i_1<\cdots<i_r\le m}\omega_{\alpha_1,\cdots,\alpha_r}(y(x))\left[\frac{\partial(y^{\alpha_1}\cdots y^{\alpha_r})}{\partial(x^{i_1}\cdots x^{i_r})}(x)\right]dx^{i_1}\wedge\cdots\wedge dx^{i_r} \in \wedge^r(M)\end{aligned}$$

for any $\omega(y) = \frac{1}{r!}\omega_{\alpha_1,\cdots,\alpha_r}(y)dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_r} \in \wedge^r(N)$

3. *To prove the general identity*

$$\phi^*(d\omega) = d(\phi^*\omega), \quad \forall\omega \in \wedge^r(N)$$

4. *To consider the following particular differential mapping*

$$\phi(x): \mathbb{R}^4 \supset \mathscr{D}_x \ni x = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} \mapsto \phi(x) = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{bmatrix}(x) \triangleq \begin{bmatrix} C_1x^1 \\ x^1x^2x^3x^4 \\ C_3x^3 \\ C_4x^4 \end{bmatrix} \in \mathbb{R}^4$$

where C_1*,* C_3 *and* C_4 *are constants. To calculate*

$$(\phi^*\omega)(x), \quad \omega(y) = f(y^2)dy^1\wedge dy^3\wedge dy^4 \in \wedge^3(\mathbb{R}^4)$$

5. *To calculate* $d(\phi^*\omega)(x)$

6. *To calculate* $(\phi^*d\omega)(x)$

Problem 3 (Integral on Differential Manifold) *The fundamentals of integral on differential manifold could be concluded as follows.*

1. *The integral of one p-form on the differential manifold with dimension p is defined as follows:*

$$\int_{\phi(I_p)\subset M}\omega^p \triangleq \int_{I_p}\phi^*\omega^p, \quad \omega^p \in \wedge^p(M)$$

where $\phi(x) \in \mathscr{C}(I_p, \phi(I_p))$ *is any chart. To prove that it is well-definition, that is the value of the integral is independent on the choose of the charts.*

2. *The tours in* $\mathbb{R}^3$ *could be represented as*

$$\Sigma(\theta,\phi):\,\mathscr{D}_{\theta\phi}\ni\begin{bmatrix}\theta\\ \phi\end{bmatrix}\mapsto\Sigma(\theta,\phi)=\begin{bmatrix}X^1\\ X^2\\ X^3\end{bmatrix}(\theta,\phi)\triangleq\begin{bmatrix}(R+r\cos\theta)\cdot\cos\phi\\ (R+r\cos\theta)\cdot\sin\phi\\ R+r\sin\theta\end{bmatrix}$$

where $\theta\in[0,2\pi]$ *and* $\phi\in[0,2\pi]$. *To calculate the integral*

$$\int_\Sigma dX^1\wedge dX^3$$

3. *To calculate the surface area of the torus through the following definition*

$$|\Sigma|\triangleq\int_{\mathscr{D}_{\theta\phi}}\sqrt{g}\,d\theta\wedge d\phi,\quad g\triangleq det[g_{ij}]$$

where $\{g_{ij}\}$ *is the measure on the torus.*

Problem 4 (Stokes Formula) *The fundamentals of Stokes formula on differential manifold could be concluded as follows.*

1. *To prove the following relation of integral*

$$\int_{I_{p-1}}\omega^{p-1}=(-1)^p\int_{\overline{H_p^+}}d\omega^{p-1}$$

where

$$\omega^{p-1}=\sum_{i=1}^{p}f_i(x)dx^1\wedge\cdots\wedge\overset{\circ}{dx}{}^i\wedge\cdots\wedge dx^p$$

such that $suppf_i(x^1,\cdots,x^{p-1},0)\subset I_{p-1}$ *and* $f_i(x^1,\cdots,x^{p-1},1)=0$ *for all* $1\le i\le p-1$.

2. *The general form of the Stokes formula is*

$$\int_{\partial M}\omega^{p-1}=(-1)^p\int_M d\omega^{p-1},\quad\omega^{p-1}\in\wedge^{p-1}(M)$$

where $\dim M=p$. *To calculate the following integral through Stokes formula*

$$\oint_\Sigma X^2dX^1\wedge dX^3+X^1dX^3\wedge dX^2$$

where Σ *is the torus in* $\mathbb{R}^3$

Problem 5 (Field Analysis on the Surface–Case Study) *To consider the so-called Helical-Surface*

$$\Sigma(u,v):\,\mathbb{R}^2\supset\mathscr{D}_{uv}\ni\begin{bmatrix}u\\ v\end{bmatrix}\mapsto\Sigma(u,v)=\begin{bmatrix}X^1\\ X^2\\ X^3\end{bmatrix}(u,v)\triangleq\begin{bmatrix}u\cos v\\ u\sin v\\ hv\end{bmatrix}\in\mathbb{R}^3$$

1. *To give the measure, i.e.the first form, on the surface that could be represented by the matrix* $[g_{ij}] \in \mathbb{R}^{2\times 2}$
2. *To give the second form on the surface that could be represented by the matrix* $[b_{ij}] \in \mathbb{R}^{2\times 2}$
3. *To calculate the Gauss and Mean curvatures denoted by* K_G *and* H *respectively.*
4. *To give the connection on the surface that is compatible to the given measure.*
5. *To determine the parallel-moving vector field following the trajectory*

$$\gamma(v) : [0, 2\pi] \ni v \mapsto \gamma(v) = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}(v) \triangleq \begin{bmatrix} a\cos v \\ a\sin v \\ hv \end{bmatrix} \in \Sigma$$

6. *To calculate the Riemann-Christoffel tensor* R_{ijpq} *on the surface through the Gauss equation, namely,*

$$R_{ijkl} = b_{ik}b_{jl} - b_{jk}b_{il}$$

7. *To prove the following identity*

$$R_{ijkl} = K_G(g_{ik}g_{jl} - g_{jk}g_{il})$$

that is just valid for 2 *dimensional differential surface in* $\mathbb{R}^3$*.*

Problem 6 (Field Analysis on the Surface–General Study) *To consider the general* $n-1$ *dimensional differential surface in* $\mathbb{R}^n$

1. *To deduce the so-called Gauss & Codazzi equations for*

$$\begin{aligned} R_{ijkl} &= b_{ik}b_{jl} - b_{jk}b_{il} \\ \nabla_p b_{qj} &= \nabla_q b_{pj} \end{aligned}$$

2. *To deduce the so-called Ricci identity*

$$\nabla_p\nabla_q\Phi^i_{\cdot j} - \nabla_p\nabla_q\Phi^i_{\cdot j} = R^i_{spq}\Phi^s_{\cdot j} - R^s_{jpq}\Phi^i_{\cdot s}$$

Generally, this identity could be extended to the tensor with any order.

Note: To give the deduction and calculation in detail. And as the score is considered, the reflection of the correct methodologies is oriented.