微分流形上微分学——流形上的微分运算—外微分与里积

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2016 年 4 月 21 日

1　知识要素

1.1　外微分运算

定义 1.1 (外微分运算)**.** 对 $\forall \boldsymbol{\Phi} \in \Lambda^r(TM)$, 可定义以下外微分运算:

$$\begin{aligned}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{x}) &= \mathrm{d}\left(\frac{1}{r!}\Phi_{i_1\cdots i_r}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\right)(\boldsymbol{x}) \triangleq \mathrm{d}\left(\frac{1}{r!}\Phi_{i_1\cdots i_r}\right)\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r} \\ &= \frac{1}{r!}\frac{\partial\Phi_{i_1\cdots i_r}}{\partial x^s}(\boldsymbol{x})\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r} \in \Lambda^{r+1}(TM).\end{aligned}$$

性质 1.1 (外微分运算的基本性质)**.** 外微分运算具有如下基本性质.

1. 线性性: 对 $\forall \boldsymbol{\Phi}, \boldsymbol{\Psi} \in \Lambda^r(TM), \forall \alpha, \beta \in \mathbb{R}$, 有

$$\mathrm{d}(\alpha\boldsymbol{\Phi} + \beta\boldsymbol{\Psi}) = \alpha\mathrm{d}\boldsymbol{\Phi} + \beta\mathrm{d}\boldsymbol{\Psi} \in \wedge^{r+1}(TM);$$

2. Poincare 性: 对 $\forall \boldsymbol{\Phi} \in \Lambda^r(TM)$, 有

$$\mathrm{d}^2\boldsymbol{\Phi} = \mathrm{d}(\mathrm{d}\boldsymbol{\Phi}) = 0;$$

3. 反导性: 对 $\forall \boldsymbol{\Phi} \in \Lambda^r(TM), \boldsymbol{\Psi} \in \Lambda^s(TM)$, 有

$$\mathrm{d}(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}) = \mathrm{d}\boldsymbol{\Phi}\wedge\boldsymbol{\Psi} + (-1)^r\boldsymbol{\Phi}\wedge\mathrm{d}\boldsymbol{\Psi}.$$

证明 通过直接计算, 可证明外微分运算的基本性质.

1. 线性性:

$$\begin{aligned}\mathrm{d}(\alpha\boldsymbol{\Phi} + \beta\boldsymbol{\Psi}) &= \mathrm{d}\left(\frac{\alpha}{r!}\Phi_{i_1\cdots i_r} + \frac{\beta}{r!}\Psi_{i_1\cdots i_r}\right)\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r} \\ &= \frac{1}{r!}\frac{\partial}{\partial x^s}\left(\alpha\Phi_{i_1\cdots i_r} + \beta\Psi_{i_1\cdots i_r}\right)\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r} \\ &= \frac{\alpha}{r!}\frac{\partial\Phi_{i_1\cdots i_r}}{\partial x^s}\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r} + \frac{\beta}{r!}\frac{\partial\Psi_{i_1\cdots i_r}}{\partial x^s}\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r} \\ &= \alpha\mathrm{d}\boldsymbol{\Phi} + \beta\mathrm{d}\boldsymbol{\Psi};\end{aligned}$$

2. Poincare 性:
$$\mathrm{d}\boldsymbol{\Phi}=\frac{1}{r!}\frac{\partial \Phi_{i_1\cdots i_r}}{\partial x^s}(\boldsymbol{x})\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r},$$
故有
$$\mathrm{d}^2\boldsymbol{\Phi}=\frac{1}{r!}\frac{\partial^2 \Phi_{i_1\cdots i_r}}{\partial x^t\partial x^s}(\boldsymbol{x})\mathrm{d}x^t\wedge\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}=0.$$
此处考虑到 $\dfrac{\partial^2 \Phi_{i_1\cdots i_r}}{\partial x^t\partial x^s}(\boldsymbol{x})$ 关于指标 t 和 s 的对称性, 以及 $\mathrm{d}x^t\wedge\mathrm{d}x^s$ 关于指标 t 和 s 的反对称性.

3. 反导性:
$$\begin{aligned}
\mathrm{d}(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi})&=\mathrm{d}\left(\frac{1}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s}\right)\\
&=\frac{1}{r!s!}\frac{\partial}{\partial x^t}\left(\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\right)(\boldsymbol{x})\mathrm{d}x^t\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s}\\
&=\frac{1}{r!s!}\left[\frac{\partial \Phi_{i_1\cdots i_r}}{\partial x^t}(\boldsymbol{x})\Psi_{j_1\cdots j_s}+\frac{\partial \Psi_{j_1\cdots j_s}}{\partial x^t}(\boldsymbol{x})\Phi_{i_1\cdots i_r}\right]\mathrm{d}x^t\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\\
&\quad\wedge\cdots\wedge\mathrm{d}x^{j_s}\\
&=\frac{1}{r!}\left[\frac{\partial \Phi_{i_1\cdots i_r}}{\partial x^t}(\boldsymbol{x})\mathrm{d}x^t\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\right]\wedge\left(\frac{1}{s!}\Psi_{j_1\cdots j_s}\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s}\right)\\
&\quad+(-1)^r\left(\frac{1}{r!}\Phi_{i_1\cdots i_r}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\right)\wedge\frac{1}{s!}\left[\frac{\partial \Psi_{j_1\cdots j_s}}{\partial x^t}(\boldsymbol{x})\mathrm{d}x^t\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s}\right]\\
&=\mathrm{d}\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}+(-1)^r\boldsymbol{\Phi}\wedge\mathrm{d}\boldsymbol{\Psi}.
\end{aligned}$$
□

定理 1.2 (外微分的内蕴形式)**.** 对 $\forall\boldsymbol{\Phi}\in\Lambda^r(TM)$, 有
$$\mathrm{d}\boldsymbol{\Phi}=(r+1)\mathscr{A}(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi}).$$

证明 考虑到 Christoffel 符号关于协变指标的对称性, 可有
$$\begin{aligned}
\mathrm{d}\boldsymbol{\Phi}&=\frac{1}{r!}\frac{\partial \Phi_{i_1\cdots i_r}}{\partial x^s}(\boldsymbol{x})\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\\
&=\frac{1}{r!}\left(\frac{\partial \Phi_{i_1\cdots i_r}}{\partial x^s}(\boldsymbol{x})-\Gamma_{si_1}^t\Phi_{ti_2\cdots i_r}-\cdots\Gamma_{si_r}^t\Phi_{i_1\cdots i_{r-1}t}\right)\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\\
&=\frac{1}{r!}\nabla_s\Phi_{i_1\cdots i_r}\mathrm{d}x^s\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\\
&=\frac{(r+1)!}{r!}\nabla_s\Phi_{i_1\cdots i_r}\mathscr{A}(\mathrm{d}x^s\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r})\\
&=(r+1)\mathscr{A}(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi}).
\end{aligned}$$
□

1.2 里积运算

定义 1.2 (里积运算 (Interior product))**.** 里积运算 $i_{\boldsymbol{u}}\boldsymbol{\Phi}$ 定义为
$$i_{\boldsymbol{u}}\boldsymbol{\Phi}:\mathscr{T}^r(\mathbb{R}^m)\ni\boldsymbol{\Phi}\mapsto i_{\boldsymbol{u}}\boldsymbol{\Phi}\in\mathscr{T}^{r-1}(\mathbb{R}^m),$$
此处
$$i_{\boldsymbol{u}}\boldsymbol{\Phi}(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\triangleq\boldsymbol{\Phi}(\boldsymbol{u},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\in\mathbb{R},\quad\forall\,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r\in\mathbb{R}^m.$$

性质 1.3 (里积运算的基本性质 1)**.** 里积运算的基本性质可归纳如下.

1. $i_{\boldsymbol{u}}(\alpha\boldsymbol{\Phi}+\beta\boldsymbol{\Psi})=\alpha i_{\boldsymbol{u}}\boldsymbol{\Phi}+\beta i_{\boldsymbol{u}}\boldsymbol{\Psi},\quad \forall\,\alpha,\beta\in\mathbb{R},\forall\,\boldsymbol{\Phi},\boldsymbol{\Psi}\in\mathscr{T}^r(\mathbb{R}^m)$;

2. $i_{\alpha\boldsymbol{u}+\beta\boldsymbol{v}}=\alpha i_{\boldsymbol{u}}+\beta i_{\boldsymbol{v}}$.

证明 可基于里积运算的定义, 证明其基本性质 1.

1. 根据定义, 有

$$\begin{aligned}i_{\boldsymbol{u}}(\alpha\boldsymbol{\Phi}+\beta\boldsymbol{\Psi})(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)&\triangleq(\alpha\boldsymbol{\Phi}+\beta\boldsymbol{\Psi})(\boldsymbol{u},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\\&=\alpha\boldsymbol{\Phi}(\boldsymbol{u},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)+\beta\boldsymbol{\Psi}(\boldsymbol{u},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\\&=(\alpha i_{\boldsymbol{u}}\boldsymbol{\Phi}+\beta i_{\boldsymbol{u}}\boldsymbol{\Psi})(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r),\end{aligned}$$

   即有 $i_{\boldsymbol{u}}(\alpha\boldsymbol{\Phi}+\beta\boldsymbol{\Psi})=\alpha i_{\boldsymbol{u}}\boldsymbol{\Phi}+\beta i_{\boldsymbol{u}}\boldsymbol{\Psi}$.

2. 根据定义, 有

$$\begin{aligned}i_{\alpha\boldsymbol{u}+\beta\boldsymbol{v}}\boldsymbol{\Phi}(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)&\triangleq\boldsymbol{\Phi}(\alpha\boldsymbol{u}+\beta\boldsymbol{v},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\\&=\alpha\boldsymbol{\Phi}(\boldsymbol{u},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)+\beta\boldsymbol{\Phi}(\boldsymbol{v},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\\&=(\alpha i_{\boldsymbol{u}}\boldsymbol{\Phi}+\beta i_{\boldsymbol{v}}\boldsymbol{\Phi})(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r),\end{aligned}$$

   即有 $i_{\alpha\boldsymbol{u}+\beta\boldsymbol{v}}\boldsymbol{\Phi}=\alpha i_{\boldsymbol{u}}\boldsymbol{\Phi}+\beta i_{\boldsymbol{v}}\boldsymbol{\Phi}$, 亦即有

$$i_{\alpha\boldsymbol{u}+\beta\boldsymbol{v}}=\alpha i_{\boldsymbol{u}}+\beta i_{\boldsymbol{v}}.$$

至此证毕. □

性质 1.4 (里积运算的基本性质 2)**.**

1. $i_{\boldsymbol{u}}(\boldsymbol{\theta}_1\wedge\cdots\wedge\boldsymbol{\theta}_r)=\sum_{i=1}^{r}(-1)^{i+1}(i_{\boldsymbol{u}}\boldsymbol{\theta}_i)\boldsymbol{\theta}_1\wedge\cdots\wedge\mathring{\boldsymbol{\theta}}_i\wedge\cdots\wedge\boldsymbol{\theta}_r$

   此处 $i_{\boldsymbol{u}}\boldsymbol{\theta}_i=\boldsymbol{\theta}_i(\boldsymbol{u}),\ \forall\,\boldsymbol{\theta}_1,\cdots,\boldsymbol{\theta}_r\in T^*M,\ \boldsymbol{u}\in TM$, 其中 $\boldsymbol{\theta}_1\wedge\cdots\wedge\mathring{\boldsymbol{\theta}}_i\wedge\cdots\wedge\boldsymbol{\theta}_r$ 表示在作用过程中去掉带圈的项.

2. 反导性: 对 $\forall\,\boldsymbol{\Phi}\in\Lambda^r(TM),\forall\,\boldsymbol{\Psi}\in\Lambda^s(TM)$, 有

$$i_{\boldsymbol{u}}(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi})=i_{\boldsymbol{u}}\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}+(-1)^r\boldsymbol{\Phi}\wedge i_{\boldsymbol{u}}\boldsymbol{\Psi}.$$

证明 可基于里积运算的定义以及基本性质 1, 证明其基本性质 2.

1. 根据定义, 有

$$
\begin{aligned}
i_{\boldsymbol{u}}(\boldsymbol{\theta}_1\wedge\cdots\wedge\boldsymbol{\theta}_r)(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r) &\triangleq \boldsymbol{\theta}_1\wedge\cdots\wedge\boldsymbol{\theta}_r(\boldsymbol{u},\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r)\\
&= \begin{vmatrix}
\boldsymbol{\theta}_1(\boldsymbol{u}) & \boldsymbol{\theta}_1(\boldsymbol{v}_2) & \cdots & \boldsymbol{\theta}_1(\boldsymbol{v}_r)\\
\vdots & \vdots & & \vdots\\
\boldsymbol{\theta}_i(\boldsymbol{u}) & \boldsymbol{\theta}_i(\boldsymbol{v}_2) & \cdots & \boldsymbol{\theta}_i(\boldsymbol{v}_r)\\
\vdots & \vdots & & \vdots\\
\boldsymbol{\theta}_r(\boldsymbol{u}) & \boldsymbol{\theta}_r(\boldsymbol{v}_2) & \cdots & \boldsymbol{\theta}_r(\boldsymbol{v}_r)
\end{vmatrix}\\
&= \sum_{i=1}^{r}(-1)^{i+1}\boldsymbol{\theta}_i(\boldsymbol{u})(\boldsymbol{\theta}_1\wedge\cdots\wedge\overset{\circ}{\boldsymbol{\theta}}_i\wedge\cdots\wedge\boldsymbol{\theta}_r)(\boldsymbol{v}_2,\cdots,\boldsymbol{v}_r),
\end{aligned}
$$

即有

$$
i_{\boldsymbol{u}}(\boldsymbol{\theta}_1\wedge\cdots\wedge\boldsymbol{\theta}_r)=\sum_{i=1}^{r}(-1)^{i+1}(i_{\boldsymbol{u}}\boldsymbol{\theta}_i)\boldsymbol{\theta}_1\wedge\cdots\wedge\overset{\circ}{\boldsymbol{\theta}}_i\wedge\cdots\wedge\boldsymbol{\theta}_r.
$$

2. 基于性质 (1), 计算

$$
\begin{aligned}
i_{\boldsymbol{u}}(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}) &= i_{\boldsymbol{u}}\left(\frac{1}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s}\right)\\
&= \frac{1}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}i_{\boldsymbol{u}}(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s})\\
&= \frac{1}{r!s!}\Phi_{i_1\cdots i_r}\Psi_{j_1\cdots j_s}\Bigg[\sum_{p=1}^{r}(-1)^{p+1}(i_{\boldsymbol{u}}\mathrm{d}x^{i_p})\mathrm{d}x^{i_1}\wedge\cdots\wedge\overset{\circ}{\mathrm{d}x^{i_p}}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\wedge\cdots\\
&\qquad\wedge\mathrm{d}x^{j_s}+\sum_{q=1}^{s}(-1)^{r+q+1}(i_{\boldsymbol{u}}\mathrm{d}x^{j_q})\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\overset{\circ}{\mathrm{d}x^{j_q}}\wedge\cdots\\
&\qquad\wedge\mathrm{d}x^{j_s}\Bigg]\\
&= \left[\frac{1}{r!}\Phi_{i_1\cdots i_r}\sum_{p=1}^{r}(-1)^{p+1}(i_{\boldsymbol{u}}\mathrm{d}x^{i_p})\mathrm{d}x^{i_1}\wedge\cdots\wedge\overset{\circ}{\mathrm{d}x^{i_p}}\wedge\cdots\wedge\mathrm{d}x^{i_r}\right]\\
&\qquad\wedge\left(\frac{1}{s!}\Psi_{j_1\cdots j_s}\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_s}\right)+(-1)^r\left(\frac{1}{r!}\Phi_{i_1\cdots i_r}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\right)\\
&\qquad\wedge\left[\frac{1}{s!}\Psi_{j_1\cdots j_s}\sum_{q=1}^{s}(-1)^{q+1}(i_{\boldsymbol{u}}\mathrm{d}x^{j_q})\mathrm{d}x^{j_1}\wedge\cdots\wedge\overset{\circ}{\mathrm{d}x^{j_q}}\wedge\cdots\wedge\mathrm{d}x^{j_s}\right]\\
&= i_{\boldsymbol{u}}\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}+(-1)^r\boldsymbol{\Phi}\wedge i_{\boldsymbol{u}}\boldsymbol{\Psi}.
\end{aligned}
$$

□

定理 1.5 (外微分计算式)**.** 对 $\forall\boldsymbol{\Phi}\in\Lambda^r(TM)$, 有

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{r+1}) &= \sum_{i=1}^{r+1}(-1)^{i+1}\boldsymbol{u}_i(\boldsymbol{\Phi}(\boldsymbol{u}_1,\cdots,\overset{\circ}{\boldsymbol{u}}_i,\cdots,\boldsymbol{u}_{r+1}))\\
&\quad+\sum_{1\leqslant i<j\leqslant r+1}(-1)^{i+j}\boldsymbol{\Phi}([\boldsymbol{u}_i,\boldsymbol{u}_j],\boldsymbol{u}_1,\cdots,\overset{\circ}{\boldsymbol{u}}_i,\cdots,\overset{\circ}{\boldsymbol{u}}_j,\cdots,\boldsymbol{u}_{r+1}).
\end{aligned}
$$

证明 利用数学归纳法, 设对 $r-1$ 阶反对称张量成立关系式, 以下考虑 $\boldsymbol{\Phi}\in\Lambda^r(TM)$. 计算

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{r+1}) &= (i_{\boldsymbol{u}_1}\circ\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{u}_2,\cdots,\boldsymbol{u}_{r+1})\\
&= L_{\boldsymbol{u}_1}\boldsymbol{\Phi}(\boldsymbol{u}_2,\cdots,\boldsymbol{u}_{r+1})-(\mathrm{d}\circ i_{\boldsymbol{u}_1}\boldsymbol{\Phi})(\boldsymbol{u}_2,\cdots,\boldsymbol{u}_{r+1})\\
&= \boldsymbol{u}_1(\boldsymbol{\Phi}(\boldsymbol{u}_2,\cdots,\boldsymbol{u}_{r+1}))-\sum_{i=2}^{r+1}\boldsymbol{\Phi}(\boldsymbol{u}_2,\cdots,[\boldsymbol{u}_1,\boldsymbol{u}_i],\cdots,\boldsymbol{u}_{r+1})\\
&\quad -\sum_{i=2}^{r+1}(-1)^i\boldsymbol{u}_i\left(i_{\boldsymbol{u}_1}\boldsymbol{\Phi}(\boldsymbol{u}_2,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\boldsymbol{u}_{r+1})\right)\\
&\quad -\sum_{2\leqslant i<j\leqslant r+1}(-1)^{i+j}i_{\boldsymbol{u}_1}\boldsymbol{\Phi}\left([\boldsymbol{u}_i,\boldsymbol{u}_j],\boldsymbol{u}_2,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\mathring{\boldsymbol{u}}_j,\cdots,\boldsymbol{u}_{r+1}\right)\\
&= \boldsymbol{u}_1(\boldsymbol{\Phi}(\boldsymbol{u}_2,\cdots,\boldsymbol{u}_{r+1}))+\sum_{i=2}^{r+1}(-1)^{i+1}\boldsymbol{u}_i\left(\boldsymbol{\Phi}(\boldsymbol{u}_1,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\boldsymbol{u}_{r+1})\right)\\
&\quad +\sum_{i=2}^{r+1}(-1)^{i+1}\boldsymbol{\Phi}\left([\boldsymbol{u}_1,\boldsymbol{u}_i],\boldsymbol{u}_2,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\boldsymbol{u}_{r+1}\right)\\
&\quad +\sum_{2\leqslant i<j\leqslant r+1}(-1)^{i+j}\boldsymbol{\Phi}\left([\boldsymbol{u}_i,\boldsymbol{u}_j],\boldsymbol{u}_1,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\mathring{\boldsymbol{u}}_j,\cdots,\boldsymbol{u}_{r+1}\right)\\
&= \sum_{i=1}^{r+1}(-1)^{i+1}\boldsymbol{u}_i\left(\boldsymbol{\Phi}\left(\boldsymbol{u}_1,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\boldsymbol{u}_{r+1}\right)\right)\\
&\quad +\sum_{1\leqslant i<j\leqslant r+1}(-1)^{i+j}\boldsymbol{\Phi}\left([\boldsymbol{u}_i,\boldsymbol{u}_j],\boldsymbol{u}_1,\cdots,\mathring{\boldsymbol{u}}_i,\cdots,\mathring{\boldsymbol{u}}_j,\cdots,\boldsymbol{u}_{r+1}\right).
\end{aligned}
$$

考虑 $\omega\in\Lambda^1(TM)$, 则有

$$
\mathrm{d}\omega(\boldsymbol{X},\boldsymbol{Y})=\boldsymbol{X}(\omega(\boldsymbol{Y}))-\boldsymbol{Y}(\omega(\boldsymbol{X}))-\omega([\boldsymbol{X},\boldsymbol{Y}]),\quad \forall\,\boldsymbol{X},\boldsymbol{Y}\in\mathscr{C}^\infty(TM).
$$

此式亦可通过直接计算获得, 如下所示:

$$
\begin{aligned}
\mathrm{d}\omega(\boldsymbol{X},\boldsymbol{Y})=\mathrm{d}(\omega_i\mathrm{d}x^i)&=\frac{\partial\omega_i}{\partial x^j}(\boldsymbol{x})\mathrm{d}x^j\wedge\mathrm{d}x^i(\boldsymbol{X},\boldsymbol{Y})\\
&=\frac{\partial\omega_i}{\partial x^j}(\boldsymbol{x})(\mathrm{d}x^j\otimes\mathrm{d}x^i-\mathrm{d}x^i\otimes\mathrm{d}x^j)(\boldsymbol{X},\boldsymbol{Y})\\
&=\frac{\partial\omega_i}{\partial x^j}(\boldsymbol{x})(X^jY^i-X^iY^j).
\end{aligned}
$$

等式右端为

$$
\begin{aligned}
&\boldsymbol{X}(\omega(\boldsymbol{Y}))-\boldsymbol{Y}(\omega(\boldsymbol{X}))-\omega([\boldsymbol{X},\boldsymbol{Y}])\\
&\quad=\boldsymbol{X}(\omega_iY^i)-\boldsymbol{Y}(\omega_iX^i)-\omega\left(\left[X^i\frac{\partial}{\partial x^i},Y^j\frac{\partial}{\partial x^j}\right]\right)\\
&\quad=X^k\frac{\partial}{\partial x^k}(\omega_iY^i)(\boldsymbol{x})-Y^k\frac{\partial}{\partial x^k}(\omega_iX^i)(\boldsymbol{x})-\omega\left(X^i\frac{\partial Y^j}{\partial x^i}(\boldsymbol{x})\frac{\partial}{\partial x^j}-Y^j\frac{\partial X^i}{\partial x^j}(\boldsymbol{x})\frac{\partial}{\partial x^i}\right)\\
&\quad=X^k\left(\frac{\partial\omega_i}{\partial x^k}Y^i+\omega_i\frac{\partial Y^i}{\partial x^k}\right)(\boldsymbol{x})-Y^k\left(\frac{\partial\omega_i}{\partial x^k}X^i+\omega_i\frac{\partial X^i}{\partial x^k}\right)(\boldsymbol{x})\\
&\quad\quad-\omega_i\left(X^j\frac{\partial Y^i}{\partial x^j}-Y^j\frac{\partial X^i}{\partial x^j}\right)\\
&\quad=\frac{\partial\omega_i}{\partial x^k}(\boldsymbol{x})(X^kY^i-X^iY^k).
\end{aligned}
$$

□

2 应用事例

3 建立路径