## 1

计算行列式$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \cdots & \cdots & \cdots & \cdots \\ a & a & \cdots & x \end{vmatrix}$的值.

解: 将第一行乘以-1分别加到其余各行,得

$$D_n = \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix},$$

再将各列都加到第一列上, 得

$$D_n = \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & x & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix} = [x+(n-1)a](x-a)^{n-1}.$$

□

## 2

证明

$$D_n = \begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1 \end{vmatrix} = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n.$$

证明: 用数学归纳法证明.

当$n=2$时, $D_2 = \begin{vmatrix} x & -1 \\ a_2 & x+a_1 \end{vmatrix} = x^2 + a_1 x + a_2$, 命题成立.

假设对于$(n-1)$阶行列式命题成立, 即

$$D_{n-1} = x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-2}x + a_{n-1},$$

则$D_n$按第一列展开, 有

$$
\begin{aligned}
D_n &= xD_{n-1} + a_n(-1)^{n+1} \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & x & -1 \end{vmatrix} \\
&= xD_{n-1} + a_n \\
&= x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.
\end{aligned}
$$

因此, 对于$n$阶行列式命题成立. □

## 3

计算行列式$D_{2n} = \begin{vmatrix} a_n & & & & & b_n \\ & \ddots & & & \iddots & \\ & & a_1 & b_1 & & \\ & & c_1 & d_1 & & \\ & \iddots & & & \ddots & \\ c_n & & & & & d_n \end{vmatrix}$的值.

解:

$$D_{2n} = \begin{vmatrix} a_n & & & & & b_n \\ & \ddots & & & \iddots & \\ & & a_1 & b_1 & & \\ & & c_1 & d_1 & & \\ & \iddots & & & \ddots & \\ c_n & & & & & d_n \end{vmatrix} \text{(按第一行展开)}$$

$$= a_n \begin{vmatrix} a_{n-1} & & & & & b_{n-1} & 0 \\ & \ddots & & & \iddots & & \\ & & a_1 & b_1 & & & \\ & & c_1 & d_1 & & & \\ & \iddots & & & \ddots & & \\ c_{n-1} & & & & & d_{n-1} & 0 \\ 0 & & & & & 0 & d_n \end{vmatrix}$$

$$+ (-1)^{2n+1} b_n \begin{vmatrix} 0 & a_{n-1} & & & & & b_{n-1} \\ & & \ddots & & & \iddots & \\ & & & a_1 & b_1 & & \\ & & & c_1 & d_1 & & \\ & & \ddots & & & \iddots & \\ & c_{n-1} & & & & & d_{n-1} \\ c_n & & & & & & 0 \end{vmatrix}.$$

再按最后一行展开得递推公式

$$D_{2n} = a_n d_n D_{2n-2} - b_n c_n D_{2n-2},$$

即

$$D_{2n} = (a_n d_n - b_n c_n) D_{2n-2}.$$

于是

$$D_{2n} = \prod_{i=2}^{n} (a_i d_i - b_i c_i) D_2.$$

而

$$D_2 = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} = a_1 d_1 - b_1 c_1,$$

所以

$$D_{2n} = \prod_{i=1}^{n} (a_i d_i - b_i c_i).$$

□

# 4

计算行列式$D = \det(a_{ij})$的值, 其中$a_{ij} = |i - j|$.

解: $a_{ij} = |i - j|$,

$$\begin{aligned}
D_n &= \det(a_{ij}) \\
&= \begin{vmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 1 & 0 & 1 & 2 & \cdots & n-2 \\ 2 & 1 & 0 & 1 & \cdots & n-3 \\ 3 & 2 & 1 & 0 & \cdots & n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & -1 & 1 & \cdots & 1 \\ -1 & -1 & -1 & -1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & -2 & 0 & 0 & \cdots & 0 \\ -1 & -2 & -2 & 0 & \cdots & 0 \\ -1 & -2 & -2 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n-1 & 2n-3 & 2n-4 & 2n-5 & \cdots & n-1 \end{vmatrix} \\
&= (-1)^{n-1}(n-1)2^{n-2}.
\end{aligned}$$

□

# 5

计算行列式$\begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix}$的值.

解:

$$\begin{aligned} \begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix} &= adf \begin{vmatrix} -b & c & e \\ b & -c & e \\ b & c & -e \end{vmatrix} \\ &= adfbce \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \\ &= 4abcdef. \end{aligned}$$

□