Discrete Mathematics (II) Spring 2012

Lecture 6: Truth Assignments and Valuations

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1 Overview

In this lecture, we define truth assignments and valuations in order to get rid of truth table, which is tedious. Finally, a truth valuation can be determined uniquely by a truth assignment. Sometimes, we call it the semantics of propositional logic. Correspondingly, well-defined proposition is called the syntax of propositional logic.

Given a set of propositions and a proposition, we can bind them in the point of view of truth valuation. Here we only connect them by truth valuation but syntax.

2 Assignments and Valuations

Propositional letters are the simplest propositions. There is no constraint between each other. We just define an operation, called assignment, which assigns a value *true* or *false* on every propositional letter.

Definition 1 (Assignment)**.** *A truth assignment $\mathcal{A}$ is a function that assigns to each **propositional letter** A a unique truth value $\mathcal{A}(A) \in \{T, F\}$.*

Generally, a proposition is a sequence of symbols constructed according to some rules determined in previous lecture. Whether it is true or false can not be simply assigned like propositional letters.

Consider an example in Figure 1.

Example 1. *Truth assignment of α and β and valuation of $(\alpha \vee \beta)$.*

α	β	$(\alpha \vee \beta)$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 1: Truth assignment and valuation

We can infer from the example that truth valuation of a propostion is determined by those propositions which it is based on.

We define the following term to guarantee the truth of a compound proposition.

Definition 2 (Valuation)**.** *A truth valuation $\mathcal{V}$ is a function that assigns to each* ***proposition*** *α a unique truth value $\mathcal{V}(\alpha)$ so that its value on a compound proposition is determined in accordance with the appropriate truth tables.*

Here, we should remember that truth valuation determines all propositions generated according to definition. Especially, when α is a propositional letter we have $\mathcal{V}(\alpha) = \mathcal{A}(\alpha)$ for some $\mathcal{A}$.

Generally, we have the following theorem:

Theorem 3. *Given a truth assignment $\mathcal{A}$ there is a unique truth valuation $\mathcal{V}$ such that $\mathcal{V}(\alpha) = \mathcal{A}(\alpha)$ for every propositional letter α.*

Proof. The proof can be divided into two step.

1. Construct a $\mathcal{V}$ from $\mathcal{A}$ by induction on the depth of the associated formation tree.
2. Prove the uniqueness of $\mathcal{V}$ with the same $\mathcal{A}$ by induction bottom-up.

□

which show us the relation between truth assignment and truth valuation.

We now consider a specific proposition α. There is a corollary.

Corollary 4. *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two valuations that agree on the support of α, the finite set of propositional letters used in the construction of the proposition of the proposition α, then $\mathcal{V}_1(\alpha) = \mathcal{V}_2(\alpha)$.*

Given a proposition, there is a case that it is always true whatever the truth valuation is.

Definition 5. *A proposition σ of propositional logic is said to be valid if for any valuation $\mathcal{V}, \mathcal{V}(\sigma) = T$. Such a proposition is also called a tautology.*

Example 2. *$\alpha \vee \neg\alpha$ is a tautology.*

Solution:

α	$\neg\alpha$	$\alpha \vee \neg\alpha$
T	F	T
F	T	T

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

α	β	$\neg\alpha$	$\neg\alpha \vee \beta$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	T	T

Figure 2: Logically equivalent propositions

□

Consider the following example.

Example 3. *$\alpha \to \beta \equiv \neg\alpha \vee \beta$.*

Proof. Prove by truth table in Figure 2. □

Although, they have different formation tree. But they are the same if they are only characterized by truth valuation.

Definition 6. *Two proposition α and β such that, for every valuation $\mathcal{V}, \mathcal{V}(\alpha) = \mathcal{V}(\beta)$ are called logically equivalent. We denote this by $\alpha \equiv \beta$.*

3 Consequence

In practice, we often mention a pattern that a result can be inferred from some facts. We now consider this pattern from the point of view of semantics.

Definition 7. *Let Σ be a (possibly infinite) set of propositions. We say that σ is a consequence of Σ (and write as $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,*

$$(\mathcal{V}(\tau) = T \text{ for all } \tau \in \Sigma) \Rightarrow \mathcal{V}(\sigma) = T.$$

Example 4.
1. *Let $\Sigma = \{A, \neg A \vee B\}$, we have $\Sigma \models B$.*
2. *Let $\Sigma = \{A, \neg A \vee B, C\}$, we have $\Sigma \models B$.*
3. *Let $\Sigma = \{\neg A \vee B\}$, we have $\Sigma \not\models B$.*

Definition 8. *We say that a valuation $\mathcal{V}$ is a model of Σ if $\mathcal{V}(\sigma) = T$ for every $\sigma \in \Sigma$. We denote by $\mathcal{M}(\Sigma)$ the set of all models of Σ.*

Example 5. *Let $\Sigma = \{A, \neg A \vee B\}$, we have models:*

1. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T$

2. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T, \mathcal{A}(C) = T$.
3. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T, \mathcal{A}(C) = F, \mathcal{A}(D) = F, \ldots$.

Definition 9. *We say that propositions Σ is* satisfiable *if it has some model. Otherwise it is called* invalid.

Proposition 10. *Let $\Sigma, \Sigma_1, \Sigma_2$ be sets of propositions. Let $Cn(\Sigma)$ denote the set of consequence of Σ and $Taut$ the set of tautologies.*

1. $\Sigma_1 \subseteq \Sigma_2 \Rightarrow Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$.
2. $\Sigma \subseteq Cn(\Sigma)$.
3. $Taut \subseteq Cn(\Sigma) = Cn(Cn(\Sigma))$.
4. $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \mathcal{M}(\Sigma_2) \subseteq \mathcal{M}(\Sigma_1)$.
5. $Cn(\Sigma) = \{\sigma | \mathcal{V}(\sigma) = T \text{ for all } \mathcal{V} \in \mathcal{M}(\Sigma)\}$.
6. $\sigma \in Cn(\{\sigma_1, \ldots, \sigma_n\}) \Leftrightarrow \sigma_1 \to (\sigma_2 \ldots \to (\sigma_n \to \sigma) \ldots) \in Taut$.

Theorem 11. *For any propositions φ, ψ, $\Sigma \cup \{\psi\} \models \varphi \Leftrightarrow \Sigma \models \psi \to \varphi$ holds.*

Proof. Prove by the definition of consequence. □

With this Theorem 11, we can prove result 6 in Proposition 10 by induction.

Exercises

1. Check whether the following propositions are valid or not
   (a) $(A \to B) \leftrightarrow ((\neg B) \to (\neg A))$
   (b) $A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$
2. Prove or refute each of the following assertions:
   (a) If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.
   (b) If $\Sigma \models (\alpha \wedge \beta)$, then both $\Sigma \models \alpha$ and $\Sigma \models \beta$.
3. Prove the following assertion:
   (a) $Cn(\Sigma) = Cn(Cn(\Sigma))$.
   (b) $\Sigma_1 \subset \Sigma_2 \Rightarrow \mathcal{M}(\Sigma_2) \subset \mathcal{M}(\Sigma_1)$.
   (c) $Cn(\Sigma) = \{\sigma \,|\, \mathcal{V}(\sigma) = T \text{ for all } \mathcal{V} \in \mathcal{M}(\Sigma)\}$.

(d) $\sigma \in Cn(\{\sigma_1, \ldots, \sigma_n\}) \Leftrightarrow \sigma_1 \to (\sigma_2 \ldots \to (\sigma_n \to \sigma)\ldots) \in Taut$.

4. Suppose we have two assertions, where α and β both are propostions and Σ is a set of propositions:

   (a) If $\Sigma \models A$, then $\Sigma \models B$.

   (b) $\Sigma \models (A \to B)$.

   Show the relation between them. It means whether one can imply another.