2013-2014 学年第二学期—连续介质力学基础—答案

谢锡麟
复旦大学　力学与工程科学系

2014 年 12 月 7 日

问题 1 (二点形式张量场微分学).

解. 1. 首先，由于

$$\boldsymbol{\Phi}(\boldsymbol{\xi}) = \boldsymbol{\Phi}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi}))$$

故有

$$\frac{\partial}{\partial \xi^L}\boldsymbol{\Phi}(\boldsymbol{\xi}) = \frac{\partial \boldsymbol{\Phi}}{\partial \xi^L}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi})) + \frac{\partial x^l}{\partial \xi^L}(\boldsymbol{\xi})\frac{\partial \boldsymbol{\Phi}}{\partial x^l}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi}))$$

再分别考虑

$$\begin{aligned}
\frac{\partial \boldsymbol{\Phi}}{\partial \xi^L}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi})) &= \frac{\partial \Phi^{A\cdot j}_{\cdot i \cdot B}}{\partial \xi^L}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B + \Phi^{A\cdot j}_{\cdot i \cdot B}\frac{\partial \boldsymbol{G}_A}{\partial \xi^L} \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&\quad + \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \frac{\partial \boldsymbol{G}^B}{\partial \xi^L} \\
&= \frac{\partial \Phi^{A\cdot j}_{\cdot i \cdot B}}{\partial \xi^L}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B + \Phi^{A\cdot j}_{\cdot i \cdot B}\Gamma^S_{LA}\boldsymbol{G}_S \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&\quad - \Phi^{A\cdot j}_{\cdot i \cdot B}\Gamma^B_{LS}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^S \\
&= \left(\frac{\partial \Phi^{A\cdot j}_{\cdot i \cdot B}}{\partial \xi^L} + \Phi^{S\cdot j}_{\cdot i \cdot B}\Gamma^A_{LS} - \Phi^{A\cdot j}_{\cdot i \cdot S}\Gamma^S_{LB}\right)\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&= \nabla_L \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
\frac{\partial \boldsymbol{\Phi}}{\partial x^l}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi})) &= \frac{\partial \Phi^{A\cdot j}_{\cdot i \cdot B}}{\partial x^l}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B + \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \frac{\partial \boldsymbol{g}^i}{\partial x^l} \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&\quad + \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \frac{\partial \boldsymbol{g}_j}{\partial x^l} \otimes \boldsymbol{G}^B \\
&= \frac{\partial \Phi^{A\cdot j}_{\cdot i \cdot B}}{\partial x^l}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B - \Phi^{A\cdot j}_{\cdot i \cdot B}\Gamma^i_{ls}\boldsymbol{G}_A \otimes \boldsymbol{g}^s \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&\quad + \Phi^{A\cdot j}_{\cdot i \cdot B}\Gamma^s_{lj}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_s \otimes \boldsymbol{G}^B \\
&= \left(\frac{\partial \Phi^{A\cdot j}_{\cdot i \cdot B}}{\partial x^l} - \Phi^{A\cdot j}_{\cdot s \cdot B}\Gamma^s_{li} + \Phi^{A\cdot s}_{\cdot i \cdot B}\Gamma^j_{ls}\right)\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&= \nabla_l \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B
\end{aligned}$$

所以有

$$
\begin{aligned}
\frac{\partial}{\partial \xi^L}\boldsymbol{\Phi}(\boldsymbol{\xi}) &= \frac{\partial \boldsymbol{\Phi}}{\partial \xi^L}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi})) + \frac{\partial x^l}{\partial \xi^L}(\boldsymbol{\xi})\frac{\partial \boldsymbol{\Phi}}{\partial x^l}(\boldsymbol{\xi}, \boldsymbol{x}(\boldsymbol{\xi})) \\
&= \nabla_L \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B + \frac{\partial x^l}{\partial \xi^L}(\boldsymbol{\xi})\nabla_l \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \\
&= \square_L \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B
\end{aligned}
$$

2. 将相应的基转换系数记作

$$
\begin{cases}
\boldsymbol{g}_i(\boldsymbol{x}) = C_i^{(j)}\boldsymbol{g}_{(j)}(\boldsymbol{x}) \\
\boldsymbol{g}^i(\boldsymbol{x}) = C_{(j)}^i \boldsymbol{g}^{(j)}(\boldsymbol{x})
\end{cases}
, \quad
\begin{cases}
\boldsymbol{G}_A(\boldsymbol{\xi}) = C_A^{(B)}\boldsymbol{G}_{(B)}(\boldsymbol{\xi}) \\
\boldsymbol{G}^A(\boldsymbol{\xi}) = C_{(B)}^A \boldsymbol{G}^{(B)}(\boldsymbol{\xi})
\end{cases}
$$

其中的系数分别满足

$$
C_i^{(j)} C_{(j)}^k = \delta_i^k, \quad C_A^{(B)} C_{(B)}^D = \delta_A^D
$$

于是可有

$$
\begin{aligned}
\boldsymbol{\Phi} \otimes \mathring{\square} = \frac{\partial \boldsymbol{\Phi}}{\partial \xi^L}(\boldsymbol{\xi}) \otimes \boldsymbol{G}^L &= \square_L \Phi^{A\cdot j}_{\cdot i \cdot B}\boldsymbol{G}_A \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{G}^B \otimes \boldsymbol{G}^L \\
&=: \square_{(S)} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)} \boldsymbol{G}_{(D)} \otimes \boldsymbol{g}^{(k)} \otimes \boldsymbol{g}_{(m)} \otimes \boldsymbol{G}^{(E)} \otimes \boldsymbol{G}^{(S)}
\end{aligned}
$$

所以

$$
\begin{aligned}
\square_{(S)} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)} &= \square_L \Phi^{A\cdot j}_{\cdot i \cdot B} C_A^{(D)} C_{(k)}^i C_j^{(m)} C_{(E)}^B C_{(S)}^L \\
&= \left(\nabla_L \Phi^{A\cdot j}_{\cdot i \cdot B} + \frac{\partial x^l}{\partial \xi^L}(\boldsymbol{\xi})\nabla_l \Phi^{A\cdot j}_{\cdot i \cdot B}\right) C_A^{(D)} C_{(k)}^i C_j^{(m)} C_{(E)}^B C_{(S)}^L
\end{aligned}
$$

下面分别考虑

$$
\begin{aligned}
\nabla_L \Phi^{A\cdot j}_{\cdot i \cdot B} C_A^{(D)} C_{(k)}^i C_j^{(m)} C_{(E)}^B C_{(S)}^L &= \nabla_{(S)} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)} \\
\frac{\partial x^l}{\partial \xi^L}\nabla_l \Phi^{A\cdot j}_{\cdot i \cdot B} C_A^{(D)} C_{(k)}^i C_j^{(m)} C_{(E)}^B C_{(S)}^L &= \frac{\partial x^l}{\partial \xi^L}\nabla_s \Phi^{A\cdot j}_{\cdot i \cdot B} C_A^{(D)} C_{(k)}^i C_j^{(m)} C_{(E)}^B C_{(S)}^L \delta_l^s \\
&= \frac{\partial x^l}{\partial \xi^L}\nabla_s \Phi^{A\cdot j}_{\cdot i \cdot B} C_A^{(D)} C_{(k)}^i C_j^{(m)} C_{(E)}^B C_{(S)}^L C_{(n)}^s C_l^{(n)} \\
&= \frac{\partial x^l}{\partial \xi^L} C_{(S)}^L C_l^{(n)} \nabla_{(n)} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)} \\
&=: \frac{\partial x^{(n)}}{\partial \xi^{(S)}} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)}
\end{aligned}
$$

其中

$$
\frac{\partial x^{(n)}}{\partial \xi^{(S)}} = \frac{\partial x^l}{\partial \xi^L} C_{(S)}^L C_l^{(n)}
$$

综上，有

$$
\square_{(S)} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)} = \nabla_{(S)} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)} + \frac{\partial x^{(n)}}{\partial \xi^{(S)}} \Phi^{(D)\cdot\ (m)}_{\cdot\ (k)\cdot\ (E)}
$$

3. 该方程即为

$$G^{AL}\Box_L \Phi^i{}_{\cdot A} \boldsymbol{g}_i = \boldsymbol{0}$$

其分量形式即为

$$G^{AL}\Box_L \Phi^i{}_{\cdot A} = \Box_L \Phi^{iL} = 0$$

其中

$$\Box_L \Phi^{iL} = \nabla_L \Phi^{iL} + \frac{\partial x^l}{\partial \xi^L}(\boldsymbol{\xi}) \nabla_l \Phi^{iL}$$

在此非完整基下，可有

$$\Box\langle L\rangle \Phi\langle iL\rangle = \nabla\langle L\rangle \Phi\langle iL\rangle + \frac{\partial x^{(l)}}{\partial \xi^{(L)}} \nabla\langle l\rangle \Phi\langle iL\rangle$$

其中

$$\begin{aligned}
\nabla\langle L\rangle \Phi\langle iL\rangle &= \partial\langle L\rangle \Phi\langle iL\rangle + \Gamma\langle LSL\rangle \Phi\langle iS\rangle \\
&= \sum_{L=1}^{3} \frac{1}{\sqrt{G_{LL}}} \frac{\partial}{\partial \xi^L} \Phi\langle iL\rangle + \Gamma\langle LSL\rangle \Phi\langle iS\rangle \\
\nabla\langle l\rangle \Phi\langle iL\rangle &= \partial\langle l\rangle \Phi\langle iL\rangle + \Gamma\langle lsi\rangle \Phi\langle iL\rangle \\
&= \frac{1}{\sqrt{g_{ll}}} \frac{\partial}{\partial x^l} \Phi\langle iL\rangle + \Gamma\langle lsi\rangle \Phi\langle iL\rangle \\
\frac{\partial x^{(l)}}{\partial \xi^{(L)}} &= \frac{\partial x^k}{\partial \xi^D} C_k^{(l)} C_{(L)}^D = \frac{\sqrt{g_{ll}}}{\sqrt{G_{LL}}} \frac{\partial x^l}{\partial \xi^L}
\end{aligned}$$

球坐标系满足

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta$$
$$\Gamma\langle 212\rangle = -\Gamma\langle 221\rangle = \frac{1}{r}, \quad \Gamma\langle 313\rangle = -\Gamma\langle 331\rangle = \frac{1}{r}, \quad \Gamma\langle 323\rangle = -\Gamma\langle 332\rangle = \frac{\cot\theta}{r}$$

其余皆为零，而对柱坐标系则有

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1$$
$$\Gamma\langle 212\rangle = -\Gamma\langle 221\rangle = \frac{1}{r}$$

其余皆为零。将上述数据代入，即可得最终结论。

问题 2 (体积形态连续介质的有限变形理论)**.**

解. 1. 例如边界可变形的钝体绕流问题。

2. 速度

$$\boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}(\boldsymbol{x}(\boldsymbol{\xi}, t), t) = \frac{\partial \boldsymbol{X}}{\partial x^i}(\boldsymbol{x}, t) \frac{\partial x^i}{\partial t}(\boldsymbol{\xi}, t) + \frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{x}, t) = \dot{x}^i \boldsymbol{g}_i(\boldsymbol{x}, t) + \frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{x}, t)$$

3. 对任意张量场 $\boldsymbol{\Phi}$，其物质导数为

$$\begin{aligned}\dot{\boldsymbol{\Phi}} &= \frac{\partial \boldsymbol{\Phi}}{\partial x^i}(\boldsymbol{x},t)\frac{\partial x^i}{\partial t}(\boldsymbol{\xi},t) + \frac{\partial \boldsymbol{\Phi}}{\partial t}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{\Phi}}{\partial t}(\boldsymbol{x},t) + \dot{x}^i \frac{\partial \boldsymbol{\Phi}}{\partial x^i}(\boldsymbol{x},t) \\ &= \frac{\partial \boldsymbol{\Phi}}{\partial t}(\boldsymbol{x},t) + \left(\dot{x}^i \boldsymbol{g}_i\right) \cdot \left[\boldsymbol{g}^l \otimes \frac{\partial \boldsymbol{\Phi}}{\partial x^i}(\boldsymbol{x},t)\right] \\ &= \frac{\partial \boldsymbol{\Phi}}{\partial t}(\boldsymbol{x},t) + \left[\boldsymbol{V} - \frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{x},t)\right] \cdot \left(\Box \otimes \boldsymbol{\Phi}\right)\end{aligned}$$

4. 质量守恒方程的积分形式为

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}} \rho \mathrm{d}\tau = 0$$

根据变形梯度的定义，可有

$$\int_{\overset{t}{V}} \rho \mathrm{d}\tau = \int_{\overset{\circ}{V}} \rho |\boldsymbol{F}| \mathrm{d}\tau$$

因此即有

$$\int_{\overset{\circ}{V}} \dot{\overline{\rho|\boldsymbol{F}|}} \mathrm{d}\tau = 0$$

即可得 Lagrange 型质量守恒微分方程

$$\dot{\overline{\rho|\boldsymbol{F}|}} = 0$$

或者

$$\rho|\boldsymbol{F}| = \overset{\circ}{\rho}$$

其中 $\overset{\circ}{\rho} = \overset{\circ}{\rho}(\boldsymbol{\xi})$ 是初始物理构型中的密度分布。

5. 动量守恒的积分形式可以表示为

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}} \rho \boldsymbol{V} \mathrm{d}\tau = \oint_{\partial \overset{t}{V}} \boldsymbol{t} \cdot \boldsymbol{n} \mathrm{d}\sigma + \int_{\overset{t}{V}} \rho \boldsymbol{f}_m \mathrm{d}\tau$$

考虑左端，即

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}} \rho \boldsymbol{V} \mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{\circ}{V}} \rho \boldsymbol{V} |\boldsymbol{F}| \mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{\circ}{V}} \overset{\circ}{\rho} \boldsymbol{V} \mathrm{d}\tau = \int_{\overset{\circ}{V}} \overset{\circ}{\rho} \boldsymbol{a} \mathrm{d}\tau$$

此处利用了 (4) 中的 Lagrange 型质量守恒方程。再考虑右端第一项，根据曲面积分的计算式，可有

$$\begin{aligned}\oint_{\partial \overset{t}{V}} \boldsymbol{t} \cdot \boldsymbol{n} \mathrm{d}\sigma &= \int_{D_{\lambda\mu}} \boldsymbol{t} \cdot \left(\frac{\partial \overset{t}{\boldsymbol{\Sigma}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{\Sigma}}}{\partial \mu}\right) \mathrm{d}\lambda \mathrm{d}\mu = \int_{D_{\lambda\mu}} \boldsymbol{t} \cdot \left[|\boldsymbol{F}|\boldsymbol{F}^{-*} \cdot \left(\frac{\partial \overset{\circ}{\boldsymbol{\Sigma}}}{\partial \lambda} \times \frac{\partial \overset{\circ}{\boldsymbol{\Sigma}}}{\partial \mu}\right)\right] \mathrm{d}\lambda \mathrm{d}\mu \\ &= \oint_{\partial \overset{\circ}{V}} \left(|\boldsymbol{F}|\boldsymbol{t} \cdot \boldsymbol{F}^{-*}\right) \cdot \boldsymbol{N} \mathrm{d}\sigma = \int_{\overset{\circ}{V}} \left(|\boldsymbol{F}|\boldsymbol{t} \cdot \boldsymbol{F}^{-*}\right) \cdot \overset{\circ}{\Box} \mathrm{d}\tau\end{aligned}$$

此处利用了有向面元之间的关系和 Nanson 公式。最后考虑右端第二项，可有

$$\int_{\overset{t}{V}} \rho \boldsymbol{f}_m \mathrm{d}\tau = \int_{\overset{\circ}{V}} \rho \boldsymbol{f}_m |\boldsymbol{F}| \mathrm{d}\tau = \int_{\overset{\circ}{V}} \overset{\circ}{\rho} \boldsymbol{f}_m \mathrm{d}\tau$$

由此可得

$$\int_{\overset{\circ}{V}} \overset{\circ}{\rho} \boldsymbol{a} \mathrm{d}\tau = \int_{\overset{\circ}{V}} \left(|\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}\right) \cdot \overset{\circ}{\Box} \mathrm{d}\tau + \int_{\overset{\circ}{V}} \overset{\circ}{\rho} \boldsymbol{f}_m \mathrm{d}\tau$$

即有 Lagrange 型动量守恒微分方程

$$\overset{\circ}{\rho} \boldsymbol{a} = \left(|\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}\right) \cdot \overset{\circ}{\Box} + \overset{\circ}{\rho} \boldsymbol{f}_m$$

利用第一类 Piola-Kirchhoff 应力张量为 $\boldsymbol{\tau} = |\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}$，第二类 Piola-Kirchhoff 应力张量为 $\boldsymbol{T} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} = |\boldsymbol{F}| \boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-*}$，则 Lagrange 型动量守恒微分方程可以分别表示为

$$\begin{cases} \overset{\circ}{\rho} \boldsymbol{a} = \boldsymbol{\tau} \cdot \overset{\circ}{\Box} + \overset{\circ}{\rho} \boldsymbol{f}_m \\ \overset{\circ}{\rho} \boldsymbol{a} = \left(\boldsymbol{F} \cdot \boldsymbol{T}\right) \cdot \overset{\circ}{\Box} + \overset{\circ}{\rho} \boldsymbol{f}_m \end{cases}$$

6. 根据线积分的计算方法，可有

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\overset{t}{\Gamma}} \boldsymbol{\Phi} \mathrm{d}l &= \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \boldsymbol{\Phi} \left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3} (\lambda) \mathrm{d}\lambda = \int_a^b \left(\dot{\boldsymbol{\Phi}} \left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3} + \boldsymbol{\Phi} \overline{\left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3}}^{\boldsymbol{\cdot}} \right) \mathrm{d}\lambda \\ &= \int_a^b \left(\dot{\boldsymbol{\Phi}} \left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3} + \boldsymbol{\Phi} (\boldsymbol{\tau} \cdot \boldsymbol{D} \cdot \boldsymbol{\tau}) \left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3} \right) \mathrm{d}\lambda \\ &= \int_{\overset{t}{\Gamma}} \left[\dot{\boldsymbol{\Phi}} + \boldsymbol{\Phi} (\boldsymbol{\tau} \cdot \boldsymbol{D} \cdot \boldsymbol{\tau}) \right] \mathrm{d}l \end{aligned}$$

此处利用了

$$\overline{\left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3}}^{\boldsymbol{\cdot}} = (\boldsymbol{\tau} \cdot \boldsymbol{D} \cdot \boldsymbol{\tau}) \left| \frac{\mathrm{d}\overset{t}{\boldsymbol{X}}}{\mathrm{d}\lambda} \right|_{\mathbb{R}^3}$$

问题 3 (曲面形态连续介质的变形运动学).

解. 1. 考虑当前物理构型中的两个有向线元

$$\begin{aligned} &\boldsymbol{\Sigma}(\boldsymbol{\xi}_\Sigma + \Delta\boldsymbol{\xi}_\Sigma, t) - \boldsymbol{\Sigma}(\boldsymbol{\xi}_\Sigma, t) = \frac{\partial \boldsymbol{\Sigma}}{\partial x_\Sigma^i}(\boldsymbol{x}_\Sigma, t) \frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}(\boldsymbol{\xi}_\Sigma, t) \Delta\xi_\Sigma^A \\ &= \frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}(\boldsymbol{\xi}_\Sigma, t) \boldsymbol{g}_i(\boldsymbol{x}_\Sigma, t) \Delta\xi_\Sigma^A \\ &= \left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}(\boldsymbol{\xi}_\Sigma, t) \boldsymbol{g}_i(\boldsymbol{x}_\Sigma, t) \otimes \boldsymbol{G}^A(\boldsymbol{x}_\Sigma) \right] \cdot \left[\Delta\xi_\Sigma^B \boldsymbol{G}_B(\boldsymbol{x}_\Sigma) \right] \\ &= \left[\frac{\partial x_\Sigma^i}{\partial \xi_\Sigma^A}(\boldsymbol{\xi}_\Sigma, t) \boldsymbol{g}_i(\boldsymbol{x}_\Sigma, t) \otimes \boldsymbol{G}^A(\boldsymbol{x}_\Sigma) \right] \cdot \left[\overset{\circ}{\boldsymbol{\Sigma}}(\boldsymbol{\xi}_\Sigma + \Delta\boldsymbol{\xi}_\Sigma) - \overset{\circ}{\boldsymbol{\Sigma}}(\boldsymbol{\xi}_\Sigma) \right] \\ &= \boldsymbol{F} \cdot \left[\overset{\circ}{\boldsymbol{\Sigma}}(\boldsymbol{\xi}_\Sigma + \Delta\boldsymbol{\xi}_\Sigma) - \overset{\circ}{\boldsymbol{\Sigma}}(\boldsymbol{\xi}_\Sigma) \right] \end{aligned}$$

因此，变形梯度可理解为初始物理构型中有向线元同当前物理构型中有向线元之间的线性变换。

2. 根据 $\overset{t}{\boldsymbol{\Sigma}}(\lambda,\mu)=\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma(\boldsymbol{\xi}_\Sigma(\lambda,\mu),t),t)$，可有

$$\begin{aligned}\left(\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu)&=\left(\boldsymbol{F}\cdot\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}(\lambda,\mu)\right)\times\left(\boldsymbol{F}\cdot\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right)\\&=\left(F^{i}_{\cdot A}\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\boldsymbol{g}_i\right)\times\left(F^{j}_{\cdot B}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\boldsymbol{g}_j\right)\\&=\left(F^{i}_{\cdot A}F^{j}_{\cdot B}\right)\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\varepsilon_{ij3}\overset{t}{\boldsymbol{n}}\\&=\sqrt{g_\Sigma}\left(e_{ij3}F^{i}_{\cdot A}F^{j}_{\cdot B}\right)\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\overset{t}{\boldsymbol{n}}\\&=\sqrt{g_\Sigma}\det\left(F^{i}_{\cdot L}\right)\left(e_{AB3}\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\right)\overset{t}{\boldsymbol{n}}\\&=\frac{\sqrt{g_\Sigma}}{\sqrt{G_\Sigma}}\det\left(F^{i}_{\cdot L}\right)\left(\varepsilon_{AB3}\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\right)\overset{t}{\boldsymbol{n}}\\&=\det\boldsymbol{F}\cdot\left|\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu)\cdot\overset{t}{\boldsymbol{n}}\end{aligned}$$

3. 考虑

$$g:=\det\left(g_{ij}\right)=\sum_{s=1}^{m}\Delta_{is}g_{is},\quad\forall\, i=1,2,\cdots,m$$

此处 Δ_{is} 表示度量矩阵 $\left(g_{ij}\right)$ 的伴随矩阵第 i 行第 s 列的元素（即元素 g_{si} 的代数余子式）。由此可知，如果 g_{ij} 包含在 g 的表达式中，则有 $\Delta_{ij}\neq 0$；并且如果 g 的表达式不包含 g_{ij}，则有 $\Delta_{ij}=0$。由此可有

$$\begin{aligned}\frac{\partial g}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)&=\sum_{\text{包含}g_{ij}}\frac{\partial g}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)=\sum_{\text{包含}g_{ij}}\Delta_{ij}\frac{\partial g_{ij}}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)\\&=\sum_{p,q=1}^{m}\Delta_{pq}\frac{\partial g_{pq}}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)=gg^{pq}\frac{\partial g_{pq}}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)\end{aligned}$$

由此即有

$$\frac{1}{g}\frac{\partial g}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)=g^{ij}\frac{\partial g_{ij}}{\partial x^l_\Sigma}(\boldsymbol{x}_\Sigma)$$

4. 计算变形梯度行列式的物质导数，即

$$\dot{\overline{|\boldsymbol{F}|}}=\dot{\overline{\frac{\sqrt{g_\Sigma}}{\sqrt{G_\Sigma}}\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)}}=\frac{1}{\sqrt{G_\Sigma}}\dot{\overline{\sqrt{g_\Sigma}}}\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)+\frac{\sqrt{g_\Sigma}}{\sqrt{G_\Sigma}}\dot{\overline{\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)}}$$

其中

$$
\begin{aligned}
\dot{\overline{\sqrt{g_\Sigma}(\boldsymbol{x}_\Sigma,t)}} &= \frac{\partial\sqrt{g_\Sigma}}{\partial t}(\boldsymbol{x}_\Sigma,t)+\dot{x}^s_\Sigma\frac{\partial\sqrt{g_\Sigma}}{\partial x^s_\Sigma}(\boldsymbol{x}_\Sigma,t) \\
&= \sqrt{g_\Sigma}\left[\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial t}(\boldsymbol{x}_\Sigma,t)+\dot{x}^s_\Sigma\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial x^s_\Sigma}(\boldsymbol{x}_\Sigma,t)\right] \\
&= \sqrt{g_\Sigma}\left[\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial t}(\boldsymbol{x}_\Sigma,t)+\mathit{\Gamma}^i_{is}\dot{x}^s_\Sigma\right]
\end{aligned}
$$

其中利用了关系式

$$
\mathit{\Gamma}^i_{is}=\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial x^s_\Sigma}(\boldsymbol{x}_\Sigma,t)
$$

再考虑

$$
\begin{aligned}
\dot{\overline{\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)}} &= \sum_{\sigma\in P_m}\mathrm{sgn}\sigma\left[\dot{\overline{\frac{\partial x^1_\Sigma}{\partial\xi^{\sigma(1)}_\Sigma}}}\frac{\partial x^2_\Sigma}{\partial\xi^{\sigma(2)}_\Sigma}\cdots\frac{\partial x^m_\Sigma}{\partial\xi^{\sigma(m)}_\Sigma}+\cdots+\frac{\partial x^1_\Sigma}{\partial\xi^{\sigma(1)}_\Sigma}\cdots\frac{\partial x^{m-1}_\Sigma}{\partial\xi^{\sigma(m-1)}_\Sigma}\dot{\overline{\frac{\partial x^m_\Sigma}{\partial\xi^{\sigma(m)}_\Sigma}}}\right] \\
&= \sum_{i=1}^m\sum_{\sigma\in P_m}\mathrm{sgn}\sigma\left[\frac{\partial x^1_\Sigma}{\partial\xi^{\sigma(1)}_\Sigma}\cdots\frac{\partial\dot{x}^i_\Sigma}{\partial\xi^{\sigma(i)}_\Sigma}\cdots\frac{\partial x^m_\Sigma}{\partial\xi^{\sigma(m)}_\Sigma}\right] \\
&= \sum_{i=1}^m\frac{\partial\dot{x}^i_\Sigma}{\partial x^s_\Sigma}\sum_{\sigma\in P_m}\mathrm{sgn}\sigma\left[\frac{\partial x^1_\Sigma}{\partial\xi^{\sigma(1)}_\Sigma}\cdots\frac{\partial x^s_\Sigma}{\partial\xi^{\sigma(i)}_\Sigma}\cdots\frac{\partial x^m_\Sigma}{\partial\xi^{\sigma(m)}_\Sigma}\right]
\end{aligned}
$$

上式中求和结果为一行列式，只有当 $s=i$ 时行列式的值才非零，否则此行列式将有两行相同而自然为零，所以有

$$
\dot{\overline{\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)}}=\sum_{i=1}^m\frac{\partial\dot{x}^i_\Sigma}{\partial x^i_\Sigma}\sum_{\sigma\in P_m}\mathrm{sgn}\sigma\left[\frac{\partial x^1_\Sigma}{\partial\xi^{\sigma(1)}_\Sigma}\cdots\frac{\partial x^i_\Sigma}{\partial\xi^{\sigma(i)}_\Sigma}\cdots\frac{\partial x^m_\Sigma}{\partial\xi^{\sigma(m)}_\Sigma}\right]=\frac{\partial\dot{x}^i_\Sigma}{\partial x^i_\Sigma}\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)
$$

所以有

$$
\dot{\overline{|\boldsymbol{F}|}}=|\boldsymbol{F}|\left[\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial t}+\frac{\partial\dot{x}^s_\Sigma}{\partial x^s_\Sigma}+\mathit{\Gamma}^s_{sl}\dot{x}^l_\Sigma\right]=|\boldsymbol{F}|\left[\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial t}+\nabla_s\dot{x}^s_\Sigma\right]
$$

计算速度的散度

$$
\begin{aligned}
\boldsymbol{V}\cdot\overset{\Sigma}{\nabla} &= \frac{\partial}{\partial x^l_\Sigma}\left(\frac{\partial\boldsymbol{\Sigma}}{\partial t}+\dot{x}^s_\Sigma\boldsymbol{g}_s\right)\cdot\boldsymbol{g}^l=\frac{\partial\boldsymbol{g}_l}{\partial t}\cdot\boldsymbol{g}^l+\nabla_s\dot{x}^s_\Sigma=g^{lk}\boldsymbol{g}_k\cdot\frac{\partial\boldsymbol{g}_l}{\partial t}+\nabla_s\dot{x}^s_\Sigma \\
&= \frac{1}{2}g^{lk}\frac{\partial g_{lk}}{\partial t}+\nabla_s\dot{x}^s_\Sigma=\frac{1}{2}\frac{1}{g_\Sigma}\frac{\partial g_\Sigma}{\partial t}+\nabla_s\dot{x}^s_\Sigma=\frac{1}{\sqrt{g_\Sigma}}\frac{\partial\sqrt{g_\Sigma}}{\partial t}+\nabla_s\dot{x}^s_\Sigma
\end{aligned}
$$

此处利用了 (3) 中结论。综上，有

$$
\dot{\overline{|\boldsymbol{F}|}}=\theta|\boldsymbol{F}|,\quad \theta=\boldsymbol{V}\cdot\overset{\Sigma}{\nabla}
$$

5. 根据第一类曲面积分的计算方法，有

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{\Sigma}}\boldsymbol{\Phi}\mathrm{d}\sigma &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_{\lambda\mu}}\boldsymbol{\Phi}\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}\mathrm{d}\lambda\mathrm{d}\mu \\ &= \int_{D_{\lambda\mu}}\left[\dot{\boldsymbol{\Phi}}\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}+\boldsymbol{\Phi}\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}\right]\mathrm{d}\lambda\mathrm{d}\mu \\ &= \int_{D_{\lambda\mu}}\left(\dot{\boldsymbol{\Phi}}+\boldsymbol{\Phi}\theta\right)\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}\mathrm{d}\lambda\mathrm{d}\mu \\ &= \int_{\overset{t}{\Sigma}}\left(\dot{\boldsymbol{\Phi}}+\theta\boldsymbol{\Phi}\right)\mathrm{d}\sigma\end{aligned}$$

6. 质量守恒定律的积分形式为

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{\Sigma}}\rho\mathrm{d}\sigma = 0$$

根据 (5) 中结论可有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{\Sigma}}\rho\mathrm{d}\sigma = \int_{\overset{t}{\Sigma}}\left(\dot{\rho}+\theta\rho\right)\mathrm{d}\sigma = 0$$

由此可有质量守恒定律的微分形式为

$$\dot{\rho}+\theta\rho = 0$$

计算速度的散度

$$\begin{aligned}\theta = \overset{\Sigma}{\nabla}\cdot\boldsymbol{V} &= \left(\frac{\partial}{\partial x_{\Sigma}^{l}}\boldsymbol{g}^{l}\right)\cdot\left(V^{i}\boldsymbol{g}_{i}+V^{3}\boldsymbol{n}\right) \\ &= \boldsymbol{g}^{l}\cdot\left(\frac{\partial V^{i}}{\partial x_{\Sigma}^{l}}\boldsymbol{g}_{i}+V^{i}\left(\mathit{\Gamma}_{li}^{s}\boldsymbol{g}_{s}+b_{li}\boldsymbol{n}\right)\right)+\boldsymbol{g}^{l}\cdot\left(\frac{\partial V^{3}}{\partial x_{\Sigma}^{l}}\boldsymbol{n}+V^{3}\frac{\partial\boldsymbol{n}}{\partial x_{\Sigma}^{l}}\right) \\ &= \frac{\partial V^{l}}{\partial x_{\Sigma}^{l}}+\mathit{\Gamma}_{ls}^{l}V^{s}-V^{3}b_{l}^{l} = \nabla_{l}V^{l}-HV^{3}\end{aligned}$$

由此质量守恒微分方程的展开形式为

$$\frac{\partial\rho}{\partial t}+\dot{x}_{\Sigma}^{i}\frac{\partial\rho}{\partial x_{\Sigma}^{i}}+\rho(\nabla_{l}V^{l}-HV^{3}) = 0$$

7. 式中的力学-几何耦合量为平均曲率，当曲面为柱面的时候，该曲率为零。

问题 4 (曲面形态连续介质的守恒律方程).

解. 1. 固定曲面上二维不可压缩流动的连续性方程即为

$$\theta = \overset{\Sigma}{\nabla}\cdot\boldsymbol{V} = 0$$

其分量形式为

$$\nabla_{l}V^{l}-HV^{3} = 0$$

2. 三维不可压缩连续性方程为

$$\theta = \nabla \cdot \boldsymbol{V} = 0$$

其分量形式为

$$\nabla_\alpha V^\alpha = 0$$

固定曲面上的二维流动连续性方程中包含了曲面的平均曲率这一几何量，而三维不可压缩流动连续性方程中则没有这一项。

3. 根据第二类广义 Stokes 公式，可有

$$\oint_{\partial \overset{t}{\Sigma}} (\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{t} \mathrm{d}l = \int_{\overset{t}{\Sigma}} \left(\overset{\Sigma}{\nabla} \cdot \boldsymbol{t} + H\boldsymbol{n} \cdot \boldsymbol{t} \right) \mathrm{d}\sigma$$

考虑到 $\boldsymbol{n} \cdot \boldsymbol{t} = \boldsymbol{0}$，因此有

$$\oint_{\partial \overset{t}{\Sigma}} (\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{t} \mathrm{d}l = \int_{\overset{t}{\Sigma}} \overset{\Sigma}{\nabla} \cdot \boldsymbol{t} \mathrm{d}\sigma$$

将上式代入一般形式的动量守恒方程积分形式

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\overset{t}{\Sigma}} \rho \boldsymbol{V} \mathrm{d}\sigma = \oint_{\partial \overset{t}{\Sigma}} (\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{t} \mathrm{d}l + \int_{\overset{t}{\Sigma}} \rho \boldsymbol{f} \mathrm{d}\sigma$$

中，可得一般形式的动量守恒微分方程为

$$\rho \boldsymbol{a} = \overset{\Sigma}{\nabla} \cdot \boldsymbol{t} + \rho \boldsymbol{f}$$

4. 动量矩守恒的积分形式可表示为

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\overset{t}{\Sigma}} \rho \boldsymbol{V} \times \boldsymbol{\Sigma} \mathrm{d}\sigma = \oint_{\partial \overset{t}{\Sigma}} \left[(\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{t} \right] \times \boldsymbol{\Sigma} \mathrm{d}l + \int_{\overset{t}{\Sigma}} \boldsymbol{f} \times \boldsymbol{\Sigma} \mathrm{d}\sigma - \int_{\overset{t}{\Sigma}} \boldsymbol{m}_\Sigma \mathrm{d}\sigma$$

考虑到 $\left[(\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{t} \right] \times \boldsymbol{\Sigma} = (\boldsymbol{\tau} \times \boldsymbol{n}) \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma})$，可有

$$\begin{aligned}
\oint_{\partial \overset{t}{\Sigma}} \left[(\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{t} \right] \times \boldsymbol{\Sigma} \mathrm{d}l &= \oint_{\partial \overset{t}{\Sigma}} (\boldsymbol{\tau} \times \boldsymbol{n}) \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma}) \mathrm{d}l \\
&= \int_{\overset{t}{\Sigma}} \left(\overset{\Sigma}{\nabla} \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma}) + H\boldsymbol{n} \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma}) \right) \mathrm{d}\sigma \\
&= \int_{\overset{t}{\Sigma}} \left(\overset{\Sigma}{\nabla} \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma}) + (H\boldsymbol{n} \cdot \boldsymbol{t}) \times \boldsymbol{\Sigma} \right) \mathrm{d}\sigma \\
&= \int_{\overset{t}{\Sigma}} \overset{\Sigma}{\nabla} \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma}) \mathrm{d}\sigma
\end{aligned}$$

由于

$$\begin{aligned}
\overset{\Sigma}{\nabla} \cdot (\boldsymbol{t} \times \boldsymbol{\Sigma}) = \boldsymbol{g}^l \cdot \frac{\partial}{\partial x_\Sigma^l} (\boldsymbol{t} \times \boldsymbol{\Sigma}) &= \boldsymbol{g}^l \cdot \left(\frac{\partial \boldsymbol{t}}{\partial x_\Sigma^l} \times \boldsymbol{\Sigma} \right) + \boldsymbol{g}^l \cdot \left(\boldsymbol{t} \times \frac{\partial \boldsymbol{\Sigma}}{\partial x_\Sigma^l} \right) \\
&= \left(\overset{\Sigma}{\nabla} \cdot \boldsymbol{t} \right) \times \boldsymbol{\Sigma} + \boldsymbol{g}^l \cdot (\boldsymbol{t} \times \boldsymbol{g}_l)
\end{aligned}$$

由此动量矩守恒方程的微分形式为

$$\begin{aligned}\rho\boldsymbol{a}\times\boldsymbol{\Sigma} &= \left(\overset{\Sigma}{\nabla}\cdot\boldsymbol{t}\right)\times\boldsymbol{\Sigma}+\boldsymbol{g}^{l}\cdot(\boldsymbol{t}\times\boldsymbol{g}_{l})+\boldsymbol{f}\times\boldsymbol{\Sigma}-\boldsymbol{m}_{\Sigma}\\ &= \left(\overset{\Sigma}{\nabla}\cdot\boldsymbol{t}+\boldsymbol{f}\right)\times\boldsymbol{\Sigma}+\boldsymbol{g}^{l}\cdot(\boldsymbol{t}\times\boldsymbol{g}_{l})-\boldsymbol{m}_{\Sigma}\end{aligned}$$

根据 (3) 中动量守恒微分方程，可有动量矩守恒的表达形式应为

$$\boldsymbol{g}^{l}\cdot(\boldsymbol{t}\times\boldsymbol{g}_{l})=\boldsymbol{m}_{\Sigma}$$