复旦大学力学与工程科学系

2009 ～ 2010 学年第一学期期末考试试卷答案

□A 卷　　□B 卷

课程名称：连续介质力学基础　　课程代码：MECH130105

开课院系：力学与工程科学系　　考试形式：开卷/闭卷/课程论文

姓名：________ 学号：________ 专业：________

题 号	1	2/(1)	2/(2)	2/(3)	2/(4)	2/(5)	3/(1)	3/(2)
得 分								
题 号	3/(3)	4	5/(1)	5/(2)	5/(3)			总分
得 分								

Please do Your Possible to Accomplish the Following Problems in English.

Problem 1 (Field Analysis in general curvilinear coordinates) (10%) *Prove the following identity*

$$I \overset{\times}{\times} (\nabla \times \Phi \times \nabla) = \nabla \otimes (\Phi \cdot \nabla) + (\nabla \cdot \Phi) \otimes \nabla - \nabla \otimes \nabla (tr\Phi) - \triangle \Phi^*, \quad \forall \Phi \in \mathbf{Lin}$$

in the point of view of general curvilinear coordinates, where

$$\Phi \overset{\times}{\times} \Psi = (\Phi^i_{\cdot j}\, g_i \otimes g^j) \overset{\times}{\times} (\Psi^p_{\cdot q}\, g_p \otimes g^q) \triangleq \Phi^i_{\cdot j} \Psi^p_{\cdot q}\, (g_i \times g^q) \otimes (g^j \times g_p)$$

$$\begin{aligned}
\nabla \times \Phi \times \nabla &= \left(\frac{\partial}{\partial x^p} g^p\right) \times \left(\Phi_{ij} g^i \otimes g^j\right) \times \nabla \\
&= \left(\nabla_p \Phi_{ij} \in^{pis} g_s \otimes g^j\right) \times \left(\frac{\partial}{\partial x^q} g^q\right) \\
&= \nabla_q \left(\nabla_p \Phi_i j\right) \epsilon^{pis} \epsilon^{jqt} g_s \otimes g_t
\end{aligned}$$

$$
\begin{aligned}
I^{\times}_{\times}\left(\nabla\times\Phi\times\nabla\right) &= \left(g^{mn}g_m\otimes g_n\right)\,{}^{\times}_{\times}\left(\nabla_q\left(\nabla_p\Phi_{ij}\right)\epsilon^{pis}\epsilon^{jqt}g_s\otimes g_t\right)\\
&= g^{mn}\nabla_q\left(\nabla_p\Phi_{ij}\right)\epsilon^{pis}\epsilon^{jqt}\epsilon_{mtk}\epsilon_{nsl}g^k\otimes g^l\\
&= g^{mn}\nabla_q\left(\nabla_p\Phi_{ij}\right)\left(\epsilon^{jqt}\epsilon_{mkl}\right)\left(\epsilon^{pis}\epsilon_{nls}\right)g^k\otimes g^l\\
&= g^{mn}\nabla_q\left(\nabla_p\Phi_{ij}\right)\left(\delta^j_m\delta^q_k-\delta^q_m\delta^j_k\right)\left(\delta^p_n\delta^i_l-\delta^i_n\delta^p_l\right)g^k\otimes g^l\\
&= g^{mn}\nabla_q\left(\nabla_p\Phi_{ij}\right)\left(\delta^j_m\delta^q_k\delta^p_n\delta^i_l-\delta^j_n\delta^q_k\delta^i_n\delta^p_l-\delta^q_m\delta^j_k\delta^p_n\delta^i_l+\delta^q_m\delta^j_k\delta^i_n\delta^p_l\right)g^k\otimes g^l\\
&= g^{mn}\nabla_k\left(\nabla_n\Phi_{lm}\right)g^k\otimes g^l-g^{mn}\nabla_k\left(\nabla_l\Phi_{nm}\right)g^k\otimes g^l-g^{mn}\nabla_m\left(\nabla_n\Phi_{lk}\right)g^k\otimes g^l\\
&\quad+g^{mn}\nabla_m\left(\nabla_l\Phi_{nk}\right)g^k\otimes g^l\\
&= \nabla_k\left(\nabla_n\Phi_l{}^n\right)g^k\otimes g^l-\nabla_k\left(\nabla_l\Phi_n{}^n\right)g^k\otimes g^l-\nabla^n\left(\nabla_n\Phi_{lk}\right)g^k\otimes g^l\\
&\quad+\nabla_m\left(\nabla_l\Phi^m{}_k\right)g^k\otimes g^l
\end{aligned}
$$

$$
\begin{aligned}
\nabla\otimes\left(\Phi\cdot\nabla\right) &= \left(g^m\frac{\partial}{\partial x^m}\right)\otimes\left(\Phi_i{}^jg^i\otimes g_j\right)\cdot\left(\frac{\partial}{\partial x^n}g^n\right)\\
&= \nabla_m\left(\nabla_n\Phi_i{}^n\right)g^m\otimes g^i=\nabla_k\left(\nabla_n\Phi_l{}^n\right)g^k\otimes g^l\\
\left(\nabla\cdot\Phi\right)\otimes\nabla &= \left(g^m\frac{\partial}{\partial x^m}\right)\cdot\left(\Phi^i{}_jg_i\otimes g^j\right)\otimes\left(\frac{\partial}{\partial x^n}g^n\right)\\
&= \nabla_n\left(\nabla_m\Phi^m{}_j\right)g^j\otimes g^n=\nabla_l\left(\nabla_m\Phi^m{}_k\right)g^k\otimes g^l=\nabla_m\left(\nabla_l\Phi^m{}_k\right)g^k\otimes g^l\\
\nabla\otimes\nabla\left(tr\Phi\right) &= \nabla_k\left(\nabla_l\Phi_n{}^n\right)g^k\otimes g^l\\
\Delta\Phi^* &= \nabla^n\left(\nabla_n\Phi_{lk}\right)g^k\otimes g^l
\end{aligned}
$$

Problem 2 (the Meaning of the Gradient a Tensor Field) *The norm of the general tensor space $\mathscr{T}^r(\mathbb{R}^m)$ is defined as*

$$
|\Phi|_{\mathscr{T}^r}\triangleq\sqrt{\Phi\odot\Phi}=\sqrt{\Phi^{i_1\cdots i_r}\Phi_{i_1\cdots i_r}},\quad\forall\,\Phi\in\mathscr{T}^r(\mathbb{R}^m)
$$

1. (10%) *To prove the following conclusion*

$$
|a\otimes b\otimes c|_{\mathscr{T}^r}=|a|_{\mathbb{R}^m}\cdot|b|_{\mathbb{R}^m}\cdot|c|_{\mathbb{R}^m},\quad\forall\,a,b,c\in\mathbb{R}^m
$$

$$
\begin{aligned}
|a\otimes b\otimes c|_{\mathscr{T}^r} &= \sqrt{a\otimes b\otimes c\odot a\otimes b\otimes c}\\
&= \sqrt{a^ib^jc^ka_ib_jc_k}\\
&= \sqrt{a^ia_ib^jb_jc^kc_k}\\
&= \sqrt{a^ia_i}\cdot\sqrt{b^jb_j}\cdot\sqrt{c^kc_k}\\
&= |a|_{\mathbb{R}^m}\cdot|b|_{\mathbb{R}^m}\cdot|c|_{\mathbb{R}^m}
\end{aligned}
$$

2. (10%) *To prove one of the following relations, either Lagrangian or Eulerian, that is available for any tensor field represented in two-point form*

$$\begin{aligned} &\textit{Lagrange} \quad \Phi(x+\delta x,t) = \Phi(x,t) + (\Phi\otimes\square)(x,t)\cdot\delta x + o(|\delta x|_{\mathbb{R}^m}) \\ &\textit{Euler} \qquad \Phi(\xi+\delta\xi,t) = \Phi(\xi,t) + (\Phi\otimes\mathring{\square})(\xi,t)\cdot\delta\xi + o(|\delta\xi|_{\mathbb{R}^m}) \end{aligned}$$

where

$$(\Phi\otimes\mathring{\square})(\xi,t) \triangleq \mathring{\square}_L\Phi^i_{\cdot A}(\xi,t)\, g_i\otimes G^A\otimes G^L, \quad \mathring{\square}_L\Phi^i_{\cdot A}(\xi,t) \triangleq \mathring{\nabla}_L\Phi^i_{\cdot A}(\xi,x,t) + \frac{\partial x^l}{\partial\xi^L}(\xi,t)\nabla_l\Phi^i_{\cdot A}(\xi,x,t)$$

$$(\Phi\otimes\square)(x,t) \triangleq \square_l\Phi^i_{\cdot A}(x,t)\, g_i\otimes G^A\otimes g^l, \quad \square_l\Phi^i_{\cdot A}(x,t) \triangleq \nabla_l\Phi^i_{\cdot A}(\xi,x,t) + \frac{\partial\xi^L}{\partial x^l}(x,t)\mathring{\nabla}_L\Phi^i_{\cdot A}(\xi,x,t)$$

through which the gradient of a tensor field can be interpreted. Certainly, the affine tensor is adopted here as the representation of any order tensor field represented in the two-point form.

$$\begin{aligned}
\Phi(\xi,t) &= \Phi^i{}_A\big((x(\xi,t),\xi,t\big)g_i(x)\otimes G^A(\xi) \\
\frac{\partial\Phi}{\partial\xi^L}(\xi,t) &= \frac{\partial}{\partial\xi^L}\left[\Phi^i{}_A(x,\xi,t)g_i(x)\otimes G^A(\xi)\right] \\
&= \left[\frac{\partial\Phi^i{}_A}{\partial x^l}\frac{\partial x^l}{\partial\xi^L} + \frac{\partial\Phi^i{}_A}{\partial\xi^L}\right]g_i\otimes G^A + \Phi^i{}_A\left[\frac{\partial g_i}{\partial x^l}\frac{\partial x^l}{\partial\xi^L}\right]\otimes G^A + \Phi^i{}_A g_i\otimes\frac{\partial G^A}{\partial\xi^L} \\
&= \frac{\partial x^l}{\partial\xi^L}\left(\frac{\partial\Phi^i{}_A}{\partial x^l}g_i\otimes G^A + \Phi^i{}_A\frac{\partial g_i}{\partial x^l}\otimes G^A\right) + \frac{\partial\Phi^i{}_A}{\partial\xi^L}g_i\otimes G^A + \Phi^i{}_A g_i\otimes\frac{\partial G^A}{\partial\xi^L} \\
&= \frac{\partial x^l}{\partial\xi^L}\left(\frac{\partial\Phi^i{}_A}{\partial x^l}g_i\otimes G^A + \Phi^i{}_A\Gamma^s_{li}g_s\otimes G^A\right) + \frac{\partial\Phi^i{}_A}{\partial\xi^L}g_i\otimes G^A - \Phi^i{}_A g_i\otimes\Gamma^A_{LS}G^S \\
&= \frac{\partial x^l}{\partial\xi^L}\left[\left(\frac{\partial\Phi^i{}_A}{\partial x^l} + \Gamma^i_{ls}\Phi^s{}_A\right)g_i\otimes G^A\right] + \left(\frac{\partial\Phi^i{}_A}{\partial\xi^L} - \Gamma^S_{LA}\Phi^i{}_S\right)g_i\otimes G^A \\
&= \frac{\partial x^l}{\partial\xi^L}\overset{>}{\nabla}_l\Phi^i_A g_i\otimes G^A + \overset{<}{\nabla}_L\Phi^i{}_A g_i\otimes G^A \\
&= \left(\frac{\partial x^l}{\partial\xi^L}\overset{>}{\nabla}_l\Phi^i{}_A + \overset{<}{\nabla}_L\Phi^i{}_A\right)g_i\otimes G^A \\
&=: \mathring{\square}_L\Phi^i{}_A g_i\otimes G^A
\end{aligned}$$

3. (10%) *To prove one of the following relations*

$$\mathring{\square}_L\, g^i_A(\xi) = 0, \quad \square_l\, g^i_A(x) = 0$$

that can be considered as the extension of the Ricci's lemma.

$$
\begin{aligned}
\overset{\circ}{\Box}_L\, g^i_A &= \overset{<}{\nabla}_L g^i_A + \frac{\partial x^l}{\partial \xi^L}\overset{>}{\nabla}_l g^i_A \\
&= \frac{\partial g^i_A}{\partial \xi^L} - \Gamma^S_{LA} g^i_S + \frac{\partial x^l}{\partial \xi^L}\left(\frac{\partial g^i_A}{\partial x^l} + \Gamma^i_{lt} g^t_A\right) \\
&= \frac{\partial}{\partial \xi^L}\left(g^i, G_A\right) - \Gamma^S_{LA} g^i_S + \frac{\partial x^l}{\partial \xi^L}\left(\frac{\partial}{\partial x^l}\left(g^i, G_A\right) + \Gamma^i_{lt} g^t_A\right) \\
&= \left(g^i, \frac{\partial}{\partial \xi^L} G_A\right) - \Gamma^S_{LA} g^i_S + \frac{\partial x^l}{\partial \xi^L}\left[\left(\frac{\partial}{\partial x^l} g^t, G_A\right) + \Gamma^i_{lt} g^t_A\right] \\
&= \left(g^i, \Gamma^S_{LA} G_S\right) - \Gamma^S_{LA} g^i_S + \frac{\partial x^l}{\partial \xi^L}\left[\left(-\Gamma^i_{lt} g^t, G_A\right) + \Gamma^i_{lt} g^t_A\right] \\
&= \Gamma^S_{LA} g^i_S - \Gamma^S_{LA} g^i_S + \frac{\partial x^l}{\partial \xi^L}\left(-\Gamma^i_{lt} g^t_A + \Gamma^i_{lt} g^t_A\right) \\
&= 0
\end{aligned}
$$

4. (10%) *To prove the relation of the basis-transformation, i.e., if it is valid that*

$$
g_i(x) =: \beta^{(j)}_i(x)\, g_j(x), \quad G_A(\xi) =: \beta^{(B)}_A(\xi)\, G_B(\xi)
$$

then

$$
\overset{\circ}{\Box}_{(F)} \Phi^{(r)}_{\cdot\,(E)} = \beta^L_{(F)} \beta^{(r)}_i \beta^A_{(E)}\, \overset{\circ}{\Box}_L \Phi^i_{\cdot A}(\xi, t) = \overset{\circ}{\nabla}_{(F)} \Phi^{(r)}_{\cdot\,(E)} + \frac{\partial x^{(l)}}{\partial \xi^{(F)}} \cdot \nabla_{(l)} \Phi^{(r)}_{\cdot\,(E)}
$$

with

$$
\frac{\partial x^{(l)}}{\partial \xi^{(F)}} := \beta^{(l)}_q(x) \beta^L_{(F)}(\xi)\, \frac{\partial x^q}{\partial \xi^L}(\xi, t)
$$

where $\overset{\circ}{\nabla}_{(F)} \Phi^{(r)}_{\cdot\,(E)}, \nabla_{(l)} \Phi^{(r)}_{\cdot\,(E)}$ *could be the "covariant derivatives" defined in the corresponding anholonomic bases with respect to* $\{g_i(x)\}$ *and* $\{G_A(\xi)\}$ *respectively.*

$$
\begin{aligned}
\overset{\circ}{\Box}_L \Phi^i{}_A\, g_i \otimes G^A \otimes G^L &= \overset{\circ}{\Box}_L \Phi^i{}_A \left(\beta^{(r)}_i g_{(r)}\right) \otimes \left(\beta^A_{(E)} G^{(E)}\right) \otimes \left(\beta^L_{(F)} G^{(F)}\right) \\
&= \beta^{(r)}_i \beta^A_{(E)} \beta^L_{(F)} \overset{\circ}{\Box}_L \Phi^i{}_A\, g_{(r)} \otimes G^{(E)} \otimes G^{(F)} \\
\because\ \overset{\circ}{\Box}_L \Phi^i{}_A g_i \otimes G^A \otimes G^L &= \overset{\circ}{\Box}_{(F)} \Phi^{(r)}{}_{(E)}\, g_{(r)} \otimes G^{(E)} \otimes G^{(F)} \\
\therefore\ \overset{\circ}{\Box}_{(F)} \Phi^{(r)}{}_{(E)} &= \beta^{(r)}_i \beta^A_{(E)} \beta^L_{(F)} \overset{\circ}{\Box}_L \Phi^i{}_A \\
&= \beta^{(r)}_i \beta^A_{(E)} \beta^L_{(F)} \overset{<}{\nabla}_L \Phi^i_{\cdot\,A} + \beta^{(r)}_i \beta^A_{(E)} \beta^L_{(F)}\, \frac{\partial x^q}{\partial \xi^L} \overset{>}{\nabla}_q \Phi^i_{\cdot\,A} \\
&= \beta^{(r)}_i \beta^A_{(E)} \beta^L_{(F)} \overset{<}{\nabla}_L \Phi^i_{\cdot\,A} + \beta^L_{(F)} \beta^{(r)}_i \beta^A_{(E)} \beta^q_{(l)} \beta^{(l)}_q\, \frac{\partial x^q}{\partial \xi^L} \overset{>}{\nabla}_q \Phi^i_{\cdot\,A} \\
&= \overset{<}{\nabla}_{(F)} \Phi^{(r)}{}_{(E)} + \beta^L_{(F)} \beta^{(l)}_q \frac{\partial x^q}{\partial \xi^L}\, \overset{>}{\nabla}_{(l)} \Phi^{(r)}{}_{(E)} \\
&=: \overset{<}{\nabla}_{(F)} \Phi^{(r)}{}_{(E)} + \frac{\partial x^{(l)}}{\partial \xi^{(F)}} \overset{>}{\nabla}_{(l)} \Phi^{(r)}{}_{(E)}
\end{aligned}
$$

5. (10%) *Let*

$$\Phi = \Phi^{i\cdot\cdot A}_{\cdot j\cdot B}(\xi, x, t)\, g_i \otimes g^j \otimes G_A \otimes G^B$$

its left gradient with respect to Lagrange coordinate is

$$\mathring{\Box} \otimes \Phi = \mathring{\Box}_L\, \Phi^{i\cdot\cdot A}_{\cdot j\cdot B}\, G^L \otimes g_i \otimes g^j \otimes G_A \otimes G^B$$

Let the anholonomic bases with respect to the initial and current configurations are the general cylindrical and sphere orthonormal bases, to deuce the representation of $(\mathring{\Box} \otimes \Phi)\langle 31232\rangle$.

$$\begin{aligned}
&(\mathring{\Box} \otimes \Phi)\langle 31232\rangle \\
=\ &(\mathring{\Box} \otimes \Phi)\langle zr\theta z\psi\rangle \\
=\ &\partial_{(z)}\Phi\langle r\theta z\psi\rangle \\
=\ &\partial\langle z\rangle\Phi\langle r\theta z\psi\rangle + \Gamma\langle zTz\rangle\Phi\langle r\theta T\psi\rangle + \Gamma\langle zT\psi\rangle\Phi\langle r\theta zT\rangle \\
&+ \frac{\partial x^{(l)}}{\partial \xi^{(3)}}\Big(\partial\langle l\rangle\Phi\langle r\theta z\psi\rangle + \Gamma\langle lsr\rangle\Phi\langle s\theta z\psi\rangle + \Gamma\langle ls\theta\rangle\Phi\langle rsz\psi\rangle\Big) \\
=\ &\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial z} + 0 \\
&+ \beta^{(l)}_l\beta^3_{(3)}\frac{\partial x^l}{\partial\xi^3}\Big(\partial\langle l\rangle\Phi\langle r\theta z\psi\rangle + \Gamma\langle lsr\rangle\Phi\langle s\theta z\psi\rangle + \Gamma\langle ls\theta\rangle\Phi\langle rsz\psi\rangle\Big)\ \text{(Here, } \beta^{(l)}_l = \sqrt{g_{ll}},\ \beta^3_{(3)} = 1.) \\
=\ &\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial z} \\
&+ 1\cdot\frac{\partial r}{\partial z}\left(1\cdot\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial r} + \Gamma\langle rsr\rangle\Phi\langle s\theta z\psi\rangle + \Gamma\langle rs\theta\rangle\Phi\langle rsz\psi\rangle\right) \\
&+ r\cdot\frac{\partial\theta}{\partial z}\left(\frac{1}{r}\cdot\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial\theta} + \Gamma\langle\theta sr\rangle\Phi\langle s\theta z\psi\rangle + \Gamma\langle\theta s\theta\rangle\Phi\langle rsz\psi\rangle\right) \\
&+ r\sin\theta\cdot\frac{\partial\varphi}{\partial z}\left(\frac{1}{r\sin\theta}\cdot\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial\varphi} + \Gamma\langle\varphi sr\rangle\Phi\langle s\theta z\psi\rangle + \Gamma\langle\varphi s\theta\rangle\Phi\langle rsz\psi\rangle\right) \\
=\ &\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial z} \\
&+ \frac{\partial r}{\partial z}\left(\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial r} + 0\right) \\
&+ r\frac{\partial\theta}{\partial z}\left(\frac{1}{r}\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial\theta} - \frac{1}{r}\Phi\langle\theta\theta z\psi\rangle + \frac{1}{r}\Phi\langle rrz\psi\rangle\right) \\
&+ r\sin\theta\frac{\partial\varphi}{\partial z}\left(\frac{1}{r\sin\theta}\frac{\partial\Phi\langle r\theta z\psi\rangle}{\partial\varphi} - \frac{1}{r}\Phi\langle\varphi\theta z\psi\rangle - \frac{1}{r}\cot\theta\cdot\Phi\langle r\varphi z\psi\rangle\right)
\end{aligned}$$

Problem 3 (Primary Properties of the Deformation Gradient tensor) *The deformation gradient tensor* $F \triangleq \frac{\partial x^i}{\partial \xi^A}(\xi, t) g_i \otimes G^A$ *has the following properties*

1. (20%)

$$|F| = \frac{\sqrt{g}}{\sqrt{G}} det \left[\frac{\partial x^i}{\partial \xi^A}\right](\xi, t)$$

$$F = \frac{\partial x^i}{\partial \xi^A}(\xi,t) g_i \otimes G^A = F^i_{.A}\, g_i \otimes G^A = F^i_{.A} g^A_j\, g_i \otimes g^j = F^i_{.j}\, g_i \otimes g^i$$

$$\begin{aligned}
\therefore detF &= det[F^i_{.j}] \\
&= det[F^i_{.A}] \cdot det[g^A_j] \\
&= det\left[\frac{\partial x^i}{\partial \xi^A}\right](\xi,t) \cdot det\Big(\begin{bmatrix} G^{1\mathsf{T}} \\ \vdots \\ G^{m\mathsf{T}} \end{bmatrix} [g_1, \cdots, g_m] \Big) \\
&= det\left[\frac{\partial x^i}{\partial \xi^A}\right](\xi,t) \cdot \frac{\sqrt{g}}{\sqrt{G}}
\end{aligned}$$

$$\begin{aligned}
where,\ \sqrt{g} &= det[g_1, \cdots, g_m] = det\, \mathscr{D}\, X(x) \\
\sqrt{G} &= det[G_1, \cdots, G_m] = det\, \mathscr{D}\, \overset{\circ}{X}(\xi)
\end{aligned}$$

2. (20%)

$$\dot{F} = (v \otimes \nabla) \cdot F \quad , \quad \dot{\overline{F^-}} = -F^- \cdot (v \otimes \nabla)$$

$$\begin{aligned}
\dot{F} &= \dot{\overline{\frac{\partial x^i}{\partial \xi^A}(\xi,t)\, g_i \otimes G^A(\xi)}} \\
&= \dot{\overline{\frac{\partial x^i}{\partial \xi^A}}}(\xi,t)\, g_i \otimes G^A(\xi) + \frac{\partial x^i}{\partial \xi^A}(\xi,t) \dot{g}_i \otimes G^A \\
&= \frac{\partial v^i}{\partial \xi^A}(\xi,t)\, g_i \otimes G^A(\xi) + \frac{\partial x^i}{\partial \xi^A}(\xi,t) \left(\frac{\partial g_i}{\partial x^j} v^j\right) \otimes G^A \\
&= \frac{\partial v^k}{\partial x^j} \frac{\partial x^j}{\partial \xi^A}\, g_k \otimes G^A + \frac{\partial x^i}{\partial \xi^A} v^j \Gamma^k_{ji}\, g_k \otimes G^A \\
&= \frac{\partial v^k}{\partial x^i} \frac{x^i}{\xi^A}\, g_k \otimes G^A + \frac{\partial x^i}{\partial \xi^A} v^j \Gamma^k_{ij}\, g_k \otimes G^A \\
&= \frac{\partial x^i}{\partial \xi^A} \left(\frac{\partial v^k}{\partial x^i} + \Gamma^k_{ij} v^j\right) g_k \otimes G^A \\
&= \frac{\partial x^i}{\partial \xi^A} \Big(\nabla_i v^k(x,t)\Big)\, g_k \otimes G^A \\
&= \frac{\partial x^i}{\partial \xi^A} \nabla_i v^k\, g_k \otimes G^A \\
&= \Big(\nabla_i v^k\, g_k \otimes g^i\Big) \cdot \left(\frac{\partial x^j}{\partial \xi^A}\, g_j \otimes G^A\right) \\
&= (v \otimes \nabla) \cdot F
\end{aligned}$$

3. (20%)

$$F \cdot \mathring{\Box} \times (b \cdot F) = |F|\Box \times b, \quad \forall b(x,t) \in \mathbb{R}^3$$

$$
\begin{aligned}
F \cdot \mathring{\Box} \times (b \cdot F) &= F \cdot \mathring{\Box} \times \left(b_i \frac{\partial x^i}{\partial \xi^A} G^A \right) \\
&= F \cdot \left[\left(\frac{\partial}{\partial \xi^L} G^L \right) \times \left(b_i \frac{\partial x^i}{\partial \xi^A} G^A \right) \right] \\
&= F \cdot \left[G^L \times \mathring{\Box}_L \left(b_i \frac{\partial x^i}{\partial \xi^A} \right) G^A \right] \\
&= F \cdot \left[\mathring{\Box}_L \left(b_i \frac{\partial x^i}{\partial \xi^A} \right) \epsilon^{LAB} G_B \right] \\
&= \left(\frac{\partial x^j}{\partial \xi^D} g_j \otimes G^D \right) \cdot \left[\mathring{\Box}_L \left(b_i \frac{\partial x^i}{\partial \xi^A} \right) \epsilon^{LAB} G_B \right] \\
&= \frac{\partial x^j}{\partial \xi^B} \mathring{\Box}_L \left(b_i \frac{\partial x^i}{\partial \xi^A} \right) \epsilon^{LAB} g_j \\
&= \frac{\partial x^j}{\partial \xi^B} \left[\left(\mathring{\Box}_L b_i \right) \frac{\partial x^i}{\partial \xi^A} + b_i \mathring{\Box}_L \left(\frac{\partial x^i}{\partial \xi^A} \right) \right] \epsilon^{LAB} g_j \\
&= \frac{\partial x^j}{\partial \xi^B} \left[\frac{\partial x^l}{\partial \xi^L} \nabla_l b_i \frac{\partial x^i}{\partial \xi^A} + b_i \left(\stackrel{<}{\nabla}_L \left(\frac{\partial x^i}{\partial \xi^A} \right) + \frac{\partial x^l}{\partial \xi^L} \stackrel{>}{\nabla}_l \left(\frac{\partial x^i}{\partial \xi^A} \right) \right) \right] \epsilon^{LAB} g_j
\end{aligned}
$$

$$
\begin{aligned}
b_i \left(\nabla_L \frac{\partial x^i}{\partial \xi^A} \right) \epsilon^{LAB} &= b_i \left(\frac{\partial^2 x^i}{\partial \xi^L \partial \xi^A} - \Gamma^C_{LA} \frac{\partial x^i}{\partial \xi^C} \right) \epsilon^{LAB} \\
&= b_i \left(\frac{\partial^2 x^i}{\partial \xi^A \partial \xi^L} - \Gamma^C_{AL} \frac{\partial x^i}{\partial \xi^C} \right) \epsilon^{ALB} \\
&= -b_i \left(\frac{\partial^2 x^i}{\partial \xi^A \partial \xi^L} - \Gamma^C_{AL} \frac{\partial x^i}{\partial \xi^C} \right) \epsilon^{LAB} \\
&= -b_i \left(\frac{\partial^2 x^i}{\partial \xi^L \partial \xi^A} - \Gamma^C_{LA} \frac{\partial x^i}{\partial \xi^C} \right) \epsilon^{LAB} \\
&= -b_i \left(\nabla_L \frac{\partial x^i}{\partial \xi^A} \right) \epsilon^{LAB} \\
&= 0 \\
b_i \frac{\partial x^l}{\partial \xi^L} \nabla_l \left(\frac{\partial x^i}{\partial \xi^A} \right) \epsilon^{LAB} &= b_i \frac{\partial x^l}{\partial \xi^L} \left(\frac{\partial}{\partial x^l} \left(\frac{\partial x^i}{\partial \xi^A} \right) + \Gamma^i_{lk} \frac{\partial x^k}{\partial \xi^A} \right) \epsilon^{LAB} \\
&= b_i \frac{\partial^2 x^i}{\partial \xi^L \xi^A} \epsilon^{LAB} + b_i \frac{\partial x^l}{\partial \xi^L} \Gamma^i_{lk} \frac{\partial x^k}{\partial \xi^A} \epsilon^{LAB} \\
&= b_i \frac{\partial^2 x^i}{\partial \xi^L \xi^A} \epsilon^{LAB} + b_i \frac{\partial x^k}{\partial \xi^A} \Gamma^i_{kl} \frac{\partial x^l}{\partial \xi^L} \epsilon^{ALB} \\
&= b_i \frac{\partial^2 x^i}{\partial \xi^A \xi^L} \epsilon^{ALB} + b_i \frac{\partial x^k}{\partial \xi^A} \Gamma^i_{lk} \frac{\partial x^l}{\partial \xi^L} \epsilon^{ALB} \\
&= -b_i \frac{\partial^2 x^i}{\partial \xi^L \xi^A} \epsilon^{LAB} - b_i \frac{\partial x^k}{\partial \xi^A} \Gamma^i_{kl} \frac{\partial x^l}{\partial \xi^L} \epsilon^{ALB} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\therefore F\cdot \overset{\circ}{\Box}\times(b\cdot F) &= \frac{\partial x^j}{\partial \xi^B}\frac{\partial x^l}{\partial \xi^L}\frac{\partial x^i}{\partial \xi^A}\nabla_l b_i \epsilon^{LAB} g_j \\
&= \frac{e^{LAB}}{\sqrt{G}}\frac{\partial x^j}{\partial \xi^B}\frac{\partial x^l}{\partial \xi^L}\frac{\partial x^i}{\partial \xi^A}\nabla_l b_i g_j \\
&= \frac{e^{lij}}{\sqrt{G}} det\left[\frac{\partial x^i}{\partial \xi^A}\right]\nabla_l b_i g_j \\
&= |F|\frac{e^{lij}}{\sqrt{g}}\nabla_l b_i g_j \\
&= |F|\epsilon^{lij}\nabla_l b_i g_j \\
&= |F|\Box\times b
\end{aligned}
$$

4.

$$
\dot{\overline{|F|}} = \theta|F|
$$

To prove the properties marked as (1),(2) and (3).

Problem 4 (Some Application of the deformation gradient tensor) *Let $v(x,t)$ be the velocity field in the form of Euclian Coordinate,then one has the identify*

$$
F\cdot\overset{\circ}{\Box}\times(v\cdot F) = |F|\Box\times v
$$

Let $\omega \triangleq \Box\times v,\quad U\triangleq v\cdot F,\quad \Omega\triangleq \overset{\circ}{\Box}\times U,$ *the above identify is just*

$$
F\cdot\Omega = |F|\omega \Longrightarrow \omega = \frac{1}{|F|}F\cdot\Omega
$$

(10%) *To prove the following identify*

$$
\overset{\circ}{\omega} = \omega\cdot(\Box\otimes v) - \theta\omega + \nabla\times a
$$

where $a = \overset{\circ}{v}$ is the acceleration field. It is just the governing equation of the vorticity.

$$
\begin{aligned}
\because \omega &= \frac{1}{|F|}F\cdot\Omega \\
\therefore \frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{\dot{\overline{|F|}}}{|F|^2}F\cdot\Omega + \frac{1}{|F|}\dot{F}\cdot\Omega + \frac{1}{|F|}F\cdot\dot{\Omega} \\
&= -\frac{\theta}{|F|}F\cdot\Omega + \frac{1}{|F|}(v\otimes\nabla)\cdot F\cdot\Omega + \frac{1}{|F|}F\cdot\dot{\Omega}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\Omega} &= \overline{\overset{\circ}{\Box} \times U}^{\cdot} = \overline{\overset{\circ}{\Box} \times (v \cdot F)}^{\cdot} = \overset{\circ}{\Box} \times \overline{(v \cdot F)}^{\cdot} \\
&= \overset{\circ}{\Box} \times (\dot{v} \cdot F) + \overset{\circ}{\Box} \times \left(v \cdot \dot{F}\right) \\
&= \overset{\circ}{\Box} \times (a \cdot F) + \overset{\circ}{\Box} \times \big(v \cdot (v \otimes \nabla) \cdot F\big) \\
&= F^{-1} \cdot |F| \nabla \times a + F^{-1} \cdot \Big(|F| \nabla \times \big(v \cdot (v \otimes \nabla)\big)\Big) \\
&= F^{-1} \cdot |F| \nabla \times a + F^{-1} \cdot |F| \nabla \times \left(\nabla \frac{|v|^2}{2}\right) \\
&= F^{-1} \cdot |F| \nabla \times a + 0 \\
\therefore \frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\theta\omega + \frac{(v \otimes \nabla) \cdot F \cdot \Omega}{|F|} + \frac{1}{|F|} F \cdot F^{-1} \cdot |F| \nabla \times a \\
&= -\theta\omega + \omega \cdot (\nabla \otimes v) + \nabla \times a
\end{aligned}
$$

Problem 5 (Reynolds Transport Theorem) *In continuum mechanics, the integrals on the material volume, surface and curve can be defined respectively. As soon as the material volume is considered, say,*

$$
I(t) = \int_{\overset{t}{v}} \Phi(x,t) d\tau
$$

the following identity is valid

$$
I(t) = \int_{\overset{t}{v}} \Phi(x,t) d\tau = \int_{\overset{\circ}{v}} \Phi(\overset{\circ}{X}, t_0) |F| d\tau
$$

1. (10%) *To prove*

$$
\overset{\circ}{I}(t) = \int_{\overset{t}{\mathscr{V}}} (\overset{\circ}{\Phi} + \theta\Phi) d\tau = \int_{\overset{t}{\mathscr{V}}} \frac{\partial \Phi}{\partial t}(x,t) d\tau + \oint_{\partial \overset{t}{\mathscr{V}}} \Phi(v \cdot n) d\sigma
$$

$$
\begin{aligned}
I(t) &= \int_{\mathscr{V}_t} \Phi(x,t) \mathrm{d}\tau \\
\dot{I}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{V}_{t_0}} \Phi |F| \mathrm{d}\tau = \int_{\mathscr{V}_{t_0}} \overline{\Phi |F|}^{\cdot} \mathrm{d} \\
&= \int_{\mathscr{V}_{t_0}} \left(\dot{\Phi}|F| + \Phi\theta|F|\right) \mathrm{d}\tau = \int_{\mathscr{V}_{t_0}} \left(\dot{\Phi} + \theta\Phi\right) |F| \mathrm{d}\tau = \int_{\mathscr{V}_t} \left(\dot{\Phi} + \theta\Phi\right) \mathrm{d}\tau \\
&= \int_{\mathscr{V}_t} \left[\frac{\partial \Phi}{\partial t} + (v \cdot \nabla) \otimes \Phi + (\nabla \cdot v)\Phi\right] \mathrm{d}\tau \\
&= \int_{\mathscr{V}_t} \left[\frac{\partial \Phi}{\partial t} + \nabla \cdot (v \otimes \Phi)\right] \mathrm{d}\tau \\
&= \int_{\mathscr{V}_t} \frac{\partial \Phi}{\partial t}(x,t) \mathrm{d}\tau + \oint_{\partial \mathscr{V}_t} \Phi(v \cdot n) \mathrm{d}\tau
\end{aligned}
$$

2. (20%) *To deduce the mass conservation equations in the point of view of Lagrange and Euler coordinates.*

$$
\begin{aligned}
\int_{\mathscr{V}_t} \Phi(x,t)\mathrm{d}\tau &= \int_{\mathscr{V}_{t_0}} \Phi(x,t)|F|\mathrm{d}\tau \\
\int_{\mathscr{V}_t} \Phi(x,t)\mathrm{d}\tau &= \int_{\mathscr{V}_{t_0}} \Phi(x,t_0)\mathrm{d}\tau \\
\therefore \Phi(x,t)|F|(x,t) &= \Phi(x,t_0) \\
\Rightarrow \Phi\big(x(\xi,t),t\big)|F|(\xi,t) &= \Phi\big(x(\xi,t_0),t_0\big) \\
\Rightarrow \Phi(\xi,t)|F|(\xi,t) &= \Phi(\mathring{X},t_0) \\
\therefore \rho(\xi,t)|F|(\xi,t) &= \mathring{\rho}(\xi)
\end{aligned}
$$

Eular's view:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{V}_t} \rho \mathrm{d}\tau = \int_{\mathscr{V}_t} (\dot{\rho}+\theta\rho)\,\mathrm{d}\tau = \int_{\mathscr{V}_t}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v)\right]\mathrm{d}\tau = \int_{\mathscr{V}_t}\frac{\partial\rho}{\partial t}\mathrm{d}\tau + \oint_{\partial\mathscr{V}_t}\rho(v\cdot n)\mathrm{d}\tau = 0
$$

$$
\begin{aligned}
\therefore \quad & \dot{\rho}+\theta\rho = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0 \\
\textit{or} \quad & \int_{\mathscr{V}_t}\frac{\partial\rho}{\partial t}\mathrm{d}\tau + \oint_{\partial\mathscr{V}_t}\rho(v\cdot n)\mathrm{d}\tau = 0
\end{aligned}
$$

3. (20%) *To deduce the momentum conservation equations in the point of view of Lagrange and Euler coordinates.*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{V}_t}\rho v\mathrm{d}\tau = \int_{\mathscr{V}_t}\rho\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}\tau = \oint_{\partial\mathscr{V}_t}(t\cdot n)\mathrm{d}\sigma + \int_{\mathscr{V}_t}\rho f_m\mathrm{d}\tau = \int_{\mathscr{V}_t} t\cdot\nabla\mathrm{d}\tau + \int_{\mathscr{V}_t}\rho f_m \mathrm{d}\tau
$$

$$
\therefore \quad \rho\frac{\mathrm{d}v}{\mathrm{d}t} = t\cdot\nabla + \rho f_m \qquad \rho a = t\cdot\nabla + \rho f_m
$$

Lagrange's view:

$$\because \quad \int_{\mathscr{V}_t} \rho a \mathrm{d}\tau = \oint_{\partial \mathscr{V}_t} (t \cdot n)\mathrm{d}\sigma + \int_{\mathscr{V}_t} \rho f_m \mathrm{d}\tau = \int_{\mathscr{V}_t} t \cdot \nabla \mathrm{d}\tau + \int_{\mathscr{V}_t} \rho f_m \mathrm{d}\tau$$

$$\therefore \quad \int_{\mathscr{V}_{t_0}} \rho a |F| \mathrm{d}\tau = \int_{\mathscr{V}_{t_0}} (\nabla \cdot t)|F| \mathrm{d}\tau + \int_{\mathscr{V}_{t_0}} \rho f_m |F| \mathrm{d}\tau$$

$$\because \quad \rho(\xi, t)|F| = \mathring{\rho}(\xi)$$

$$\therefore \quad LHS = \int_{\mathscr{V}_{t_0}} a(\xi, t)\mathring{\rho}(\xi)\mathrm{d}\tau$$

$$\begin{aligned}
RHS1 &= \int_{\mathscr{V}_t} t \cdot \nabla \mathrm{d}\tau = \oint_{\partial \mathscr{V}_t} t \cdot n \mathrm{d}\sigma \\
&= \int_{\mathscr{D}_{\lambda\mu}} t \cdot \left(\frac{\partial X}{\partial \lambda} \times \frac{\partial X}{\partial \mu} \right) (\lambda, \mu, t)\mathrm{d}\sigma = \int_{\mathscr{D}_{\lambda\mu}} t \cdot |F| F^{-*} \left(\frac{\partial \mathring{X}}{\partial \lambda} \times \frac{\partial \mathring{X}}{\partial \mu} \right) \mathrm{d}\sigma \\
&=: \int_{\mathscr{D}_{\lambda\mu}} \tau \cdot \left(\frac{\partial \mathring{X}}{\partial \lambda} \times \frac{\partial \mathring{X}}{\partial \mu} \right) \mathrm{d}\sigma \qquad (\text{Here: } \tau \triangleq |F| t \cdot F^{-*}. \quad) \\
&= \oint_{\partial \mathscr{V}_{t_0}} \tau \cdot \mathring{n} \mathrm{d}\sigma = \int_{\mathscr{V}_{t_0}} \tau \cdot \mathring{\nabla} \mathrm{d}\sigma \\
RHS2 &= \int_{\mathscr{V}_{t_0}} \rho f_m |F| \mathrm{d}\tau = \int_{\mathscr{V}_{t_0}} f_m(\xi, t)\big(\rho(\xi, t)|F|(\xi, t)\big)\mathrm{d}\tau = \int_{\mathscr{V}_{t_0}} f_m(\xi, t)\mathring{\rho}(\xi, t)\mathrm{d}\tau
\end{aligned}$$

$$\therefore \quad a(\xi, t)\mathring{\rho} + \tau \cdot \mathring{\nabla} = f_m(\xi, t)\mathring{\rho}(\xi) \qquad (\tau \triangleq |F| t \cdot F^{-*}) \quad \text{(Boussinesq Equation)}$$

Note: To give the deduction and calculation in detail. And as the score is considered, the reflection of the correct methodologies is oriented.