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# OPTIMAL MULTIPERIOD PORTFOLIO POLICIES\*

JAN MOSSIN†

## I. INTRODUCTION

### A. BACKGROUND

Most of the work in portfolio theory to date<sup>1</sup> has taken what may be called a mean variability approach—that is, the investor is thought of as choosing among alternative portfolios on the basis of the mean and variance of the portfolios' rate of return. A recent contribution by Arrow prepares the ground for a considerably more general approach.<sup>2</sup>

Although there would seem to be an obvious need for extending the one-period analysis to problems of portfolio management over several periods, Tobin appears to be one of the first to make an attempt in this direction.<sup>3</sup> However, as will be demonstrated in this article, the validity of portions of this analysis appears to be doubtful. The explanation is partly to be found in the fact that a for-

mulation of the decision problem (even in the one-period case) in terms of portfolio rate of return tends to obscure an important aspect of the problem, namely, the role of the absolute size of the portfolio. In a multiperiod theory the development through time of total wealth becomes crucial and must be taken into account. A formulation neglecting this can easily become misleading.

In order to bring out and resolve the problems connected with a rate-of-return formulation, it is therefore necessary to start with an analysis of the one-period problem. Thus prepared, the extension to multiperiod problems can be accomplished, essentially by means of a dynamic programming approach.

### B. RISK-AVERSION FUNCTIONS

The Pratt-Arrow measures of risk aversion are employed at various points in the analysis. They are *absolute risk aversion*,

$$R_a(Y) = -\frac{U''(Y)}{U'(Y)},$$

*relative risk aversion*,

$$R_r(Y) = -\frac{U''(Y)Y}{U'(Y)},$$

where  $U$  is a utility function representing preferences over probability distributions for wealth  $Y$ . Discussions of the significance of these functions are found in Arrow and Pratt.<sup>4</sup>

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<sup>1</sup> J. Tobin, "Liquidity Preference as Behavior towards Risk," *Review of Economic Studies* (1957-58), pp. 65-86; H. Markowitz, *Portfolio Selection* (New York: Wiley, 1959); J. Tobin, "The Theory of Portfolio Selection," in F. H. Hahn and F. P. R. Brechling (eds.), *The Theory of Interest Rates* (London: Macmillan, 1965); J. Mossin, "Equilibrium in a Capital Asset Market," *Econometrica* (1966), pp. 768-83.

<sup>2</sup> K. J. Arrow, *Aspects of the Theory of Risk-Bearing* (Yrjö Jahnsson Lectures [Helsinki: The Yrjö Jahnsson Foundation, 1965]).

<sup>3</sup> Tobin, "Theory of Portfolio Selection."

<sup>4</sup> Arrow, *op. cit.*; J. Pratt, "Risk Aversion in the Small and in the Large," *Econometrica* (1964), pp. 122-36.

## II. SINGLE-PERIOD MODELS

By a single-period model is meant a theory of the following structure: The investor makes his portfolio decision at the beginning of a period and then waits until the end of the period when the rate of return on his portfolio materializes. He cannot make any intermediate changes in the composition of his portfolio. The investor makes his decision with the objective of maximizing expected utility of wealth at the end of the period (final wealth).

## A. THE SIMPLEST CASE

In the simplest possible case there are only two assets, one of which yields a random rate of return (an interest rate) of  $X$  per dollar invested, while the other asset (call it "cash") gives a certain rate of return of zero. This model has been analyzed in some detail in Arrow.<sup>5</sup>

If the investor's initial wealth is  $A$ , of which he invests an amount  $a$  in the risky asset, his final wealth is the random variable

$$Y = A + aX. \quad (1)$$

With a preference ordering  $U(Y)$  over levels of final wealth, the optimal value of  $a$  is the one which maximizes  $E[U(Y)]$ , subject to the condition  $0 \leq a \leq A$ .

*General analysis.*—The first two derivatives of  $E[U(Y)]$  are

$$\frac{dE[U(Y)]}{da} = E[U'(Y)X]$$

and

$$\frac{d^2E[U(Y)]}{da^2} = E[U''(Y)X^2].$$

Assuming general risk aversion ( $U'' < 0$ ), the second derivative is negative, so that a unique maximum point is guaranteed. This might occur at one of the end

points  $a = 0$  or  $a = A$ ; the condition for the former is that  $dE[U(Y)]/da$  is negative at  $a = 0$ , which is seen to imply and require  $E(X) \leq 0$ . Thus, the investor will hold positive amounts of the risky asset if and only if its expected rate of return is positive.

If the maximum occurs at an interior value of  $a$ , we have at this point

$$E[U'(Y)X] = 0. \quad (2)$$

To see how such an optimal value-of  $a$  depends upon the level of initial wealth, we differentiate (2) with respect to  $A$  and obtain

$$\frac{da}{dA} = -\frac{E[U''(Y)X]}{E[U''(Y)X^2]}. \quad (3)$$

It is possible to prove that the sign of this derivative is positive, zero, or negative, according as absolute risk aversion is decreasing, constant, or increasing.

*Quadratic utility.*—In particular, one might consider preferences over probability distributions of  $Y$  being defined in terms of means and variances only. If such a preference ordering applies to arbitrary probability distributions, the utility function must clearly be of the form

$$U(Y) = Y - \alpha Y^2. \quad (4)$$

Then the optimal  $a$  is the one which maximizes

$$\begin{aligned} E[U(Y)] &= E[A + aX - \alpha(A + aX)^2] \\ &= (A - \alpha A)^2 + (1 - 2\alpha A)Ea \\ &\quad - \alpha(V + E^2)a^2, \end{aligned}$$

where  $E$  and  $V$  denote expectation and variance of  $X$ , respectively. An interior maximum is then given by

$$a = \frac{(1 - 2\alpha A)E}{2\alpha(V + E^2)}. \quad (5)$$

Thus, the optimal  $a$  depends on the level of initial wealth. The same is also true of

<sup>5</sup> Arrow, *op. cit.*

the proportion  $a/A$  of initial wealth held in the risky asset. It is seen that  $da/dA < 0$ ; this is the disconcerting property mentioned above of any utility function exhibiting increasing absolute risk aversion.

With the optimal value of  $a$  given by (5), maximum expected utility will be

$$\begin{aligned} \max E[U(Y)] &= \frac{E^2}{4a(V+E^2)} \\ &+ \frac{V}{V+E^2}(A-aA^2). \end{aligned} \quad (6)$$

*Tobin's formulation.*—Tobin's formulation is somewhat different.<sup>6</sup> He also assumes quadratic utility, but the argument of the utility function is taken as *one plus the portfolio rate of return*. Second, he takes as decision variable the *proportion* of initial wealth invested in the risky asset. If this fraction is called  $k$ , he thus wishes to maximize expected utility of the variate  $R = 1 + kX$ . In the symbols used above,

$$R = \frac{Y}{A} = \frac{A+aX}{A} = 1 + \frac{a}{A}X = 1 + kX.$$

Then with a quadratic utility function

$$V(R) = R - \beta R^2, \quad (7)$$

$k$  is determined such that  $E[V(R)]$  is a maximum:

$$\begin{aligned} E[V(R)] &= E[1 + kX - \beta(1 + kX)^2] \\ &= (1 - \beta) + (1 - 2\beta)Ek \\ &\quad - \beta(V + E^2)k^2. \end{aligned}$$

An interior maximum is given by the decision

$$k = \frac{(1 - 2\beta)E}{2\beta(V + E^2)}. \quad (8)$$

The important point to be made here is that the way (8) is written, *it seems as if the optimal  $k$  is independent of initial wealth*. In the formulation of the maxi-

mization problem, the level of initial wealth has somehow slipped out the back door. Also, the resulting maximum level of expected utility would seem to be independent of initial wealth.

So it appears to be a conflict between the two formulations. A little reflection shows, however, that when initial wealth is taken as a given, constant datum (say, 100), any level of final wealth can obviously be equivalently described either in absolute terms (say, 120) or as a rate of return (.2). But in considering a final wealth level of 120, it is immaterial to the investor whether this is a result of an initial wealth of 80 with yield .5 or an initial wealth of 100 with yield .2 (or any other combination of  $A$  and  $R$  such that  $AR = 120$ ). The explanation of the apparent conflict is now very simple: When using a quadratic utility function in  $R$ , the coefficient  $\beta$  is not independent of  $A$  if the function shall lead to consistent decisions at different levels of wealth. This is seen by observing that  $R = Y/A$ , so that

$$V(R) = V\left(\frac{Y}{A}\right) = \frac{Y}{A} - \beta\left(\frac{Y}{A}\right)^2,$$

which is equivalent, as a utility function, to  $Y - (\beta/A)Y^2$ . What this implies, then, is that *a utility function of the form  $R - \beta R^2$  cannot be used with the same  $\beta$  at different levels of initial wealth*. The appropriate value of  $\beta$  must be set such that  $\beta/A = a$ —that is,  $\beta$  must be changed in proportion to  $A$ . But when this precaution is taken, Tobin's formulation will obviously lead to the correct decision; with  $\beta = aA$  substituted in equation (8), we get

$$\frac{a}{A} = k = \frac{(1 - 2aA)E}{2aA(V + E^2)},$$

that is,

$$a = \frac{(1 - 2aA)E}{2a(V + E^2)},$$

<sup>6</sup> Tobin, "Theory of Portfolio Selection."

which is the same solution as (5). For example, if the utility function for final wealth is  $Y - (1/400)Y^2$ , it may be perfectly acceptable to maximize the expectation of  $R - \frac{1}{4}R^2$ , but *only* if initial wealth happens to be 100.

It should be kept in mind that when we are here speaking of different levels of wealth, this is to be interpreted strictly in terms of comparative statics; we are only asserting that if the investor had had an initial wealth different from  $A$ , then his optimal  $k$  would have been different from  $(1 - 2\alpha A)E/2\alpha A(V + E^2)$ . When we consider different levels of wealth *at different points in time* (in a sequence of portfolio decisions), other factors may also affect the decisions, as we shall see later. And it will also become clear that attempting to use a utility function of the form of equation (7) in such a setting may easily cause difficulties.

*Utility functions implying constant asset proportions.*—If attention is not restricted to quadratic utility functions, however, it may be possible to get investment in the risky asset strictly proportional to initial wealth.

Requiring that  $a/A = k$  is seen to be the same as requiring that choices among portfolios be based upon consideration of the probability distribution for the portfolio's rate of return independently of initial wealth: the choice of a probability distribution for  $R = 1 + kX$  consists in a choice of a value of  $k$ , this choice being made independently of  $A$ . Therefore, the problem of finding the class of utility functions with the property that  $a/A = k$  is equivalent to the problem of determining the class of utility functions with the property that choices among distributions for rate of return on the portfolio are independent of initial wealth.

If two utility functions  $U$  and  $V$  repre-

sent the same preference ordering, there exist constants  $b$  and  $c$  such that  $V = bU + c$ . Therefore, if a utility function  $U$  determines an ordering of probability distributions for *rate of return* and this ordering is identical with the ordering of probability distributions for *final wealth*, then  $U(R)$  and  $U(Y) = U(AR)$  must represent the same ordering. This must mean that  $U(R)$  and  $U(AR)$  are linear transformations of each other:

$$U(AR) = bU(R) + c. \quad (9)$$

Here  $b$  and  $c$  are independent of  $R$ , but they may depend upon  $A$ .

Differentiation of (9) with respect to  $R$  gives

$$U'(AR)A = bU'(R). \quad (10)$$

Then differentiating (10) with respect to  $A$ , we have

$$U''(AR)AR + U'(AR) = b'U'(R), \quad (11)$$

where  $b'$  denotes derivative with respect to  $A$ . From (10) the right-hand side is  $(b'A/b)U'(AR)$ , so that (11) can be written

$$U''(Y)Y + U'(Y) = \frac{b'A}{b} U'(Y)$$

or

$$-\frac{U''(Y)Y}{U'(Y)} = 1 - \frac{b'A}{b}. \quad (12)$$

Since this must hold for independent variations in  $Y$  and  $A$ , both sides are constant. This means that relative risk aversion must be constant, equal to, say,  $\gamma$ . It is easily verified that the only solutions to this condition are linear transformations of the functions

$$U(Y) = \ln Y \quad \text{if } \gamma = 1 \quad (13a)$$

and

$$U(Y) = Y^{1-\gamma} \quad \text{if } \gamma \neq 1. \quad (13b)$$

Thus, utility functions belonging to this class are the only ones permitted if constant asset proportions are to be optimal.

To see that these functions indeed satisfy our requirement, we observe that when relative risk aversion is constant, that is, when

$$\frac{U''(Y)Y}{U'(Y)} = -\gamma,$$

then

$$U''(Y)YX = -\gamma U'(Y)X,$$

and so

$$E[U''(Y)XY] = -\gamma E[U'(Y)X].$$

At an interior maximum point we have

$$E[U'(Y)X] = 0,$$

and so

$$E[U''(Y)XY] = 0,$$

or

$$AE[U''(Y)X] + aE[U''(Y)X^2] = 0;$$

thus

$$-\frac{E[U''(Y)X]}{E[U''(Y)X^2]} = \frac{a}{A}.$$

But from (3) the left-hand side is  $da/dA$ ; hence  $da/dA = a/A$ , implying  $a = kA$ .

The conclusion is therefore that there may exist preferences which can be represented by a utility function in rate of return only, but then it must be of the form  $\ln R$  or  $R^{1-\gamma}$  ( $\ln Y$  and  $Y^{1-\gamma}$  are equivalent, as utility functions, to  $\ln R$  and  $R^{1-\gamma}$ ). Other forms, like the quadratic (7) with constant  $\beta$ , are ruled out.

We note for later reference that when  $U = \ln Y$ , the maximum condition becomes

$$E\left(\frac{X}{A + aX}\right) = 0,$$

so that  $k$  is determined by the condition

$$E\left(\frac{X}{1 + kX}\right) = 0.$$

Maximum expected utility is then

$$\begin{aligned} \max E(\ln Y) &= \ln A \\ &+ E[\ln(1 + kX)]. \end{aligned} \quad (14)$$

Similarly, with  $U = Y^{1-\gamma}$ ,  $k$  will be determined by

$$E[(1 + kX)^{-\gamma}X] = 0,$$

and so correspondingly

$$\max E(Y^{1-\gamma}) = A^{1-\gamma}E[1 + kX]^{1-\gamma}. \quad (15)$$

#### B. MORE GENERAL CASES

Almost all the analysis above is easily generalized to the case where the yield on the certain asset is non-zero or to the case where the yields on both assets are random. Since the analyses are in both cases completely parallel, we shall only give the results for the more general of the two (both yields random). Results for the former case are then obtained simply by replacing the random yield  $X_2$  by a non-random variable  $r$  to represent the interest on the certain asset.

Generalization to an arbitrary number of assets would be trivial and add little of theoretical interest.

If the random rates of return on the two assets are  $X_1$  and  $X_2$ , and  $a$  is the amount invested in the first asset, then final wealth is

$$Y = (1 + X_2)A + a(X_1 - X_2).$$

By so to say substituting  $(1 + X_2)A$  for  $A$  and  $X_1 - X_2$  for  $X$  throughout, most of the conclusions from the discussion of the simplest case are readily obtained.

Thus, in the general case, an interior maximum point would be one where

$$E[U'(Y)(X_1 - X_2)] = 0, \quad (16)$$

and the corresponding expression for  $da/dA$  would be

$$\frac{da}{dA} = -\frac{E[U''(Y)(X_1 - X_2)(1 + X_2)]}{E[U''(Y)(X_1 - X_2)^2]}. \quad (17)$$

It is clear that in general nothing can be said about the sign of this derivative; for  $X_2$  non-random, the dependence on the slope of the absolute risk-aversion function is exactly as before, however.

In the case where utility is quadratic in wealth (and assuming  $X_1$  and  $X_2$  to be independently distributed), the optimal  $a$  will be given by

$$a = \frac{E_1 - E_2 - 2\alpha A [(1 + E_2)(E_1 - E_2) - V_2]}{2\alpha [V_1 + V_2 + (E_1 - E_2)^2]}, \quad (18)$$

and correspondingly

$$\begin{aligned} \max E[U(Y)] &= \frac{V_1(1 + E_2) + V_2(1 + E_1)}{V_1 + V_2 + (E_1 - E_2)^2} \\ &\times \left[ A - \frac{V_1(1 + E_2)^2 + V_2(1 + E_1)^2 + V_1V_2}{V_1(1 + E_2) + V_2(1 + E_1)} \alpha A^2 \right] + \frac{(E_1 - E_2)^2}{4\alpha [V_1 + V_2 + (E_1 - E_2)^2]}. \end{aligned} \quad (19)$$

With the Tobin formulation, however, the optimal  $k$  would be expressed as

$$k = \frac{E_1 - E_2 - 2\beta [(1 + E_2)(E_1 - E_2) - V_2]}{2\beta [V_1 + V_2 + (E_1 - E_2)^2]}, \quad (20)$$

and, again, if decisions are to be consistent at different levels of initial wealth,  $\beta$  must be proportional to  $A$ .

The derivation of the utility functions, (13a) and (13b), is clearly independent of the specific setting of the decision problem. The sufficiency part of the proof is also completely analogous.

With the utility function  $U = \ln Y$ ,  $k$  would now be determined by the condition

$$E\left(\frac{X_1 - X_2}{1 + X_2 + k(X_1 - X_2)}\right) = 0,$$

giving

$$\begin{aligned} \max E(\ln Y) &= \ln A \\ &+ E\{\ln [1 + X_2 + k(X_1 - X_2)]\}. \end{aligned} \quad (21)$$

Similarly, with  $U = Y^{1-\gamma}$ ,  $k$  is determined by

$$E\{[1 + X_2 + k(X_1 - X_2)]^{-\gamma}(X_1 - X_2)\} = 0,$$

giving

$$\begin{aligned} \max E(Y^{1-\gamma}) \\ = A^{1-\gamma} E\{[1 + X_2 + k(X_1 - X_2)]^{1-\gamma}\}. \end{aligned} \quad (22)$$

### III. MULTIPERIOD MODELS

#### A. GENERAL METHOD OF SOLUTION

By a multiperiod model is meant a theory of the following structure: The investor has determined a certain future point in time (his horizon) at which he plans to consume whatever wealth he has then available. He will still make his investment decisions with the objective of maximizing expected utility of wealth at that time. However, it is now assumed that the time between the present and his horizon can be subdivided into  $n$  periods (not necessarily of the same length), at the end of each of which return on the portfolio held during the period materializes and he can make a new decision on the composition of the portfolio to be held during the next period.

This formulation of the problem deliberately ignores possibilities for intermediate consumption. Consumption and portfolio decisions are clearly interrelated, and no defense for leaving consumption decisions out of the picture can

be offered except for the simple (but advantageous) strategy of taking one thing at a time. Such a partial analysis has been justified by picturing the investor as providing for a series of future consumption dates by dividing total wealth into separate portfolios for each consumption date, with each such portfolio to be managed independently.<sup>7</sup> This procedure must be rejected as clearly sub-optimal and hardly represents a satisfactory solution. Neither the decision on how much to consume in any given period nor the management of any given subportfolio could generally be independent of actual performance of other portfolios. The first attempts to consider squarely the interrelations between consumption and portfolio decisions appear to be represented by the still unpublished papers by Dreze and Modigliani and by Sandmo.<sup>8</sup>

In our version of the theory, an investor, starting out with a given initial wealth  $A_0$ , will make a first-period decision on the allocation of this wealth to different assets, then wait until the end of the period when a wealth level  $A_1$  materializes. He then makes a second-period decision on the allocation of  $A_1$ , and so on.

It is clear that for such a multiperiod planning problem it is rarely optimal, if at all possible, to specify a sequence of single-period decisions once and for all. Nor could it generally be optimal to simply make a first-period decision that would maximize expected utility of wealth at the end of that period while disregarding the investment opportunities in the second and later periods.

<sup>7</sup> *Ibid.*

<sup>8</sup> J. Dreze and F. Modigliani, "Consumption Decisions under Uncertainty" (manuscript in preparation); A. Sandmo, "Capital Risk, Consumption, and Portfolio Choice" (manuscript in preparation).

Rather, any sequence of portfolio decisions must be contingent upon the outcomes of previous periods and at the same time take into account information on future probability distributions. It is only when the last period is reached, and the final decision is to be taken, that the simple models of the preceding section are applicable.

At the beginning of the last period ( $n$ ) the investor's problem is simply to make a decision (call it  $d_n$ ), dividing his wealth as of that time,  $A_{n-1}$ , among the different assets such that  $E_n[U(A_n)]$  is maximized (the notation  $E_n$  indicates expectation with respect to probability distributions of yields during the  $n$ th period). But once he has thus chosen his optimal decision (depending, in general, upon  $A_{n-1}$ ), the maximum of expected utility of final wealth is determined solely in terms of  $A_{n-1}$ , that is,

$$\max_{d_n} E_n[U(A_n)] = \phi_{n-1}(A_{n-1}).$$

The function  $\phi_{n-1}$  is referred to as the "indirect" or "derived" utility function and is *the appropriate representation of preferences over probability distributions for  $A_{n-1}$* . Therefore, the optimal decision  $d_{n-1}$  to choose at the beginning of period  $n - 1$  is the one which maximizes

$$E_{n-1}[\phi_{n-1}(A_{n-1})] = E_{n-1}\{\max_{d_n} E_n[U(A_n)]\}.$$

In this way it is possible to consider the next-to-last decision as a simple one-period problem, *granted* that the objective is appropriately defined in terms of the "derived" utility function. But to do so obviously requires the investor to specify the optimal last-period decision for every possible outcome of yield during period  $n - 1$ . It is by means of such a backward-recursive procedure that it is possible to determine an optimal first-period decision.



Both for a theoretical development and for purposes of practical computation, the solution method is very much complicated if statistical dependence among yields in different periods (i.e., serial correlation) is to be allowed for. We shall therefore assume throughout that such dependence is absent, although

TABLE 1

ASSET 1		ASSET 2	
Yield	P	Yield	P
0.0	.5	-1.0	.5
0.2	.5	2.0	.5
$E_1 = .1$	$V_1 = .01$	$E_2 = .5$	$V_2 = 2.25$

Initial wealth:  $A_0 = 200$   
Utility function:  $U = A_2 - (1/1,000)A_2^2$

this certainly means some loss of generality. The basic nature of the approach is the same, however, and the conclusions we are to derive are certainly unaffected by this simplification. Dependence among yields *within* any period would be relatively easy to handle, but for a theoretical development it does not seem worth the extra trouble. Also, we shall ignore transaction costs.

#### B. A TWO-PERIOD EXAMPLE WITH QUADRATIC UTILITY

To illustrate the procedure, we shall develop in some detail a numerical example with two assets with random yields. To simplify the notation as much as possible, it is assumed that the yields  $X_1$  and  $X_2$  are independent and that their distributions are the same in both periods. We take  $a_1$  and  $a_2$  to be investment in the first asset in periods 1 and 2, respectively. The data of the example are given in Table 1.

The optimal decision for the second period is obtained from (18) as

$$a_2 = \frac{5.7A_1 - 400}{4.84}. \quad (23)$$

This expression defines the best possible decision for any value of wealth at the beginning of the second period. When this decision rule is adopted, the expected value of final wealth will be, according to (19):

$$\begin{aligned} \phi_1(A_1) &= \max_{a_2} E_2[U(A_2)] \\ &= \frac{2.94}{2.42} \left( A_1 - \frac{2.7675}{2490} A_1^2 \right) + \text{const.} \end{aligned}$$

It is the expectation of this function which is to be maximized by the first-period decision, which we achieve by

TABLE 2

$X_1$	$X_2$	$A_1$	$a_2$	$a_2$ in % of $A_1$
0.0	-1	161.2	107.2	66.5
0.2	-1	193.4	145.1	75.0
0.0	2	277.6	244.3	80.0
0.2	2	309.8	282.2	91.1

maximizing the expectation of the function in parentheses:

$$\max_{a_1} E_1 \left( A_1 - \frac{2.7675}{2490} A_1^2 \right).$$

Using formula (18) again (with  $a = 2.7675/2490$ ), we then get the optimal  $a_1$  as

$$a_1 = \frac{5.7A_0 - 360}{4.84} = 161.2. \quad (24)$$

Thus, the optimal decision to be effected immediately is to invest about 80.6 per cent in asset 1 and the remainder in asset 2.

The possible outcomes of the first-period portfolio are then as shown in Table 2 (each with probability  $\frac{1}{4}$ ).

The example illustrates several important points. First, although it is possible, and indeed necessary, to specify in advance an optimal *decision rule* for the second-period investment, it is not possible to determine the specific decision to be taken. This will depend upon the outcome of the "experiment" performed in the first period.

Second, the decision rules are generally different in different periods: the relationship in equation (23) between the optimal  $a_2$  and  $A_1$  is different from the relationship in equation (24) between the optimal  $a_1$  and  $A_0$ . The reason for this is that in period 1 the investor must take into account the probability distributions for the second period, because the "derived" utility function depends on it. More explicitly, it is seen that in period 1 he still maximizes a quadratic utility function, but the coefficient  $\alpha$  is now replaced by

$$\frac{V_1(1+E_2)^2 + V_2(1+E_1)^2 + V_1V_2}{V_1(1+E_2) + V_2(1+E_1)} \alpha$$

(cf. [19]). Thus, even if the investor happens to end up at the end of the first period with the same wealth as he had at the beginning (this is not actually possible in the example), he will make a different decision. In the example, with  $A_1 = 200$ , he would have invested a smaller amount in asset 1 ( $a_2 = 152.8$ ), thus making up for the loss of time left before the horizon by playing more boldly. Under certain conditions, it may be possible to identify such a "time effect," and we shall return to a discussion of that problem below (Sec. III D).

### C. UTILITY FUNCTIONS ALLOWING MYOPIC DECISIONS

As noted, it is generally non-optimal to make decisions for one period at a time without looking ahead. There may

be utility functions, however, for which such a procedure is optimal.

We shall say that if the investor's sequence of decisions is obtained as a series of single-period decisions (starting with the first period), where each period is treated as if it were the last one, then he behaves *myopically*. With myopia, the investor bases each period's decision on that period's initial wealth and probability distribution of yields only, with the objective of maximizing expected utility of final wealth in that period while disregarding the future completely. It is obvious that if it were optimal to make decisions in this manner, the problem of portfolio management would be greatly simplified. But, also on the theoretical level, it is interesting to isolate those utility functions for which such behavior is optimal.

The set of utility functions for which myopia is optimal will generally change according to assumptions about the nature of the asset yields. Here we shall analyze the case with one riskless asset with yield  $r$ . For arbitrary  $r$ , it will be shown that the only utility functions allowing myopic decision making are the logarithmic and power functions which we have encountered earlier. For  $r = 0$ , the set includes other functions; it is generally characterized by the condition

$$-\frac{U'(Y)}{U''(Y)} = \mu + \lambda Y.$$

It now turns out that this larger set still requires only a very modest amount of foresight even when  $r \neq 0$ . All that needs to be known about subsequent periods is the value (or values) of  $r$ , while information about the yield distribution for the risky asset is unnecessary. In such cases, the investor can make his immediate decision as if the entire resulting wealth would have to be invested at the

riskless rate(s) for all subsequent periods. We may characterize this behavior as "partial myopia." Thus, while it is not optimal to behave as if the immediate *period* were the last one (as with complete myopia), the investor can behave as if the immediate *decision* were the last one.

We shall begin by considering the decision for the next-to-last period; we can then revert to the simple notation used in Section II. Thus, we let  $Y$  represent final wealth, so that  $A$  is wealth at the end of the next-to-last period. We have here

$$Y = (1 + r)A + a(X - r). \quad (25)$$

Generally, the optimal last-period decision  $a$  is determined by the condition

$$E[U'(Y)(X - r)] = 0 \quad (26)$$

(Cf. [16]). Assuming (26) to hold, we have seen that the relevant utility function for evaluating the next-to-last decision is

$$\phi(A) = E[U(Y)].$$

If, on the other hand,  $A$  was to be invested entirely in the riskless asset, final wealth would be simply  $A^* = (1 + r)A$ . In that case, the relevant utility function for evaluating the next-to-last decision would be  $U(A^*)$ . Then, if this partial myopia is to lead to the correct decision,  $E[U(Y)]$  and  $U(A^*)$  must be equivalent as utility functions—that is, there must be constants  $b$  and  $c$  ( $b > 0$ ) such that

$$E[U(Y)] = bU(A^*) + c.$$

To eliminate the constant  $c$ , we differentiate with respect to  $A$ :

$$E \left\{ U'(Y) \left[ 1 + r + \frac{da}{dA} (X - r) \right] \right\} \\ = b(1 + r)U'(A^*),$$

or

$$(1 + r)E[U'(Y)] + E[U'(Y)(X - r)] \\ \times \frac{da}{dA} = b(1 + r)U'(A^*),$$

so in view of (26) we have the equivalent formulation

$$E[U'(Y)] = bU'(A^*). \quad (27)$$

If *complete* myopia is to be optimal, the corresponding condition becomes

$$E[U'(Y)] = bU'(A). \quad (28)$$

Of course, for the case  $r = 0$ , the two conditions coincide.

We shall take (27) first and show that a necessary and sufficient condition for (27) to hold is that the utility function is such that

$$-\frac{U'(Y)}{U''(Y)} = \mu + \lambda Y,$$

where  $\mu$  and  $\lambda$  are independent of  $Y$ . The function  $-U'(Y)/U''(Y)$  is the inverse of the absolute risk-aversion function and is sometimes referred to as the risk-tolerance function. Thus, we require risk tolerance to be linear in wealth. Apart from linear transformations, the following (and only the following) forms of utility functions satisfy this condition (for  $\mu \neq 0$ ):

$$\text{Exponential: } -e^{-Y/\mu} \quad (\text{for } \lambda = 0); \quad (29a)$$

$$\text{Logarithmic: } \ln(Y + \mu) \quad (\text{for } \lambda = 1); \quad (29b)$$

$$\text{Power: } \frac{1}{\lambda - 1} (\mu + \lambda Y)^{1-1/\lambda} \quad (29c) \\ (\text{otherwise}).$$

It is noticed that quadratic utility functions are included in this set as the special case  $\lambda = -1$ . When  $\mu = 0$  (constant relative risk aversion) we have seen that the set shrinks to  $\ln Y$  and  $Y^{1-1/\lambda}$ .

Suppose that

$$-\frac{U'(Y)}{U''(Y)} = \mu + \lambda Y, \quad (30)$$

that is,

$$\begin{aligned} -U'(Y) \\ = [\mu + \lambda A^* + \lambda a(X - r)]U''(Y). \end{aligned} \quad (31)$$

Multiplying through by  $(X - r)$  and taking expectations gives

$$\begin{aligned} -E[U'(Y)(X - r)] \\ = (\mu + \lambda A^*)E[U''(Y)(X - r)] \\ + \lambda aE[U''(Y)(X - r)^2]. \end{aligned}$$

But on the left we have zero, so that

$$-\frac{(1+r)E[U''(Y)(X-r)]}{E[U''(Y)(X-r)^2]} = \frac{(1+r)\lambda a}{\mu + \lambda A^*}.$$

Here we recognize the expression on the left as  $da/dA$  (cf. [17]); hence

$$\frac{da}{dA} = \frac{(1+r)\lambda a}{\mu + (1+r)\lambda A}.$$

This is a simple differential equation with general solution form

$$a = C[\mu + (1+r)\lambda A]. \quad (32)$$

Consequently,

$$\frac{da}{dA} = C(1+r)\lambda.$$

With this result we get

$$\begin{aligned} \mu + \lambda Y &= \mu + \lambda A^* + \lambda C(\mu + \lambda A^*)(X - r) \\ &= \frac{1}{1+r}(\mu + \lambda A^*) \\ &\quad \times \left[ 1 + r + \frac{da}{dA}(X - r) \right] \\ &= \frac{1}{1+r}(\mu + \lambda A^*)\frac{dY}{dA}. \end{aligned}$$

Therefore, from (31),

$$U'(Y) = \frac{1}{1+r} \frac{U'(A^*)}{U''(A^*)} U''(Y) \frac{dY}{dA},$$

so that

$$\begin{aligned} E[U'(Y)] &= \frac{1}{1+r} \frac{U'(A^*)}{U''(A^*)} \\ &\quad \times E \left[ U''(Y) \frac{dY}{dA} \right] \\ &= \frac{1}{1+r} \frac{U'(A^*)}{U''(A^*)} \\ &\quad \times \frac{d}{dA} E[U'(Y)]. \end{aligned}$$

This is the same as

$$\frac{d}{dA} \ln E[U'(Y)] = \frac{d}{dA} \ln U'(A^*)$$

hence

$$E[U'(Y)] = bU'(A^*),$$

where  $b$  is a constant of integration. This establishes the sufficiency of our condition for optimality of partial myopia.

To demonstrate necessity, we observe that the condition

$$E[U'(Y)] = bU'(A^*),$$

with  $b$  independent of  $A$ , can be satisfied only if there exists, for each value of  $X$ , a factor  $h(X)$  such that

$$U'(Y) = h(X)U'(A^*). \quad (33)$$

Then  $b$  is to be taken as  $E[h(X)]$ . For each outcome of yield  $X$ , the value of  $U'(Y)$  is proportional to  $U'(A^*)$  by a factor which is independent of  $A$ ; if this were not the case, the weighted sum  $b$  of such factors could not be independent of  $A$ , either.

Considering alternative outcomes of yield, we differentiate (33) with respect to  $X$ :

$$aU''(Y) = h'(X)U'(A^*). \quad (34)$$

For the particular value  $X = r$ , this becomes

$$aU''(A^*) = h'(r)U'(A^*),$$

so that

$$a = h'(r) \frac{U'(A^*)}{U''(A^*)}. \quad (35)$$

Thus, risky investment must be proportional to the risk-tolerance function. We substitute for  $a$  in (34) and get

$$U''(Y) = \frac{h'(X)}{h'(r)} U''(A^*).$$

Multiplying with  $(X - r)$  and  $(X - r)^2$  and taking expectations gives, respectively,

$$E[U''(Y)(X - r)] = k_1 U''(A^*)$$

and

$$E[U''(Y)(X - r)^2] = k_2 U''(A^*),$$

where  $k_1$  and  $k_2$  are constants. But this means that  $da/dA$  is constant or that  $a$  is a linear function of  $A$ . It then follows from (35) that the risk-tolerance function is also linear in wealth, say,

$$-\frac{U'(Y)}{U''(Y)} = \mu + \lambda Y. \quad (36)$$

The necessary and sufficient condition for *complete* myopia to be optimal (as defined by [28]) is that

$$-\frac{U'(Y)}{U''(Y)} = \lambda Y,$$

that is, relative risk-aversion should be constant. By an argument parallel to that given above, it will be found that (28) implies

$$a = h'(r) \frac{U'(A)}{U''(A^*)} \quad (37)$$

(corresponding to [35]) and further that  $-U'(A)/U''(A^*)$  is a linear function of

$A$ . This latter condition is clearly satisfied only if

$$-\frac{U'(A)}{U''(A)} = \mu + \lambda A.$$

However, from (32) we know that if

$$-\frac{U'(A)}{U''(A)} = \mu + \lambda A,$$

then  $a$  is proportional to  $-U'(A^*)/U''(A^*)$ . This must mean that (28) holds if and only if  $U'(A^*)$  and  $U'(A)$  are proportional, say,

$$U'(A^*) = v(r)U'(A).$$

Now differentiate with respect to  $A$  and  $r$ ; this gives

$$(1 + r)U''(A^*) = v(r)U''(A)$$

and

$$AU''(A^*) = v'(r)U'(A),$$

from which we get

$$\frac{U''(A)A}{U'(A)} = \frac{(1 + r)v'(r)}{v(r)}.$$

Since this must hold for independent variations in  $A$  and  $r$ , both sides must be constant, and consequently

$$-\frac{U'(A)}{U''(A)} = \lambda A,$$

proving our proposition.

We notice in addition that according to (21) and (22) the functions  $\ln Y$  and  $Y^{1-\gamma}$  also allow complete myopia when *both* yields are random. It can readily be shown that the condition is also necessary.

The results of this section are summarized in Table 3.

#### D. TIME EFFECTS AND STATIONARY PORTFOLIOS

The analysis of the preceding section was, at least in form, concerned with the relationship between the utility function

for the last period and the derived utility function for the next-to-last period. The main reason for this was notational simplicity; it is clear that what we were *really* concerned with was the derived utility functions for *any* two subsequent periods—that is, we really showed that if complete myopia is optimal, then the derived utility function for any period  $n - j$  is related to that of period  $n - j + 1$  by

$$\phi_{n-j}(A_{n-j}) \sim \phi_{n-j+1}(A_{n-j}) \quad (38)$$

(where  $\sim$  denotes equivalence as utility functions). It is obvious that if complete

where

$$z_{n-j} = \prod_{i=1}^j (1 + r_{n-j+1})$$

(with the interpretation  $z_n = 1$ )—that is, with partial myopia, the optimal decision  $a_{n-j}$ , when there are  $j + 1$  periods left to the horizon, is made as if the resulting wealth  $A_{n-j}$  were to be invested at the riskless rates for all remaining periods.

We are now in a position to discuss the question of whether an optimal portfolio policy can be *stationary* in the sense that the same proportion is invested in each asset in every period.<sup>9</sup> An investor who

TABLE 3

	NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY OF:	
	Complete Myopia	Partial Myopia
One riskless asset with zero yield.....	$-[U'(Y)/U''(Y)] = \mu + \lambda Y$	.....
One riskless asset with non-zero yield.....	$-[U'(Y)/U'''(Y)] = \lambda Y$	$-[U'(Y)/U'''(Y)] = \mu + \lambda Y$
Both assets with random yield.....	$-[U'(Y)/U'''(Y)] = \lambda Y$	.....

myopia is optimal for the next-to-last period, it is also optimal for all other periods (and conversely). For  $j = 1$ , (38) becomes

$$\phi_{n-1}(A_{n-1}) \sim U(A_{n-1})$$

so that, by induction, we have

$$\phi_{n-j}(A_{n-j}) \sim U(A_{n-j}) \quad (39)$$

Similarly, when partial myopia is optimal, the relation between derived utility functions for subsequent periods is

$$\phi_{n-j}(A_{n-j}) \sim \phi_{n-j+1}[(1 + r_{n-j+1})A_{n-j}], \quad (40)$$

and consequently also

$$\phi_{n-j}(A_{n-j}) \sim U(z_{n-j}A_{n-j}), \quad (41)$$

follows such a policy determines the proportions to be held in the different assets once and for all; at the end of each period he simply ploughs back the yield earned during the period in the same proportions.

Regardless of the investor's preferences, his optimal policy could not be stationary unless yield distributions were stationary. It is equally obvious that this condition is not sufficient. Under a stationary policy, decisions are independent of conditions during later periods, including their number. But this means that a stationary policy cannot be optimal un-

<sup>9</sup> Tobin, "Theory of Portfolio Selection," sec. 2.2.2.2.

less the investor's preferences are such that complete myopia is optimal, meaning that the utility function must be either  $\ln Y$  or  $Y^{1-\gamma}$ . But this is clearly also sufficient, because we have seen that these utility functions (and only these) allow the proportion held of each asset to be determined independently of total wealth.

In the case when one yield is zero with certainty, we have seen that other utility functions also allow complete myopia. However, for all these the optimal asset proportions depend upon the level of wealth at the beginning of the period. When this is changing from one period to the next, the asset proportions change accordingly (even if yield distributions are the same). We might, of course, conceivably happen to observe an investor with, say, a quadratic utility function holding a stationary portfolio because all yields turned out by chance to be zero and because yield distributions were the same from period to period. This would obviously be a rather exceptional occurrence. On the average, quadratic investors will, as we shall see shortly, reduce their holdings of the risky asset over time.

We thus conclude that a stationary portfolio policy is optimal if and only if both these conditions are satisfied: (1) the utility function is either  $\ln Y$  or  $Y^{1-\gamma}$ ; (2) yield distributions are identical in all periods.

On a couple of occasions allusion has been made to a "time effect" in multi-period portfolio problems. We can put the problem this way: With a given wealth and a given yield distribution for the immediate period, how does the optimal investment depend upon the number of periods left before the horizon? If complete myopia is optimal, there is, by defi-

nition, no time effect. When only partial myopia is optimal, such an effect is present, however, and is furthermore quite easy to analyze. We shall again consider the case with one riskless asset.

Using (41), we can write the equivalent of (33) as

$$U'(z_{n-j+1}A_{n-j+1}) = h_{n-j+1}(X_{n-j+1})U'(z_{n-j}A_{n-j}).$$

We will then get, corresponding to (35),

$$a_{n-j+1} = \frac{h'_{n-j+1}(r_{n-j+1})}{z_{n-j+1}} \frac{U'(z_{n-j}A_{n-j})}{U''(z_{n-j}A_{n-j})};$$

that is,

$$a_{n-j+1} = -h'_{n-j+1}(r_{n-j+1})(1 + r_{n-j+1}) \times \left[ \frac{\mu}{z_{n-j}} + \lambda A_{n-j} \right].$$

Now, if we had had  $\mu = 0$ , we know that  $a_{n-j+1}/A_{n-j}$  would have been independent of  $j$ , and therefore the factor  $h'_{n-j+1}(r_{n-j+1})(1 + r_{n-j+1})$  must be independent of  $j$ . We can therefore write

$$a_{n-j+1} = k_{n-j+1} \left[ \frac{\mu}{z_{n-j}} + \lambda A_{n-j} \right], \quad (42)$$

where  $k_{n-j+1}$  depends only upon  $r_{n-j+1}$  and the distribution of  $X_{n-j+1}$  but on nothing beyond the immediate period.

We can now see that the time effect depends upon the values of  $\mu$  and of the interest factor for the period between now and the horizon. Assuming all interest rates positive,  $z_{n-1}$  increases with  $j$ . We can then say that the time effect is of the same sign as  $\mu$  in the sense that for  $\mu > 0$  ( $\mu < 0$ ), investment in the risky asset will be larger (smaller) as the horizon is coming closer (given the characteristics of the immediate period). Thus, as indicated earlier, a quadratic utility function implies a positive time effect.

Since the sign of  $\mu$  is the same as the slope of the relative-risk-aversion function, we could alternatively say that the time effect is positive or negative according as relative risk aversion is increasing or decreasing with wealth.

A further observation on (42) is worth noting. As  $j$  increases, the importance of the value of  $\mu$  becomes smaller, or, to put it differently, even if complete myopia is not optimal, it is nearly so when the horizon is a long way off. This has a comforting practical implication: it is diffi-

cult to estimate interest rates for periods far into the future, but we now know that we do not need to worry much about it. Only when the horizon comes reasonably close do we have to be more careful in calculating optimal investments.

Do any of these results carry over to arbitrary utility functions? They seem reasonable enough, but the generalization does not appear easy to make. As one usually characterizes those problems one hasn't been able to solve oneself: this is a promising area for future research.