§1.5 曲线坐标系

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$$u_1 = f_1(x, y, z), \quad u_2 = f_2(x, y, z), \quad u_3 = f_3(x, y, z)$$

并且其反函数

$$x = x_1 = \varphi_1(u_1, u_2, u_3), \quad y = x_2 = \varphi_2(u_1, u_2, u_3), \quad z = x_3 = \varphi_3(u_1, u_2, u_3)$$

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用曲线坐标来描述空间点位置的坐标系称为一般曲线坐标系

曲线坐标系中，

位置矢量： $\vec{r} = x\,\hat{\boldsymbol{e}}_x + y\,\hat{\boldsymbol{e}}_y + z\,\hat{\boldsymbol{e}}_z = \vec{r}(x, y, z)$

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微分线元： $\mathsf{d}\,\vec{\boldsymbol{l}} \equiv \mathsf{d}\,\vec{\boldsymbol{r}} = \hat{\boldsymbol{e}}_x\,\mathsf{d}x + \hat{\boldsymbol{e}}_y\,\mathsf{d}y + \hat{\boldsymbol{e}}_z\,\mathsf{d}z$

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两曲面相交的曲线称为坐标曲线 u_1

（因为这时 u_2 和 u_3 已取定值，只有 u_1 可变）

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以 $\hat{e}_1$, $\hat{e}_2$ 和 $\hat{e}_3$ 分别表示坐标曲线 u_1，u_2 和 u_3的切线方向（方向分别沿u_1, u_2 和 u_3 增加的方向）的单位矢量。

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如果在空间任一点， $\hat{e}_1$，$\hat{e}_2$ 和 $\hat{e}_3$ 都两两垂直，则称为正交曲线坐标系。

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（h_1 用于归一化，使 $\hat{\boldsymbol{e}}_1$ 长度为1。）

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$$\begin{cases} x = \varphi_1(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \\ y = \varphi_2(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \\ z = \varphi_3(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \end{cases}$$

从其参数方程 φ_i 易求坐标曲线 u_1 切线方向的单位矢量 $\hat{\boldsymbol{e}}_1$:

$$\hat{\boldsymbol{e}}_1 = \frac{1}{h_1}\Big(\frac{\partial\varphi_1}{\partial u_1}\hat{\boldsymbol{e}}_x + \frac{\partial\varphi_2}{\partial u_1}\hat{\boldsymbol{e}}_y + \frac{\partial\varphi_3}{\partial u_1}\hat{\boldsymbol{e}}_z\Big) = \frac{1}{h_1}\Big(\frac{\partial x}{\partial u_1}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial u_1}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial u_1}\hat{\boldsymbol{e}}_z\Big)$$

$$h_1 = \Big[\Big(\frac{\partial x}{\partial u_1}\Big)^2 + \Big(\frac{\partial y}{\partial u_1}\Big)^2 + \Big(\frac{\partial z}{\partial u_1}\Big)^2\Big]^{1/2}$$

（h_1 用于归一化，使 $\hat{\boldsymbol{e}}_1$ 长度为1。）

同理：

坐标曲线 u_1 的方程:

表达为两曲面相交:

$$\begin{cases} u_2 = f_2(x,y,z) = c_2 \\ u_3 = f_3(x,y,z) = c_3 \end{cases}$$

表达为参数方程（以 u_1 为参数）：

$$\begin{cases} x = \varphi_1(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \\ y = \varphi_2(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \\ z = \varphi_3(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \end{cases}$$

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$$h_1 = \Big[\Big(\frac{\partial x}{\partial u_1}\Big)^2 + \Big(\frac{\partial y}{\partial u_1}\Big)^2 + \Big(\frac{\partial z}{\partial u_1}\Big)^2\Big]^{1/2}$$

（h_1 用于归一化，使 $\hat{\boldsymbol{e}}_1$ 长度为1。）

同理:

$$\hat{\boldsymbol{e}}_i = \frac{1}{h_i}\Big(\frac{\partial x}{\partial u_i}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial u_i}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial u_i}\hat{\boldsymbol{e}}_z\Big), \quad h_i = \Big[\Big(\frac{\partial x}{\partial u_i}\Big)^2 + \Big(\frac{\partial y}{\partial u_i}\Big)^2 + \Big(\frac{\partial z}{\partial u_i}\Big)^2\Big]^{1/2}$$

保持 u_2, u_3 不变，而 u_1 有一微小变化：$u_1 \Longrightarrow u_1 + \mathrm{d}u_1$

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微分线元： $\mathrm{d}\vec{\boldsymbol{l}}_1 = \vec{\boldsymbol{r}}(u_1 + \mathrm{d}u_1, u_2, u_3) - \vec{\boldsymbol{r}}(u_1, u_2, u_3)$

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&= \frac{\partial\, (x\, \hat{\boldsymbol{e}}_x + y\, \hat{\boldsymbol{e}}_y + z\, \hat{\boldsymbol{e}}_z)}{\partial u_1}\, \mathrm{d}u_1
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从而
$$\mathrm{d}\vec{\boldsymbol{l}}_1 = \left(\frac{\partial x}{\partial u_1}\, \hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial u_1}\, \hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial u_1}\, \hat{\boldsymbol{e}}_z \right) \mathrm{d}u_1$$

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因此
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$$
$$
h_1 = \left[\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2\right]^{1/2}
$$

类似可得

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$$\mathrm{d}\vec{\boldsymbol{l}}_i = \frac{\partial\,\vec{\boldsymbol{r}}(u_1,u_2,u_3)}{\partial u_i}\,\mathrm{d}u_i = h_i\,\hat{\boldsymbol{e}}_i\,\mathrm{d}u_i, \qquad i=1,2,3$$

类似可得

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类似可得

$$d\vec{l}_i = \frac{\partial\,\vec{r}(u_1,u_2,u_3)}{\partial u_i}\,du_i = h_i\,\hat{e}_i\,du_i, \qquad i=1,2,3$$

其中： $h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}, \qquad i=1,2,3$ 　称为度量因子

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微分线元

类似可得

$$\mathrm{d}\vec{\boldsymbol{l}}_i = \frac{\partial\,\vec{\boldsymbol{r}}(u_1,u_2,u_3)}{\partial u_i}\,\mathrm{d}u_i = h_i\,\hat{\boldsymbol{e}}_i\,\mathrm{d}u_i, \qquad i=1,2,3$$

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微分线元

$$\mathrm{d}\vec{\boldsymbol{l}} = \mathrm{d}\vec{\boldsymbol{r}} = \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_3}\,\mathrm{d}u_3$$

类似可得

$$\mathrm{d}\vec{\boldsymbol{l}}_i = \frac{\partial\,\vec{\boldsymbol{r}}(u_1,u_2,u_3)}{\partial u_i}\,\mathrm{d}u_i = h_i\,\hat{\boldsymbol{e}}_i\,\mathrm{d}u_i, \qquad i=1,2,3$$

其中： $$h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2+\left(\frac{\partial y}{\partial u_i}\right)^2+\left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}, \qquad i=1,2,3$$ 称为度量因子

微分线元

$$\mathrm{d}\vec{\boldsymbol{l}} = \mathrm{d}\vec{\boldsymbol{r}} = \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_1}\,\mathrm{d}u_1+\frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_2}\,\mathrm{d}u_2+\frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_3}\,\mathrm{d}u_3 = \vec{\boldsymbol{a}}_1\,\mathrm{d}u_1+\vec{\boldsymbol{a}}_2\,\mathrm{d}u_2+\vec{\boldsymbol{a}}_3\,\mathrm{d}u_3$$

类似可得

$$\mathrm{d}\vec{\boldsymbol{l}}_i = \frac{\partial\,\vec{\boldsymbol{r}}(u_1,u_2,u_3)}{\partial u_i}\,\mathrm{d}u_i = h_i\,\hat{\boldsymbol{e}}_i\,\mathrm{d}u_i, \qquad i=1,2,3$$

其中： $h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}, \qquad i=1,2,3$ 称为度量因子

微分线元

$$\begin{aligned}\mathrm{d}\vec{\boldsymbol{l}} = \mathrm{d}\vec{\boldsymbol{r}} &= \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_3}\,\mathrm{d}u_3 = \vec{\boldsymbol{a}}_1\,\mathrm{d}u_1 + \vec{\boldsymbol{a}}_2\,\mathrm{d}u_2 + \vec{\boldsymbol{a}}_3\,\mathrm{d}u_3 \\ &= \mathrm{d}\vec{\boldsymbol{l}}_1 + \mathrm{d}\vec{\boldsymbol{l}}_2 + \mathrm{d}\vec{\boldsymbol{l}}_3\end{aligned}$$

类似可得

$$\mathrm{d}\vec{\boldsymbol{l}}_i = \frac{\partial\,\vec{\boldsymbol{r}}(u_1,u_2,u_3)}{\partial u_i}\,\mathrm{d}u_i = h_i\,\hat{\boldsymbol{e}}_i\,\mathrm{d}u_i, \qquad i=1,2,3$$

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称为度量因子

微分线元

$$\begin{aligned}\mathrm{d}\vec{\boldsymbol{l}} = \mathrm{d}\vec{\boldsymbol{r}} &= \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial\,\vec{\boldsymbol{r}}}{\partial u_3}\,\mathrm{d}u_3 = \vec{\boldsymbol{a}}_1\,\mathrm{d}u_1 + \vec{\boldsymbol{a}}_2\,\mathrm{d}u_2 + \vec{\boldsymbol{a}}_3\,\mathrm{d}u_3 \\ &= \mathrm{d}\vec{\boldsymbol{l}}_1 + \mathrm{d}\vec{\boldsymbol{l}}_2 + \mathrm{d}\vec{\boldsymbol{l}}_3 \\ &= h_1\,\hat{\boldsymbol{e}}_1\,\mathrm{d}u_1 + h_2\,\hat{\boldsymbol{e}}_2\,\mathrm{d}u_2 + h_3\,\hat{\boldsymbol{e}}_3\,\mathrm{d}u_3\end{aligned}$$

类似可得

$$\mathrm{d}\vec{l}_i = \frac{\partial\,\vec{r}(u_1,u_2,u_3)}{\partial u_i}\,\mathrm{d}u_i = h_i\,\hat{e}_i\,\mathrm{d}u_i, \qquad i=1,2,3$$

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微分线元

$$\begin{aligned}\mathrm{d}\vec{l} = \mathrm{d}\vec{r} &= \frac{\partial\,\vec{r}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial\,\vec{r}}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial\,\vec{r}}{\partial u_3}\,\mathrm{d}u_3 = \vec{a}_1\,\mathrm{d}u_1 + \vec{a}_2\,\mathrm{d}u_2 + \vec{a}_3\,\mathrm{d}u_3 \\ &= \mathrm{d}\vec{l}_1 + \mathrm{d}\vec{l}_2 + \mathrm{d}\vec{l}_3 \\ &= h_1\,\hat{e}_1\,\mathrm{d}u_1 + h_2\,\hat{e}_2\,\mathrm{d}u_2 + h_3\,\hat{e}_3\,\mathrm{d}u_3\end{aligned}$$

$\hat{e}_i$： 坐标曲线 u_i 切线方向的单位矢量（方向沿 u_i 增加的方向）

类似可得

$$d\vec{l}_i = \frac{\partial \vec{r}(u_1, u_2, u_3)}{\partial u_i} du_i = h_i \hat{e}_i du_i, \qquad i = 1, 2, 3$$

其中： $h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}, \qquad i = 1, 2, 3$ 称为度量因子

微分线元

$$\begin{aligned} d\vec{l} = d\vec{r} &= \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = \vec{a}_1 du_1 + \vec{a}_2 du_2 + \vec{a}_3 du_3 \\ &= d\vec{l}_1 + d\vec{l}_2 + d\vec{l}_3 \\ &= h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3 \end{aligned}$$

$\hat{e}_i$: 坐标曲线 u_i 切线方向的单位矢量（方向沿 u_i 增加的方向）

微分线元： $$\boxed{d\vec{l} = h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3}$$

二、正交曲线坐标系中的梯度、散度、旋度

二、正交曲线坐标系中的梯度、散度、旋度

直角坐标系

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直角坐标系

全微分

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直角坐标系

全微分

$$\mathrm{d}\,T=\frac{\partial T}{\partial x}\mathrm{d}x+\frac{\partial T}{\partial y}\mathrm{d}y+\frac{\partial T}{\partial z}\mathrm{d}z$$

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直角坐标系 正交曲线坐标系

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$$\nabla\cdot(\vec{\boldsymbol{A}}) \quad = \nabla\cdot\left(A_1\,\hat{\boldsymbol{e}}_1 + A_2\,\hat{\boldsymbol{e}}_2 + A_3\,\hat{\boldsymbol{e}}_3\right)$$

与直角坐标系不同，这时 $\hat{\boldsymbol{e}}_i$不是常矢量

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[...]为$h_2h_3A_1$的梯度

$$= \left[\frac{1}{h_1}\frac{\partial(h_2h_3A_1)}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial(h_2h_3A_1)}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial(h_2h_3A_1)}{\partial u_3}\hat{\boldsymbol{e}}_3\right]\cdot\frac{\hat{\boldsymbol{e}}_1}{h_2h_3}$$

$$+ (h_2h_3A_1)\,[\underbrace{(\nabla\times\nabla u_2)}_{0}\cdot\nabla u_3 - \underbrace{(\nabla\times\nabla u_3)}_{0}\cdot\nabla u_2]$$

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$$+ (h_2h_3A_1)\,[\underbrace{(\nabla\times\nabla u_2)}_{0}\cdot\nabla u_3 - \underbrace{(\nabla\times\nabla u_3)}_{0}\cdot\nabla u_2]$$

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$$\nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1) = \nabla\cdot[A_1\,\underbrace{h_2h_3(\nabla u_2\times\nabla u_3)}_{\text{即为 }\hat{\boldsymbol{e}}_1}] = \nabla\cdot[h_2h_3A_1\,(\nabla u_2\times\nabla u_3)]$$

$$= [\nabla(h_2h_3A_1)]\cdot(\nabla u_2\times\nabla u_3) + (h_2h_3A_1)\,[\nabla\cdot(\nabla u_2\times\nabla u_3)]$$

[...]为$h_2h_3A_1$的梯度

$$= \left[\frac{1}{h_1}\frac{\partial(h_2h_3A_1)}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial(h_2h_3A_1)}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial(h_2h_3A_1)}{\partial u_3}\hat{\boldsymbol{e}}_3\right]\cdot\frac{\hat{\boldsymbol{e}}_1}{h_2h_3}$$

$$+ (h_2h_3A_1)\,[\underbrace{(\nabla\times\nabla u_2)}_{0}\cdot\nabla u_3 - \underbrace{(\nabla\times\nabla u_3)}_{0}\cdot\nabla u_2]$$

$$\nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1) = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3A_1)}{\partial u_1}$$

利用了 $\hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}_1 = 0$ 和 $\hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}_1 = 0$

类似可求得

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) \quad = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2}$$

类似可求得

$$\nabla \cdot (A_2\, \hat{\boldsymbol{e}}_2) = \frac{1}{h_1 h_2 h_3} \frac{\partial (h_3 h_1 A_2)}{\partial u_2} \qquad \nabla \cdot (A_3\, \hat{\boldsymbol{e}}_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial (h_1 h_2 A_3)}{\partial u_3}$$

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1) + \nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) + \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3)$$

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{A}} &= \nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\end{aligned}$$

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{A}} &= \nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\end{aligned}$$

旋度

$$\nabla\times\vec{\boldsymbol{A}} = \nabla\times(A_1\,\hat{\boldsymbol{e}}_1+A_2\,\hat{\boldsymbol{e}}_2+A_3\,\hat{\boldsymbol{e}}_3)$$

不同于直角坐标系，这里 $\hat{\boldsymbol{e}}_i$不是常矢量

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{A}} &= \nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\end{aligned}$$

旋度

$$\begin{aligned}\nabla\times\vec{\boldsymbol{A}} &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1+A_2\,\hat{\boldsymbol{e}}_2+A_3\,\hat{\boldsymbol{e}}_3)\\ &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)\end{aligned}$$

不同于直角坐标系，这里 $\hat{\boldsymbol{e}}_i$不是常矢量

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{A}} &= \nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\end{aligned}$$

旋度

$$\begin{aligned}\nabla\times\vec{\boldsymbol{A}} &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1+A_2\,\hat{\boldsymbol{e}}_2+A_3\,\hat{\boldsymbol{e}}_3)\\ &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)\end{aligned}$$

不同于直角坐标系，这里 $\hat{\boldsymbol{e}}_i$不是常矢量

利用 $\nabla u_i = \dfrac{\hat{\boldsymbol{e}}_i}{h_i}$

类似可求得

$$\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{A}} &= \nabla\cdot(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\cdot(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\cdot(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\end{aligned}$$

旋度

$$\begin{aligned}\nabla\times\vec{\boldsymbol{A}} &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1+A_2\,\hat{\boldsymbol{e}}_2+A_3\,\hat{\boldsymbol{e}}_3)\\ &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \nabla\times(A_1h_1\nabla u_1)+\nabla\times(A_2h_2\nabla u_2)+\nabla\times(A_3h_3\nabla u_3)\end{aligned}$$

不同于直角坐标系，这里 $\hat{\boldsymbol{e}}_i$不是常矢量

利用 $\nabla u_i = \dfrac{\hat{\boldsymbol{e}}_i}{h_i}$

$$\nabla \times (A_1 h_1 \nabla u_1)$$

利用 $\nabla\times(f\vec{\boldsymbol{a}})=(\nabla f)\times\vec{\boldsymbol{a}}+f\nabla\times\vec{\boldsymbol{a}}$

$\nabla\times(A_1 h_1 \nabla u_1)$

利用 $\nabla \times (f\vec{a}) = (\nabla f) \times \vec{a} + f\nabla \times \vec{a}$

$$\nabla \times (A_1 h_1 \nabla u_1) = [\,\nabla(A_1 h_1)\,] \times \nabla u_1 + (A_1 h_1)\,[\,\nabla \times (\nabla u_1)\,]$$

利用 $\nabla\times(f\vec{\boldsymbol{a}}) = (\nabla f)\times\vec{\boldsymbol{a}} + f\nabla\times\vec{\boldsymbol{a}}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\nabla u_1 + (A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}$$

Let there be light

利用 $\nabla \times (f\vec{a}) = (\nabla f) \times \vec{a} + f\nabla \times \vec{a}$

$$\nabla \times (A_1 h_1 \nabla u_1) = [\nabla(A_1 h_1)] \times \nabla u_1 + (A_1 h_1) \underbrace{[\nabla \times (\nabla u_1)]}_{0}$$

利用 $\nabla\times(f\vec{a}) = (\nabla f)\times\vec{a} + f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\underbrace{\nabla u_1} + (A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

利用 $\nabla\times(f\vec{a})=(\nabla f)\times\vec{a}+f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1)=[\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}}+(A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

利用 $\nabla\times(f\vec{a})=(\nabla f)\times\vec{a}+f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1)=[\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}}+(A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

$$\nabla(A_1h_1)=\frac{1}{h_1}\frac{\partial(A_1h_1)}{\partial u_1}\hat{e}_1+\frac{1}{h_2}\frac{\partial(A_1h_1)}{\partial u_2}\hat{e}_2+\frac{1}{h_3}\frac{\partial(A_1h_1)}{\partial u_3}\hat{e}_3$$

利用 $\nabla\times(f\vec{a})=(\nabla f)\times\vec{a}+f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{\boldsymbol{e}}_1}{h_1}}+(A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

$$\nabla(A_1h_1) = \frac{1}{h_1}\frac{\partial(A_1h_1)}{\partial u_1}\hat{\boldsymbol{e}}_1+\frac{1}{h_2}\frac{\partial(A_1h_1)}{\partial u_2}\hat{\boldsymbol{e}}_2+\frac{1}{h_3}\frac{\partial(A_1h_1)}{\partial u_3}\hat{\boldsymbol{e}}_3$$

$$\nabla\times(A_1\hat{\boldsymbol{e}}_1)=\nabla\times(A_1h_1\nabla u_1)$$

利用 $\nabla \times (f\vec{a}) = (\nabla f) \times \vec{a} + f\nabla \times \vec{a}$

$$\nabla \times (A_1 h_1 \nabla u_1) = [\nabla(A_1 h_1)] \times \underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}} + (A_1 h_1)\underbrace{[\nabla \times (\nabla u_1)]}_{0}$$

$$\nabla(A_1 h_1) = \frac{1}{h_1}\frac{\partial(A_1 h_1)}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial(A_1 h_1)}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial(A_1 h_1)}{\partial u_3}\hat{e}_3$$

$$\nabla \times (A_1\,\hat{e}_1) = \nabla \times (A_1 h_1 \nabla u_1) = [\nabla(A_1 h_1)] \times \frac{\hat{e}_1}{h_1}$$

利用 $\nabla\times(f\vec{a}) = (\nabla f)\times\vec{a} + f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}} + (A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

$$\nabla(A_1h_1) = \frac{1}{h_1}\frac{\partial(A_1h_1)}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial(A_1h_1)}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial(A_1h_1)}{\partial u_3}\hat{e}_3$$

$$\nabla\times(A_1\,\hat{e}_1) = \nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\frac{\hat{e}_1}{h_1}$$

利用 $\hat{e}_1\times\hat{e}_1 = 0$, $\hat{e}_2\times\hat{e}_1 = -\hat{e}_3$ 和 $\hat{e}_3\times\hat{e}_1 = \hat{e}_2$

利用 $\nabla\times(f\vec{a}) = (\nabla f)\times\vec{a} + f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}} + (A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

$$\nabla(A_1h_1) = \frac{1}{h_1}\frac{\partial(A_1h_1)}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial(A_1h_1)}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial(A_1h_1)}{\partial u_3}\hat{e}_3$$

$$\nabla\times(A_1\hat{e}_1) = \nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\frac{\hat{e}_1}{h_1}$$

利用 $\hat{e}_1\times\hat{e}_1 = 0,\ \hat{e}_2\times\hat{e}_1 = -\hat{e}_3$ 和 $\hat{e}_3\times\hat{e}_1 = \hat{e}_2$

$$\nabla\times(A_1\hat{e}_1) = \frac{\hat{e}_2}{h_1h_3}\frac{\partial(A_1h_1)}{\partial u_3} - \frac{\hat{e}_3}{h_1h_2}\frac{\partial(A_1h_1)}{\partial u_2}$$

利用 $\nabla\times(f\vec{a})=(\nabla f)\times\vec{a}+f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}}+(A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

$$\nabla(A_1h_1) = \frac{1}{h_1}\frac{\partial(A_1h_1)}{\partial u_1}\hat{e}_1+\frac{1}{h_2}\frac{\partial(A_1h_1)}{\partial u_2}\hat{e}_2+\frac{1}{h_3}\frac{\partial(A_1h_1)}{\partial u_3}\hat{e}_3$$

$$\nabla\times(A_1\,\hat{e}_1)=\nabla\times(A_1h_1\nabla u_1)=[\nabla(A_1h_1)]\times\frac{\hat{e}_1}{h_1}$$

利用 $\hat{e}_1\times\hat{e}_1=0,\ \hat{e}_2\times\hat{e}_1=-\hat{e}_3$ 和 $\hat{e}_3\times\hat{e}_1=\hat{e}_2$

$$\nabla\times(A_1\,\hat{e}_1)=\frac{\hat{e}_2}{h_1h_3}\frac{\partial(A_1h_1)}{\partial u_3}-\frac{\hat{e}_3}{h_1h_2}\frac{\partial(A_1h_1)}{\partial u_2}$$

类似可得

$$\nabla\times(A_2\,\hat{e}_2)=\frac{\hat{e}_3}{h_2h_1}\frac{\partial(A_2h_2)}{\partial u_1}-\frac{\hat{e}_1}{h_2h_3}\frac{\partial(A_2h_2)}{\partial u_3}$$

利用 $\nabla \times (f\vec{a}) = (\nabla f) \times \vec{a} + f\nabla \times \vec{a}$

$$\nabla \times (A_1 h_1 \nabla u_1) = [\nabla (A_1 h_1)] \times \underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}} + (A_1 h_1) \underbrace{[\nabla \times (\nabla u_1)]}_{0}$$

$$\nabla (A_1 h_1) = \frac{1}{h_1}\frac{\partial (A_1 h_1)}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial (A_1 h_1)}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial (A_1 h_1)}{\partial u_3}\hat{e}_3$$

$$\nabla \times (A_1\,\hat{e}_1) = \nabla \times (A_1 h_1 \nabla u_1) = [\nabla (A_1 h_1)] \times \frac{\hat{e}_1}{h_1}$$

利用 $\hat{e}_1 \times \hat{e}_1 = 0$, $\hat{e}_2 \times \hat{e}_1 = -\hat{e}_3$ 和 $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$

$$\nabla \times (A_1\,\hat{e}_1) = \frac{\hat{e}_2}{h_1 h_3}\frac{\partial (A_1 h_1)}{\partial u_3} - \frac{\hat{e}_3}{h_1 h_2}\frac{\partial (A_1 h_1)}{\partial u_2}$$

类似可得

$$\nabla \times (A_2\,\hat{e}_2) = \frac{\hat{e}_3}{h_2 h_1}\frac{\partial (A_2 h_2)}{\partial u_1} - \frac{\hat{e}_1}{h_2 h_3}\frac{\partial (A_2 h_2)}{\partial u_3}$$

$$\nabla \times (A_3\,\hat{e}_3) = \frac{\hat{e}_1}{h_3 h_2}\frac{\partial (A_3 h_3)}{\partial u_2} - \frac{\hat{e}_2}{h_3 h_1}\frac{\partial (A_3 h_3)}{\partial u_1}$$

Let there be light

$\nabla \times \vec{\boldsymbol{A}}$

Let there be light

$$\nabla \times \vec{\boldsymbol{A}} = \nabla \times (A_1\, \hat{\boldsymbol{e}}_1) + \nabla \times (A_2\, \hat{\boldsymbol{e}}_2) + \nabla \times (A_3\, \hat{\boldsymbol{e}}_3)$$

$$\begin{aligned}\nabla\times\vec{\boldsymbol{A}} &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}\end{aligned}$$

$$\nabla \times \vec{\boldsymbol{A}} = \nabla \times (A_1\,\hat{\boldsymbol{e}}_1) + \nabla \times (A_2\,\hat{\boldsymbol{e}}_2) + \nabla \times (A_3\,\hat{\boldsymbol{e}}_3)$$

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \longrightarrow \text{直角坐标} \quad \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla\times\vec{\boldsymbol{A}} = \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)$$

$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \longrightarrow \text{直角坐标} \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

$$
\begin{aligned}
\nabla\times\vec{\boldsymbol{A}} &= \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)\\
&= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix} \quad \longrightarrow \text{直角坐标} \quad \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}\\
\nabla\cdot\vec{\boldsymbol{A}} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\\
\nabla T &= \frac{1}{h_1}\frac{\partial T}{\partial u_1}\,\hat{\boldsymbol{e}}_1+\frac{1}{h_2}\frac{\partial T}{\partial u_2}\,\hat{\boldsymbol{e}}_2+\frac{1}{h_3}\frac{\partial T}{\partial u_3}\,\hat{\boldsymbol{e}}_3
\end{aligned}
$$

$$\nabla\times\vec{A} = \nabla\times(A_1\,\hat{e}_1)+\nabla\times(A_2\,\hat{e}_2)+\nabla\times(A_3\,\hat{e}_3)$$

$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\,\hat{e}_1 & h_2\,\hat{e}_2 & h_3\,\hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \longrightarrow \text{直角坐标} \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\,\hat{e}_1+\frac{1}{h_2}\frac{\partial T}{\partial u_2}\,\hat{e}_2+\frac{1}{h_3}\frac{\partial T}{\partial u_3}\,\hat{e}_3$$

最后

$$\nabla^2T = \nabla\cdot(\nabla T)$$

$$\nabla \times \vec{\boldsymbol{A}} = \nabla \times (A_1\,\hat{\boldsymbol{e}}_1) + \nabla \times (A_2\,\hat{\boldsymbol{e}}_2) + \nabla \times (A_3\,\hat{\boldsymbol{e}}_3)$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \longrightarrow \text{直角坐标} \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial(h_1 h_2 A_3)}{\partial u_3}\right]$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

最后

$$\nabla^2 T = \nabla \cdot (\nabla T) = \nabla \cdot \Big(\overbrace{\frac{1}{h_1}\frac{\partial T}{\partial u_1}}^{A_1}\hat{\boldsymbol{e}}_1 + \overbrace{\frac{1}{h_2}\frac{\partial T}{\partial u_2}}^{A_2}\hat{\boldsymbol{e}}_2 + \overbrace{\frac{1}{h_3}\frac{\partial T}{\partial u_3}}^{A_3}\hat{\boldsymbol{e}}_3\Big)$$

$$\nabla\times\vec{A} = \nabla\times(A_1\,\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\,\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\,\hat{\boldsymbol{e}}_3)$$

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$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\Big)+\frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\Big)+\frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\Big)\right]$$

三、球坐标系中的梯度、散度、旋度

三、球坐标系中的梯度、散度、旋度

要求梯度、散度、旋度，关键在于求

$$h_i = \left[\left(\frac{\partial x}{\partial u_i} \right)^2 + \left(\frac{\partial y}{\partial u_i} \right)^2 + \left(\frac{\partial z}{\partial u_i} \right)^2 \right]^{1/2}$$

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为求 h_i，需要知道：

$$\begin{aligned} x &= \varphi_1(u_1, u_2, u_3) = \varphi_1(r, \theta, \phi) \\ y &= \varphi_2(u_1, u_2, u_3) = \varphi_2(r, \theta, \phi) \\ z &= \varphi_3(u_1, u_2, u_3) = \varphi_3(r, \theta, \phi) \end{aligned}$$

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对球坐标系：

$u_1 = r$ 为 P 点到坐标原点的距离，

$u_2 = \theta$ 称为极角(polar angle)，

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对球坐标系，熟知

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta \end{cases}$$

三、球坐标系中的梯度、散度、旋度

要求梯度、散度、旋度，关键在于求

$$h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}$$

为求 h_i，需要知道：

$$x = \varphi_1(u_1, u_2, u_3) = \varphi_1(r, \theta, \phi)$$
$$y = \varphi_2(u_1, u_2, u_3) = \varphi_2(r, \theta, \phi)$$
$$z = \varphi_3(u_1, u_2, u_3) = \varphi_3(r, \theta, \phi)$$

对球坐标系：

$u_1 = r$ 为 P 点到坐标原点的距离，

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对球坐标系，熟知

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta \end{cases}$$

从而：

$$\begin{cases} h_1 = 1 \\ h_2 = r \\ h_3 = r\sin\theta \end{cases}$$

$$\nabla T \quad = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入}\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

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代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

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代入 $\begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$

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代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

代入 $\begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2A_r)}{\partial r} + r\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + r\frac{\partial A_\phi}{\partial\phi}\right]$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

代入 $\begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2A_r)}{\partial r} + r\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + r\frac{\partial A_\phi}{\partial\phi}\right]$$

$$\nabla\times\vec{\boldsymbol{A}}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

代入 $\begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2A_r)}{\partial r} + r\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + r\frac{\partial A_\phi}{\partial \phi}\right]$$

$$\nabla\times\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

代入 $\begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2A_r)}{\partial r} + r\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + r\frac{\partial A_\phi}{\partial\phi}\right]$$

$$\nabla\times\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\theta & r\sin\theta\,\hat{\boldsymbol{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$

代入 $\begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2A_r)}{\partial r} + r\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + r\frac{\partial A_\phi}{\partial\phi}\right]$$

$$\nabla\times\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\,\hat{\boldsymbol{e}}_\theta & r\sin\theta\,\hat{\boldsymbol{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\hat{\boldsymbol{e}}_\theta$$
$$+ \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{e}}_\phi$$

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_3 h_1}{h_2}\frac{\partial T}{\partial u_2}\Big) + \frac{\partial}{\partial u_3}\Big(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\Big)\Big]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2\sin\theta}\Big[\sin\theta\frac{\partial}{\partial r}\Big(r^2\frac{\partial T}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial T}{\partial\theta}\Big) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\phi^2}\Big]$$

$$\nabla^2 T = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\phi^2}\right]$$

四、柱坐标系中的梯度、散度、旋度

$$\nabla^2 T = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\phi^2}\right]$$

四、柱坐标系中的梯度、散度、旋度

对柱坐标系，有

$$\begin{cases} u_1 = s & \text{为 } P \text{ 点到 } z \text{ 轴的距离} \\ u_2 = \phi & \text{同球坐标的 } \phi \\ u_3 = z & \text{同直角坐标的 } z \end{cases}$$

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$

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$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$

四、柱坐标系中的梯度、散度、旋度

对柱坐标系，有

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$$\nabla^2 T = \frac{1}{h_1h_2h_3}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\Big)+\frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\Big)+\frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\Big)\Big]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2\sin\theta}\Big[\sin\theta\frac{\partial}{\partial r}\Big(r^2\frac{\partial T}{\partial r}\Big)+\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial T}{\partial\theta}\Big)+\frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\phi^2}\Big]$$

四、柱坐标系中的梯度、散度、旋度

对柱坐标系，有

$\begin{cases} u_1 = s & \text{为 } P \text{ 点到 } z \text{ 轴的距离} \\ u_2 = \phi & \text{同球坐标的 } \phi \\ u_3 = z & \text{同直角坐标的 } z \end{cases}$ 且 $\begin{cases} x = s\cos\phi, \\ y = s\sin\phi, \\ z = z \end{cases}$ 从而 $\begin{cases} h_1 = 1 \\ h_2 = s \\ h_3 = 1 \end{cases}$

利用了$h_i = \Big[\Big(\frac{\partial x}{\partial u_i}\Big)^2+\Big(\frac{\partial y}{\partial u_i}\Big)^2+\Big(\frac{\partial z}{\partial u_i}\Big)^2\Big]^{1/2}$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3$$

代入 $\begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$

$$= \frac{\partial T}{\partial s}\hat{\boldsymbol{e}}_s + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

$$= \frac{\partial T}{\partial s}\hat{\boldsymbol{e}}_s + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial(h_1 h_2 A_3)}{\partial u_3}\right]$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入}\begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

$$= \frac{\partial T}{\partial s}\hat{\boldsymbol{e}}_s + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial(h_1 h_2 A_3)}{\partial u_3}\right] \quad \text{代入}\begin{cases} A_1 = A_s \\ A_2 = A_\phi \\ A_3 = A_z \end{cases}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

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$$= \frac{1}{s}\left[\frac{\partial(s A_s)}{\partial s} + \frac{\partial(A_\phi)}{\partial \phi} + s\frac{\partial A_z}{\partial z}\right]$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入}\begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

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$$\nabla \times \vec{\boldsymbol{A}}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

$$= \frac{\partial T}{\partial s}\hat{\boldsymbol{e}}_s + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial(h_1 h_2 A_3)}{\partial u_3}\right] \quad \text{代入} \begin{cases} A_1 = A_s \\ A_2 = A_\phi \\ A_3 = A_z \end{cases}$$

$$= \frac{1}{s}\left[\frac{\partial(s A_s)}{\partial s} + \frac{\partial(A_\phi)}{\partial \phi} + s\frac{\partial A_z}{\partial z}\right]$$

$$\nabla \times \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

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$$= \frac{1}{s}\left[\frac{\partial(sA_s)}{\partial s} + \frac{\partial(A_\phi)}{\partial \phi} + s\frac{\partial A_z}{\partial z}\right]$$

$$\nabla \times \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\boldsymbol{e}}_1 & h_2 \hat{\boldsymbol{e}}_2 & h_3 \hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{s}\begin{vmatrix} \hat{\boldsymbol{e}}_s & s\,\hat{\boldsymbol{e}}_\phi & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial s} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入}\begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

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$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial(h_1 h_2 A_3)}{\partial u_3}\right] \quad \text{代入}\begin{cases} A_1 = A_s \\ A_2 = A_\phi \\ A_3 = A_z \end{cases}$$

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$$\nabla \times \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{s}\begin{vmatrix} \hat{\boldsymbol{e}}_s & s\,\hat{\boldsymbol{e}}_\phi & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial s} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix}$$

$$= \left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{e}}_s + \left[\frac{\partial A_s}{\partial z} - \frac{\partial(A_z)}{\partial s}\right]\hat{\boldsymbol{e}}_\phi + \frac{1}{s}\left[\frac{\partial(sA_\phi)}{\partial s} - \frac{\partial A_s}{\partial \phi}\right]\hat{\boldsymbol{e}}_z$$

$$\nabla^2 T = \frac{1}{h_1h_2h_3}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\Big) + \frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\Big)\Big]$$

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代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$\nabla^2 T = \frac{1}{h_1h_2h_3}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\Big) + \frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\Big)\Big]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{s}\frac{\partial}{\partial r}\Big(s\frac{\partial T}{\partial s}\Big) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$