§3.6 静电边值问题：多极矩展开

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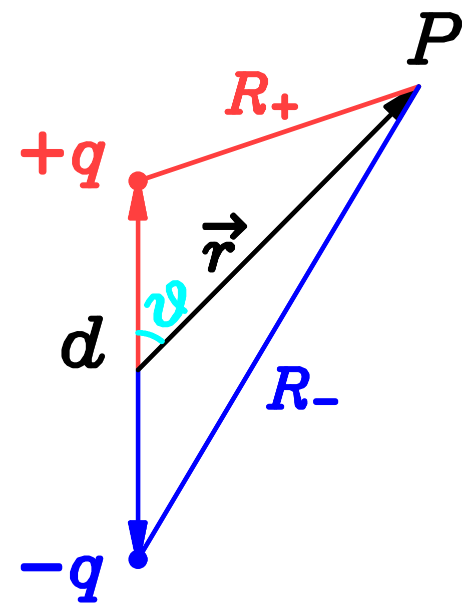

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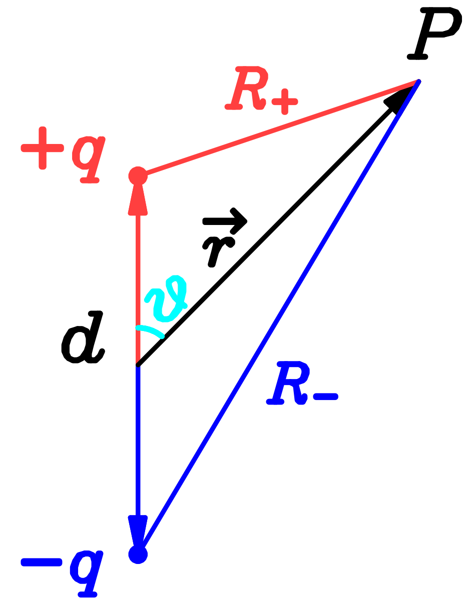

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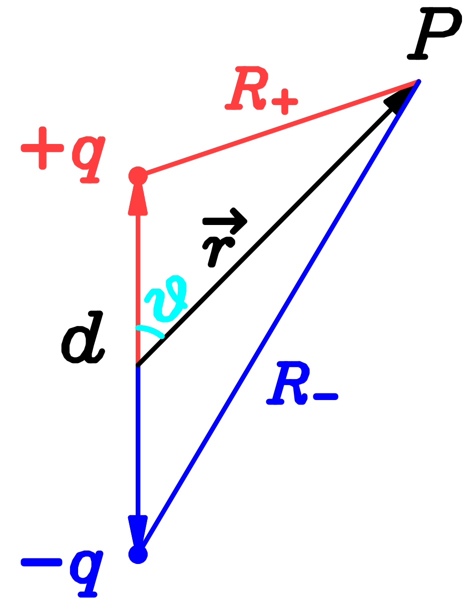

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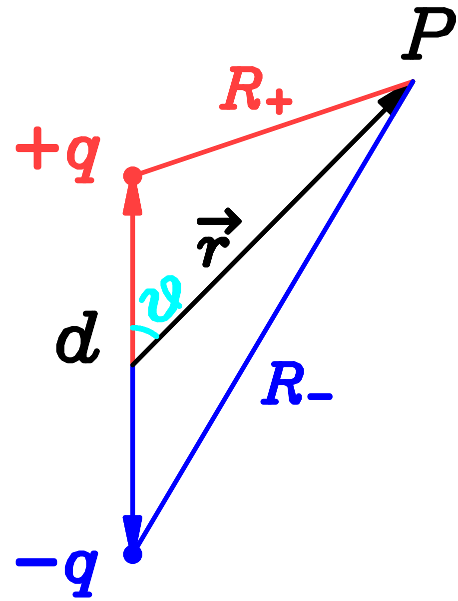

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从而电势可近似为：$\varphi(\vec{r}) \approx \dfrac{1}{4\pi\epsilon_0}\dfrac{qd\cos\theta}{r^2} \approx \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}\cdot\hat{e}_r}{r^2}$ (偶极子电势) $\sim \left(\dfrac{d}{r}\right)\dfrac{1}{r}$

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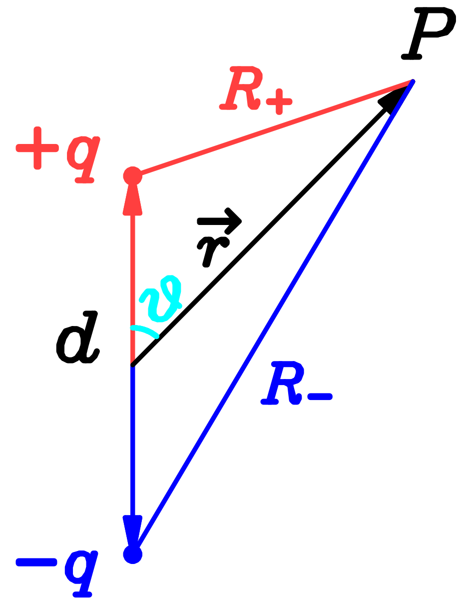

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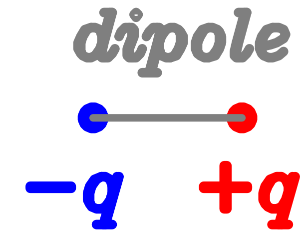
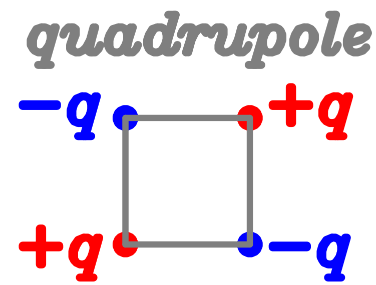
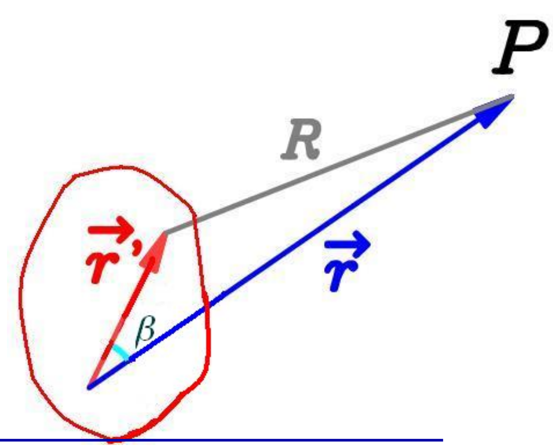
其中：$\vec{\boldsymbol{p}} = \displaystyle\sum_i q_i \vec{\boldsymbol{r}}_i'$， $\vec{\boldsymbol{r}}_i'$ 为 q_i 电荷所在位置，d 为电荷分布尺度

一个电荷构成单极子 (monopole)，$Q \neq 0$，







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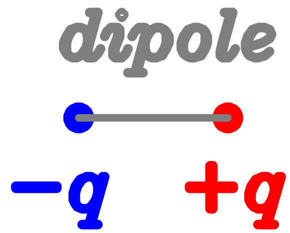

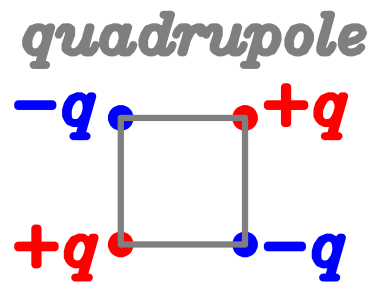

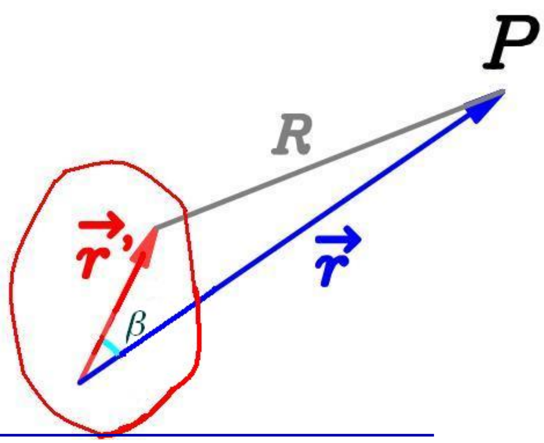

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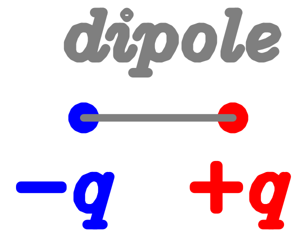

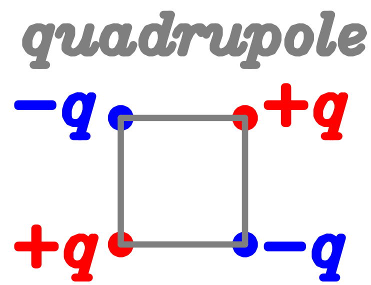

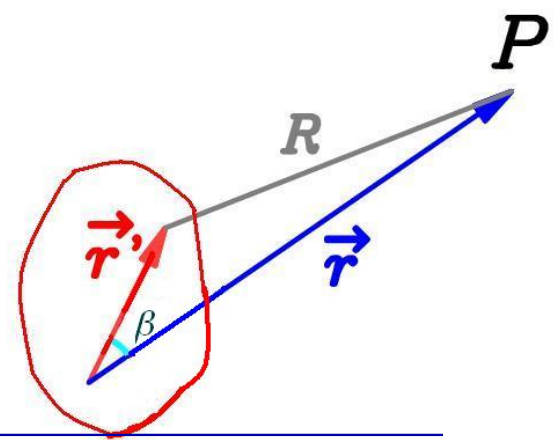

Let there be light

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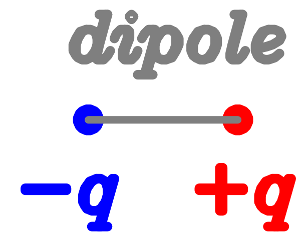

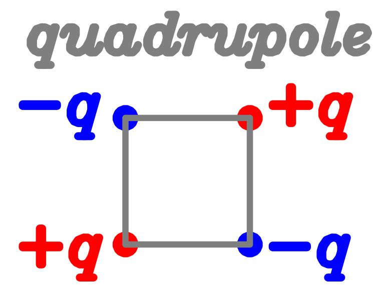

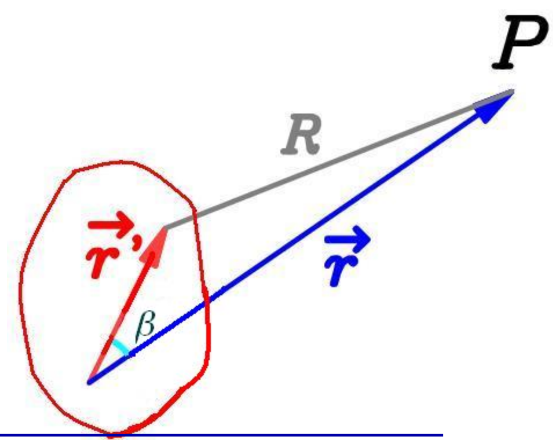

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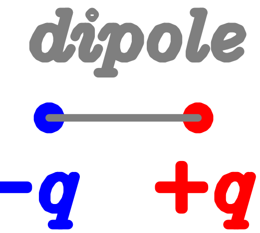

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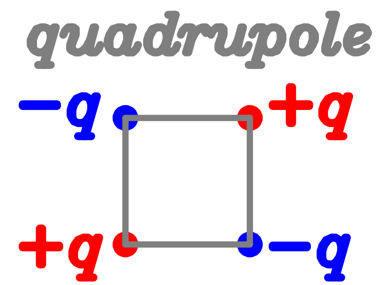

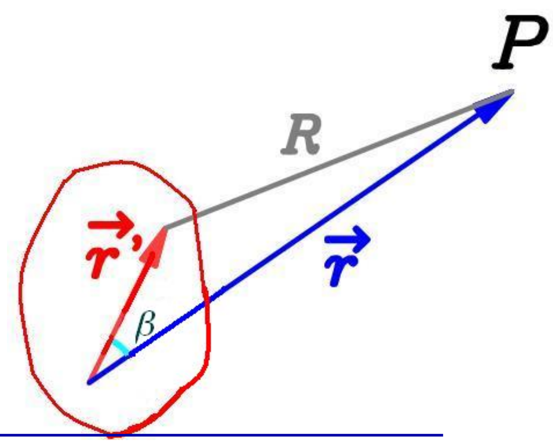

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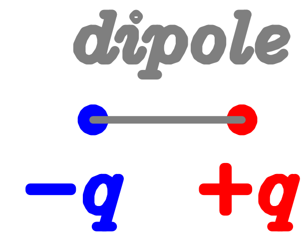

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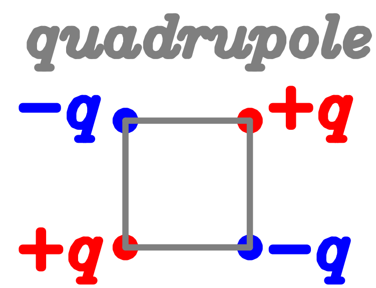

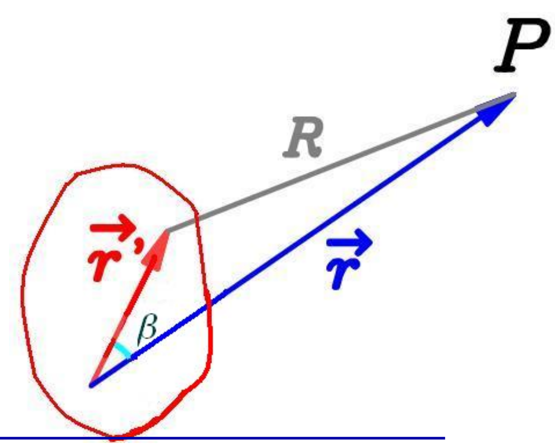

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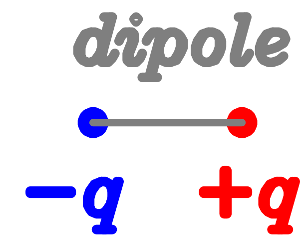

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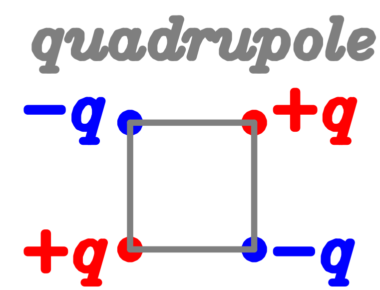

对一般的电荷分布，是否也能写成，如何写成如下形式：

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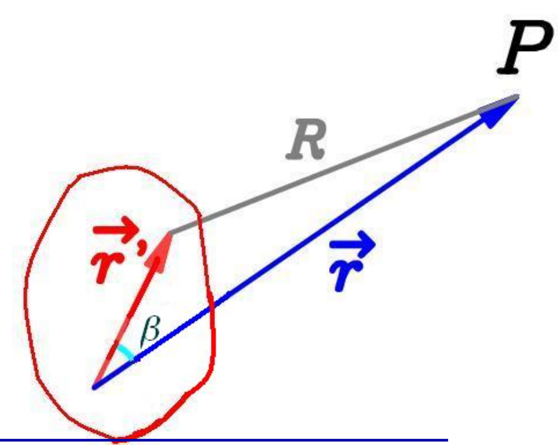

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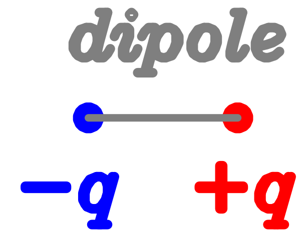

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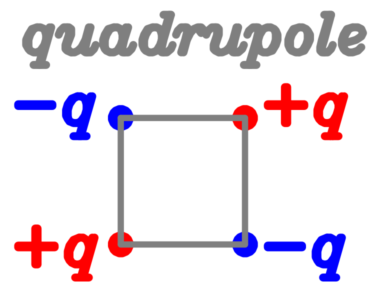

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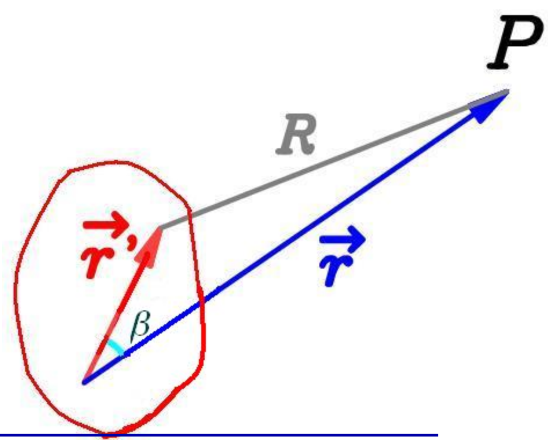
多极矩展开 (multipole expansion)



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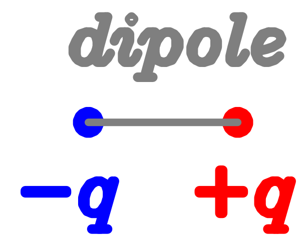

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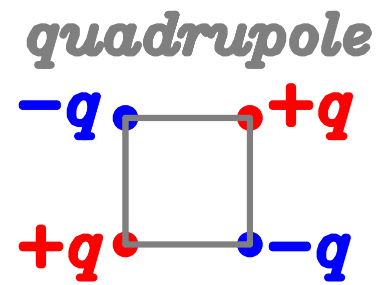

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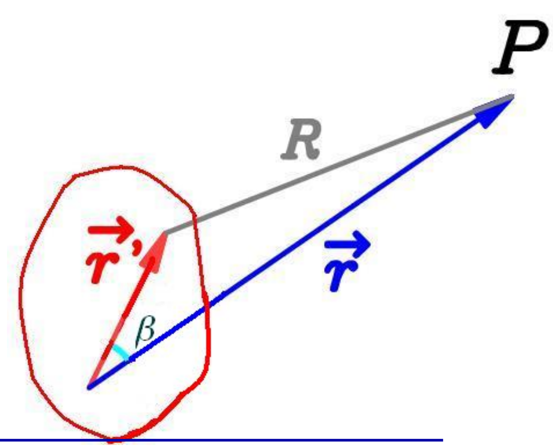
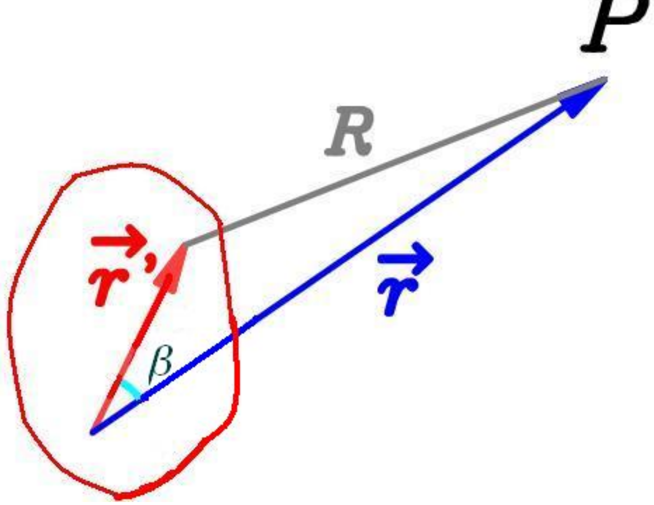


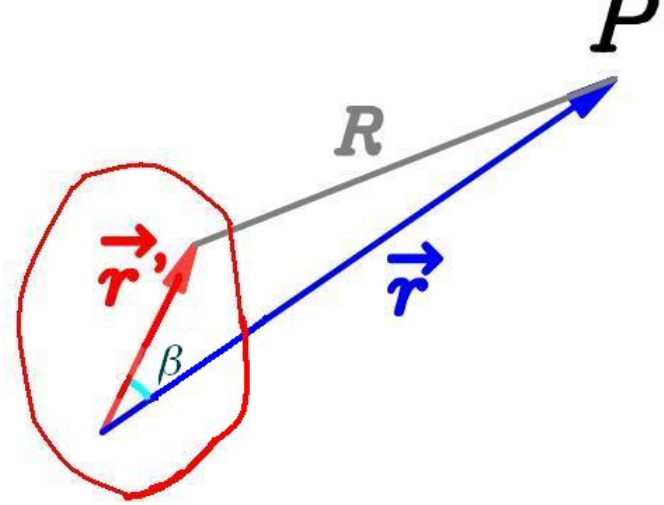
Let there be light

$$\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,\mathrm{d}\tau'$$



Let there be light

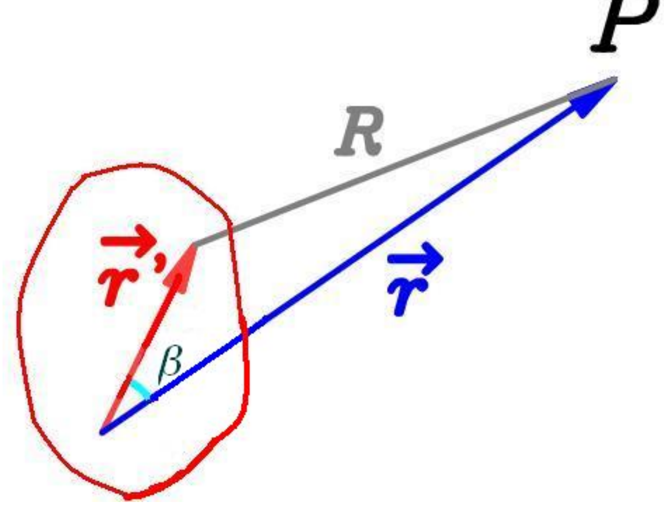
$$\begin{aligned}\varphi(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}}\end{aligned}$$



Let there be light

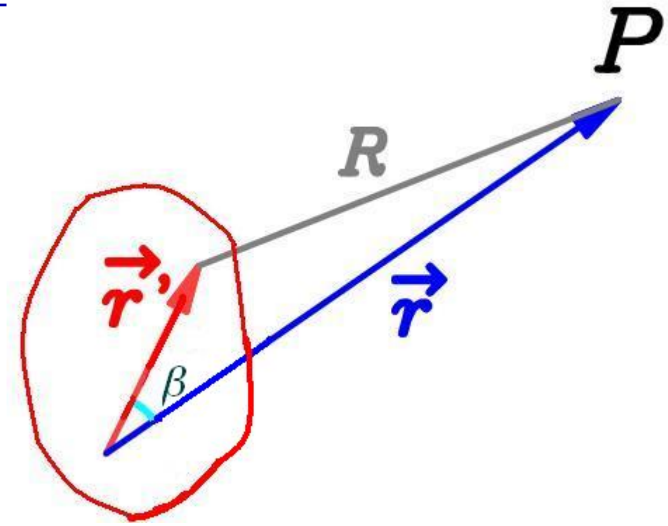
$$\begin{aligned}\varphi(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}}\end{aligned}$$

$$\frac{1}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)$$



Let there be light

$$\begin{aligned}\varphi(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}}\end{aligned}$$

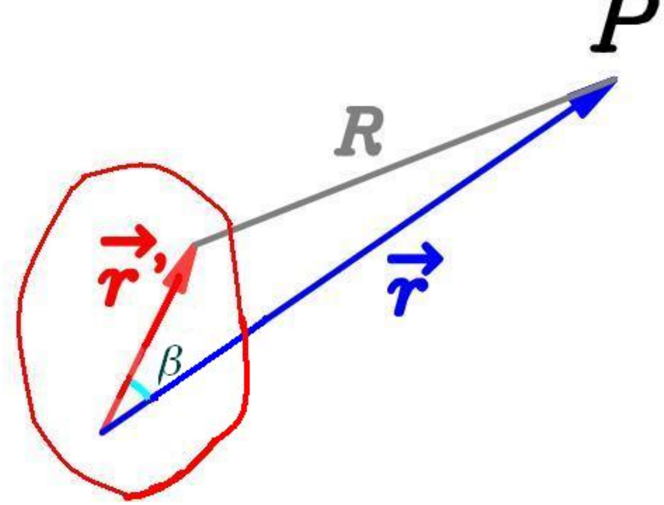


$$\frac{1}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)$$

(对 $r > r'$)

Let there be light

$$\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^2-2rr'\cos\beta}}$$



$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)$$

(对 $r > r'$)

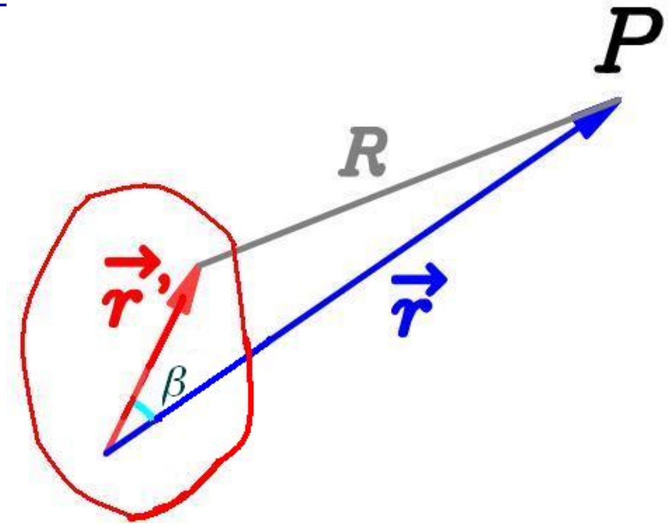
从而 $\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$

多极矩展开

$P_n(x)$ 为勒让德多项式

Let there be light

$$
\begin{aligned}
\varphi(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^2-2rr'\cos\beta}}
\end{aligned}
$$



$$
\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)
$$

(对 $r > r'$)

从而 $$\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'$$

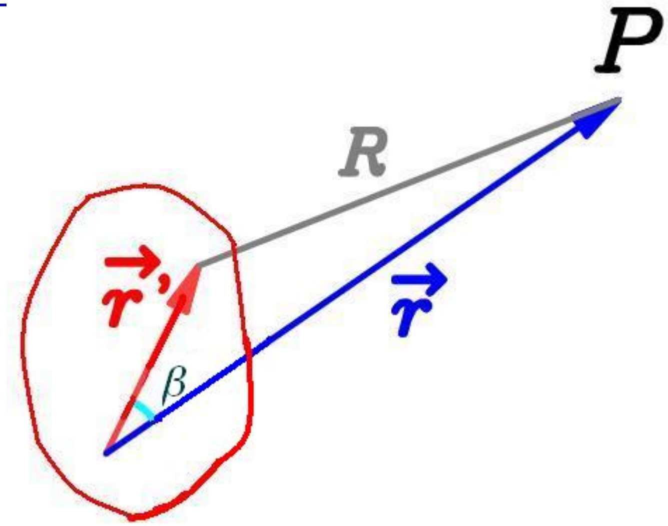
多极矩展开

$P_n(x)$ 为勒让德多项式

$$
\sim \frac{1}{r}\left[c_0 + c_1\frac{d}{r} + c_2\left(\frac{d}{r}\right)^2 + \ldots\right]
$$

Let there be light

$$\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,\mathrm{d}\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}}$$



$$\frac{1}{\sqrt{r^2+r'^{\,2}-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)$$

(对 $r > r'$)

从而 $$\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$$

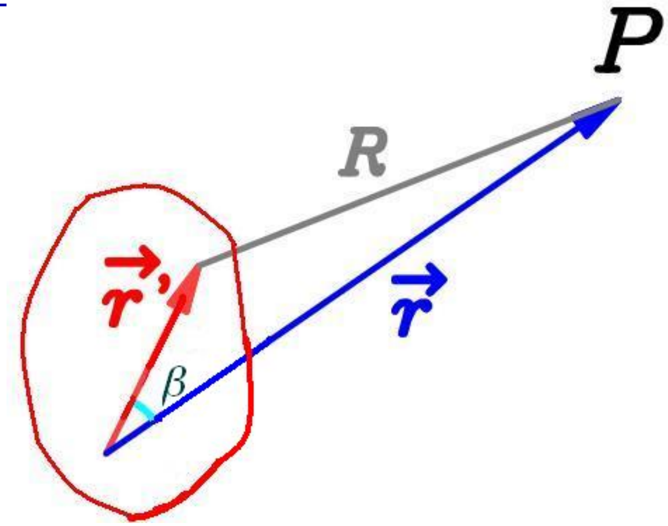
$$\sim \frac{1}{r}\left[c_0 + c_1\frac{d}{r} + c_2\left(\frac{d}{r}\right)^2 + \dots\right]$$

多极矩展开

$P_n(x)$ 为勒让德多项式

单极项 (monopole term) $n = 0$ 项， 利用 $P_0(x) = 1$

Let there be light



$$\begin{aligned}\varphi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^2-2rr'\cos\beta}}\end{aligned}$$

$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)$$

(对 $r > r'$)

从而 $$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{r}')\,\mathrm{d}\tau'$$

多极矩展开

$P_n(x)$ 为勒让德多项式

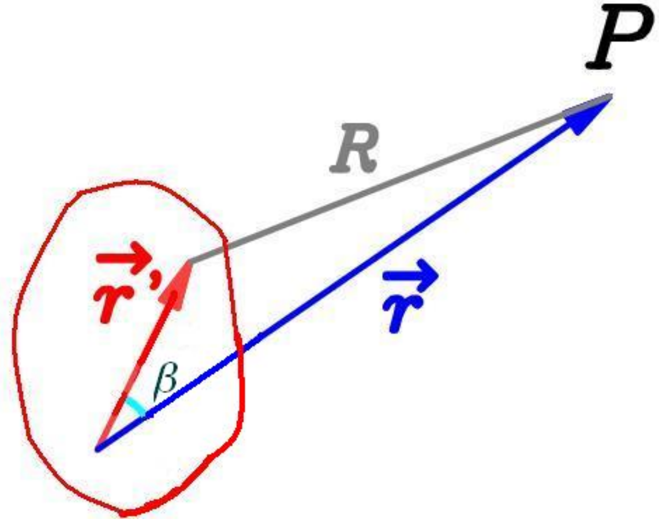
$$\sim \frac{1}{r}\left[c_0 + c_1\frac{d}{r} + c_2\left(\frac{d}{r}\right)^2 + \ldots\right]$$

单极项 (monopole term) $n=0$ 项， 利用 $P_0(x)=1$

$$\varphi_0(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int P_0(\cos\beta)\rho(\vec{r}')\,\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\,\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Let there be light

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,\mathrm{d}\tau'}{\sqrt{r^2+r'^2-2rr'\cos\beta}}$$



$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\beta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\beta)$$

(对 $r > r'$)

从而 $$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{r}\,')\,\mathrm{d}\tau'$$

$$\sim \frac{1}{r}\left[c_0 + c_1\frac{d}{r} + c_2\left(\frac{d}{r}\right)^2 + \dots\right]$$

多极矩展开

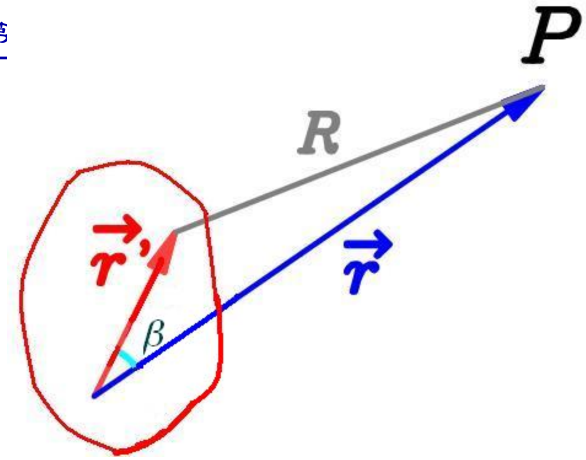
$P_n(x)$ 为勒让德多项式

单极项 (monopole term) $n = 0$ 项， 利用 $P_0(x) = 1$

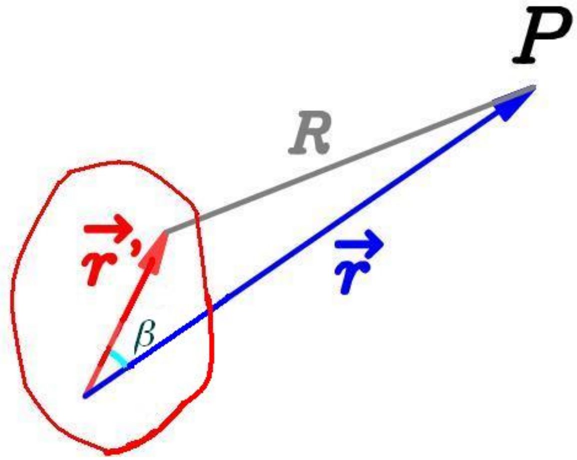
$$\varphi_0(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int P_0(\cos\beta)\rho(\vec{r}\,')\,\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\rho(\vec{r}\,')\,\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

远离有限电荷分布之处，$r \gg d$，电势确实可用点电荷近似

多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$

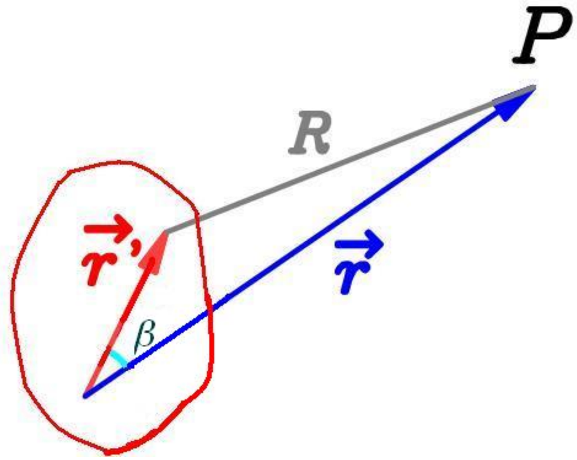


多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$



偶极项 (dipole term)　$n = 1$ 项，　利用 $P_1(x) = x$

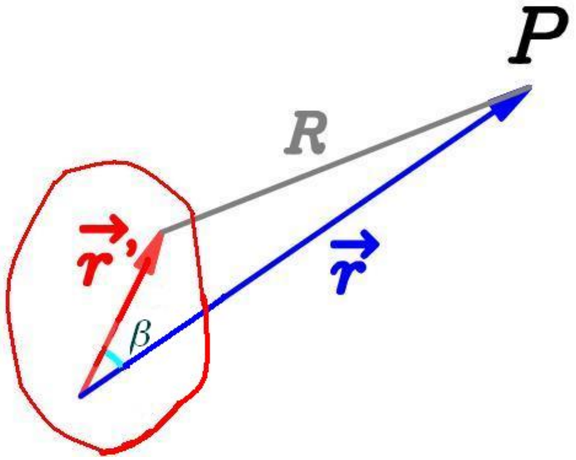
多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$



偶极项 (dipole term)　$n = 1$ 项，　利用 $P_1(x) = x$

$$\varphi_1(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$$

利用 $r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}'$



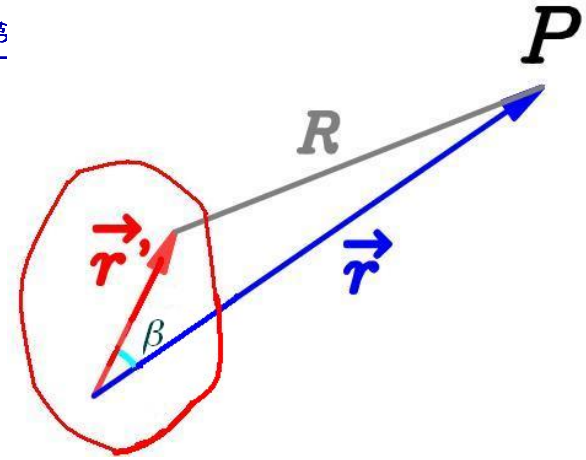
多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$

偶极项 (dipole term)　$n=1$ 项，　利用 $P_1(x)=x$

$$\begin{aligned}\varphi_1(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{e}}_r\cdot\int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'\end{aligned}$$

利用 $r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}'$

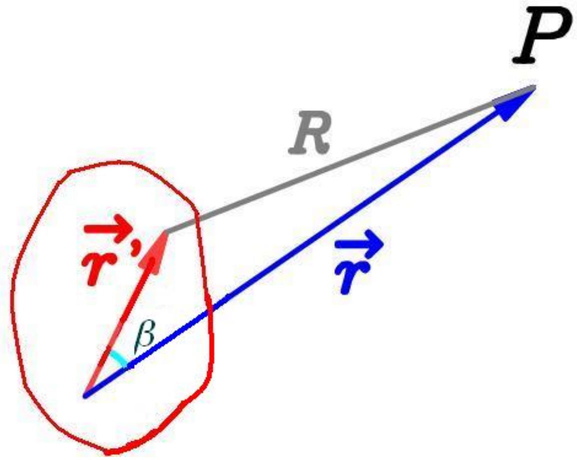
多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$



偶极项 (dipole term)　$n=1$ 项，　利用 $P_1(x) = x$

$$\begin{aligned}\varphi_1(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{e}}_r\cdot\underbrace{\int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}_{\vec{\boldsymbol{p}}}\end{aligned}$$

利用 $r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}'$

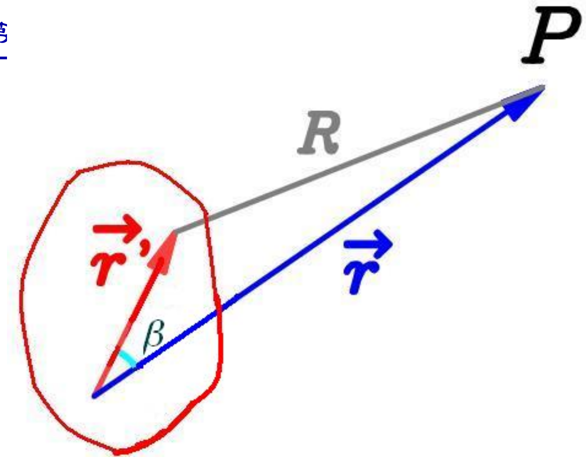


多极矩展开：$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{r}')\,\mathrm{d}\tau'$

偶极项 (dipole term)　$n=1$ 项，　利用 $P_1(x) = x$

$$\begin{aligned}\varphi_1(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{r}')\,\mathrm{d}\tau' \qquad \text{利用 } r'\cos\beta = \hat{e}_r\cdot\vec{r}' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{e}_r\cdot\underbrace{\int\vec{r}'\rho(\vec{r}')\,\mathrm{d}\tau'}_{\vec{p}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{e}_r}{r^2}\end{aligned}$$

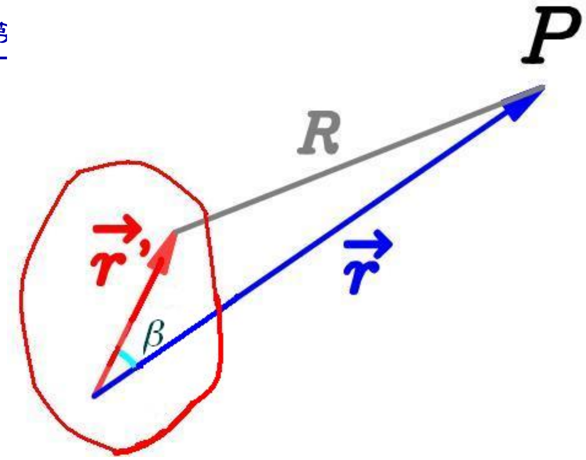
多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$



偶极项 (dipole term) $n=1$ 项， 利用 $P_1(x) = x$

$$\begin{aligned}
\varphi_1(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' \qquad \text{利用 } r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}' \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{e}}_r\cdot\underbrace{\int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}_{\vec{\boldsymbol{p}}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_r}{r^2}
\end{aligned}$$

一般电荷分布的偶极矩：$\boxed{\vec{\boldsymbol{p}} = \int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}$



多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$

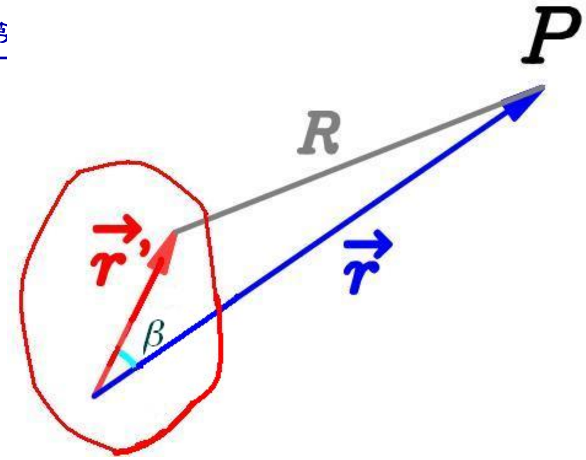
偶极项 (dipole term)　$n = 1$ 项，　利用 $P_1(x) = x$

$$\begin{aligned}\varphi_1(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{e}}_r\cdot\underbrace{\int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}_{\vec{\boldsymbol{p}}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_r}{r^2}\end{aligned}$$

利用 $r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}'$

一般电荷分布的偶极矩：$\boxed{\vec{\boldsymbol{p}} = \int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}$　对点电荷体系 $\vec{\boldsymbol{p}} = \sum_i q_i\vec{\boldsymbol{r}}_i'$

多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$



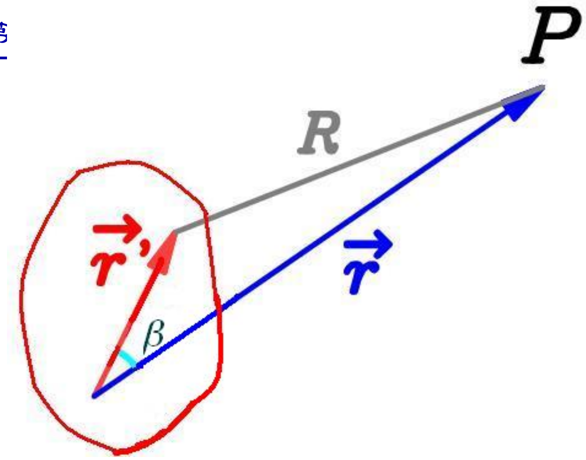
偶极项 (dipole term)　　$n=1$ 项，　　利用 $P_1(x)=x$

$$\begin{aligned}\varphi_1(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{e}}_r\cdot\underbrace{\int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}_{\vec{\boldsymbol{p}}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_r}{r^2}\end{aligned}$$

利用 $r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}'$

一般电荷分布的偶极矩：$\boxed{\vec{\boldsymbol{p}} = \int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}$　对点电荷体系 $\vec{\boldsymbol{p}} = \sum_i q_i\vec{\boldsymbol{r}}_i'$

偶极电势：$\boxed{\varphi_1(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_r}{r^2}}$



多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$

偶极项 (dipole term) $n=1$ 项， 利用 $P_1(x)=x$

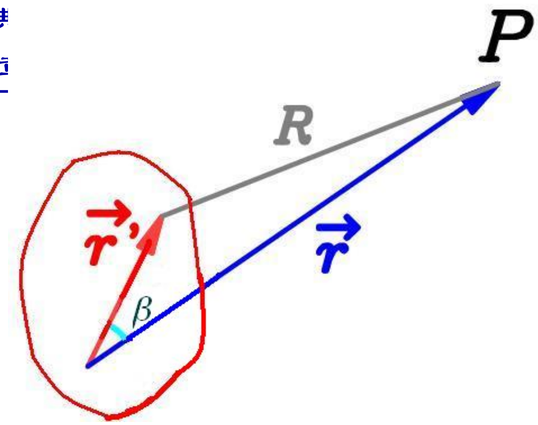
$$\begin{aligned}\varphi_1(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int\left(\frac{r'}{r}\right)\cos\beta\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' \qquad \text{利用 } r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{e}}_r\cdot\underbrace{\int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}_{\vec{\boldsymbol{p}}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_r}{r^2}\end{aligned}$$

一般电荷分布的偶极矩：$\boxed{\vec{\boldsymbol{p}} = \int\vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}$ 对点电荷体系 $\vec{\boldsymbol{p}} = \sum_i q_i\vec{\boldsymbol{r}}_i'$

偶极电势：$\boxed{\varphi_1(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_r}{r^2}}$

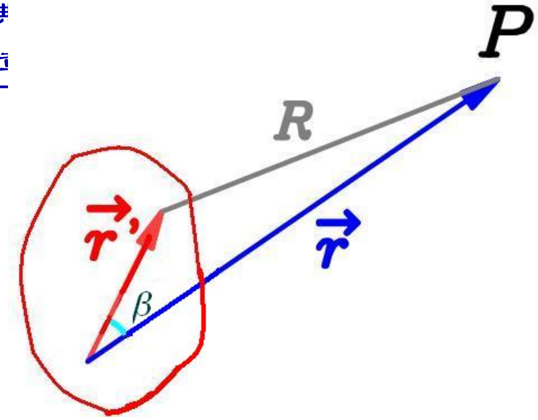
比点电荷近似更高阶的近似是偶极电势近似：$\varphi_1 \sim \left(\frac{d}{r}\right)\frac{1}{r}$

Let there be light



多极矩展开：$\varphi(\vec{\boldsymbol{r}})=\dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\sum_{n=0}^{\infty}\int\left(\dfrac{r'}{r}\right)^{n}P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'$

Let there be light



多极矩展开：$\varphi(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{r}\,')\,\mathrm{d}\tau'$

四极项 (quadrupole term), $n=2$ 利用 $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

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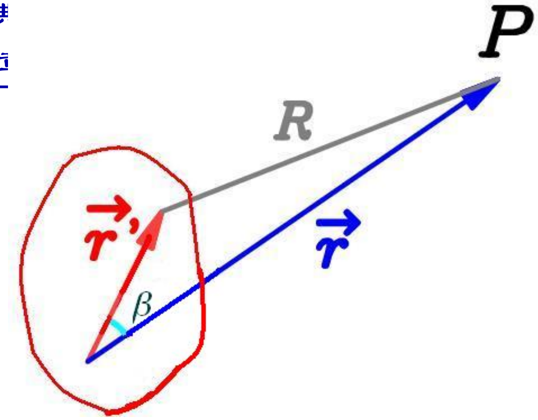

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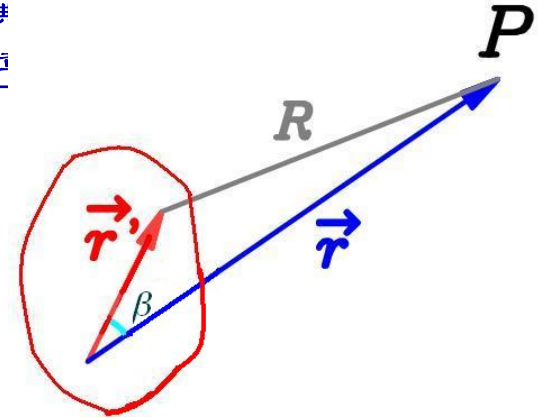

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因为：$r'\cos\beta = \hat{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{r}}'$, 故

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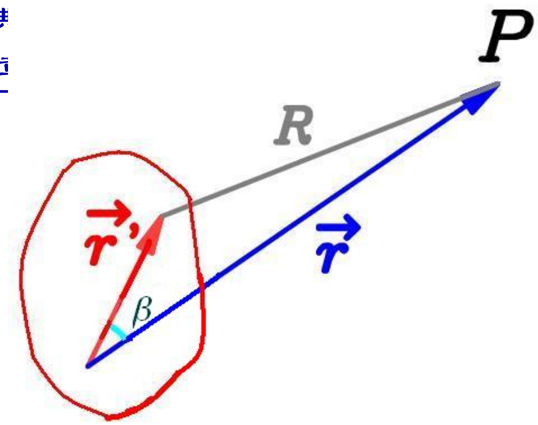

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&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\frac{1}{2}(\hat{\boldsymbol{e}}_r\,\hat{\boldsymbol{e}}_r) : \int(3\vec{\boldsymbol{r}}\,'\vec{\boldsymbol{r}}\,' - r'^2\,\overleftrightarrow{\boldsymbol{I}})\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'
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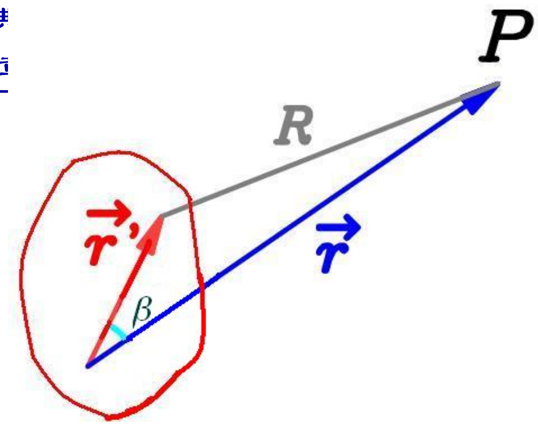

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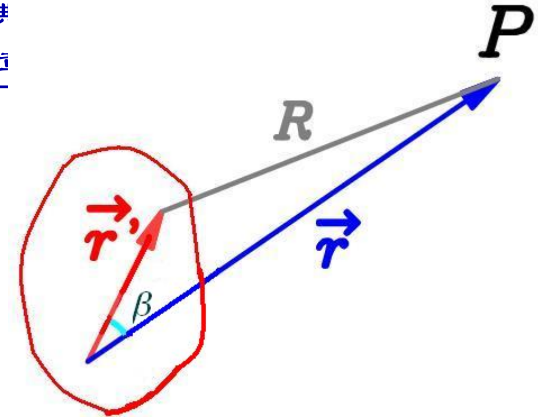

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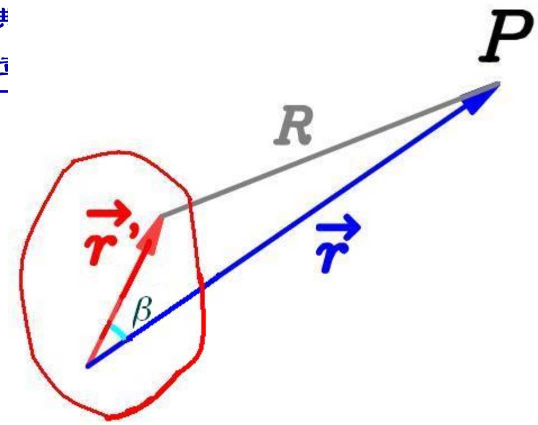

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一般电荷分布的四极矩：$\boxed{\overleftrightarrow{\boldsymbol{D}} = \int(3\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}'-r'^2\overleftrightarrow{\boldsymbol{I}})\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'}$

对点电荷体系

$$\overleftrightarrow{\boldsymbol{D}} = \sum_i q_i(3\vec{\boldsymbol{r}}_i'\vec{\boldsymbol{r}}_i' - r_i'^2\overleftrightarrow{\boldsymbol{I}})$$

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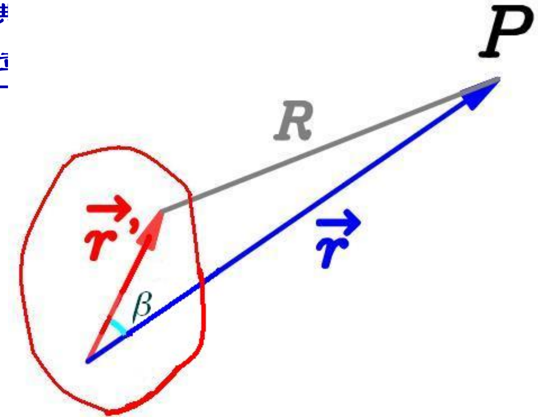

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对点电荷体系

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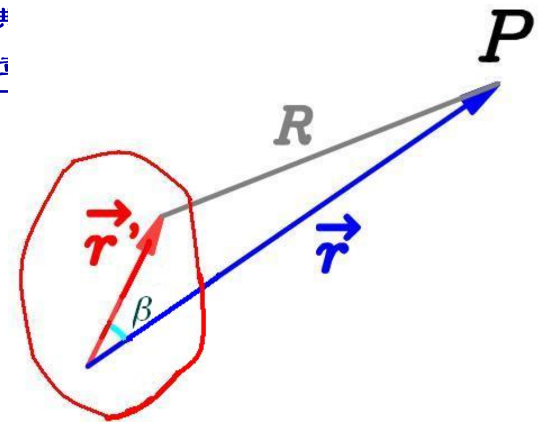
四极电势：$\boxed{\varphi_2(\vec{r}) = \frac{1}{8\pi\epsilon_0}\frac{(\hat{e}_r\hat{e}_r):\overleftrightarrow{D}}{r^3}}$



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比偶极近似更高阶的近似是四极近似：$\varphi_2 \sim \left(\dfrac{d}{r}\right)^2\dfrac{1}{r}$

电多极矩的性质

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1. 偶极矩（对应于 $n=1$） $\vec{\boldsymbol{p}}=\int\vec{\boldsymbol{r}}\,'\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'$ 是矢量（一阶张量）；

   四极矩（对应于 $n=2$） $\overleftrightarrow{\boldsymbol{D}}=\int(3\vec{\boldsymbol{r}}\,'\vec{\boldsymbol{r}}\,'-r'^{\,2}\overleftrightarrow{\boldsymbol{I}})\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'$ 是二阶张量，其迹为 0。

   2^n 极矩（对应于 $n=n$） 是 n 阶张量。

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$$\mathrm{Tr}\,\overleftrightarrow{\boldsymbol{D}} = \sum_i D_{ii} = \overleftrightarrow{\boldsymbol{D}} : \overleftrightarrow{\boldsymbol{I}} = \int \left[3(\vec{\boldsymbol{r}}{}'\vec{\boldsymbol{r}}{}') : \overleftrightarrow{\boldsymbol{I}} - r'^2 \overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{I}}\right] \rho(\vec{\boldsymbol{r}}{}')\,\mathrm{d}\tau' = 0$$

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利用了：$\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{I}} = 3$ 和 $(\vec{\boldsymbol{a}}\vec{\boldsymbol{a}}) : \overleftrightarrow{\boldsymbol{I}} = \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}^{\,2}$，从而：$[3(\vec{\boldsymbol{r}}\,'\vec{\boldsymbol{r}}\,') : \overleftrightarrow{\boldsymbol{I}} - r'^{\,2}\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{I}}] = 0$

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2. 对任何电荷的分布，其最低阶非零多极矩的值与坐标原点的选择无关。

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(证明见教材 p111)

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   (证明见教材 p111)

   如果 $Q = \int \rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau' = 0$，则 $\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{r}}\,'\,\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}\tau'$ 与坐标原点的选择无关。

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   如果 $Q=\int\rho(\vec{\boldsymbol{r}}^{\,\prime})\,\mathrm{d}\tau'=0$，则 $\vec{\boldsymbol{p}}=\int\vec{\boldsymbol{r}}^{\,\prime}\,\rho(\vec{\boldsymbol{r}}^{\,\prime})\,\mathrm{d}\tau'$ 与坐标原点的选择无关。

   如果 $Q=0$，$\vec{\boldsymbol{p}}=0$，则 $\overleftrightarrow{\boldsymbol{D}}=\int\left[3(\vec{\boldsymbol{r}}^{\,\prime}\vec{\boldsymbol{r}}^{\,\prime})-r'^{\,2}\overleftrightarrow{\boldsymbol{I}}\right]\rho(\vec{\boldsymbol{r}}^{\,\prime})\,\mathrm{d}\tau'$ 与坐标原点的选择无关。

电多极矩的性质

1. 偶极矩（对应于 $n=1$） $\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{r}}'\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$ 是矢量（一阶张量）；

   四极矩（对应于 $n=2$） $\overleftrightarrow{\boldsymbol{D}} = \int (3\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}' - r'^2\overleftrightarrow{\boldsymbol{I}})\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$ 是二阶张量，其迹为 0。

   2^n 极矩（对应于 $n=n$） 是 n 阶张量。

   证明： $$\mathrm{Tr}\,\overleftrightarrow{\boldsymbol{D}} = \sum_i D_{ii} = \overleftrightarrow{\boldsymbol{D}} : \overleftrightarrow{\boldsymbol{I}} = \int \left[3(\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}') : \overleftrightarrow{\boldsymbol{I}} - r'^2\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{I}}\right]\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' = 0$$

   利用了：$\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{I}} = 3$ 和 $(\vec{\boldsymbol{a}}\vec{\boldsymbol{a}}) : \overleftrightarrow{\boldsymbol{I}} = \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}^{\,2}$，从而：$[3(\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}') : \overleftrightarrow{\boldsymbol{I}} - r'^2\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{I}}] = 0$

2. 对任何电荷的分布，其最低阶非零多极矩的值与坐标原点的选择无关。

   (证明见教材 p111)

   如果 $Q = \int \rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau' = 0$，则 $\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{r}}'\,\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$ 与坐标原点的选择无关。

   如果 $Q=0$，$\vec{\boldsymbol{p}}=0$，则 $\overleftrightarrow{\boldsymbol{D}} = \int \left[3(\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}') - r'^2\overleftrightarrow{\boldsymbol{I}}\right]\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$ 与坐标原点的选择无关。

3. 实用性：

电多极矩的性质

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   多极矩展开，零极矩为零阶张量，偶极矩为一阶张量，四极矩为二阶张量，
   求 2^n 极距须球一 n 阶张量，再与另一 n 阶张量作 n 次点积，才能得到静电标势。

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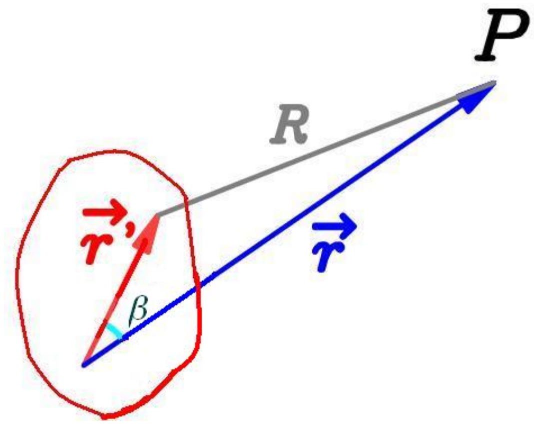
   多极矩展开是否具有实用性？

多极矩展开：$\varphi(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\int\left(\frac{r'}{r}\right)^n P_n(\cos\beta)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}\tau'$



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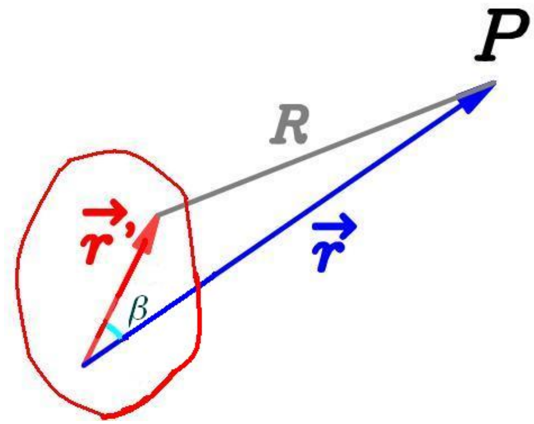

实际应用中，常再利用 Legendre 函数的加法定理 (addition theorem)

$$P_n(\cos\beta) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y^*_{nm}(\theta',\phi')Y_{nm}(\theta,\phi)$$

其中 $Y_{nm}(\theta,\phi)$ 为球谐函数：$Y_{nm} = \sqrt{\dfrac{2n+1}{4\pi}\dfrac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\theta)e^{im\phi}$

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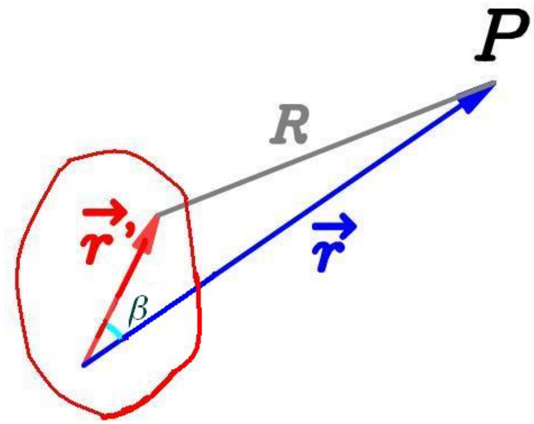

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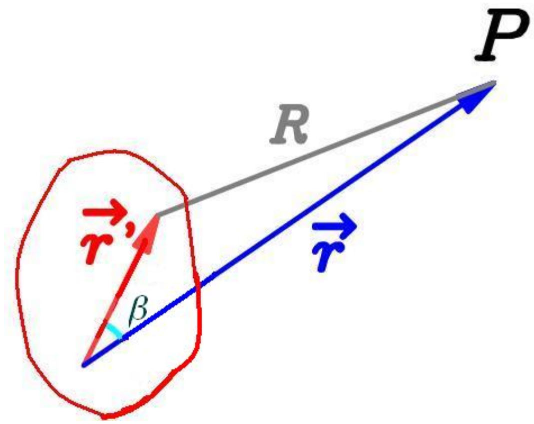

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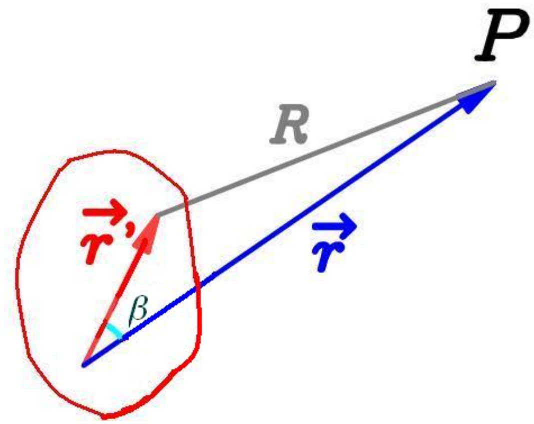

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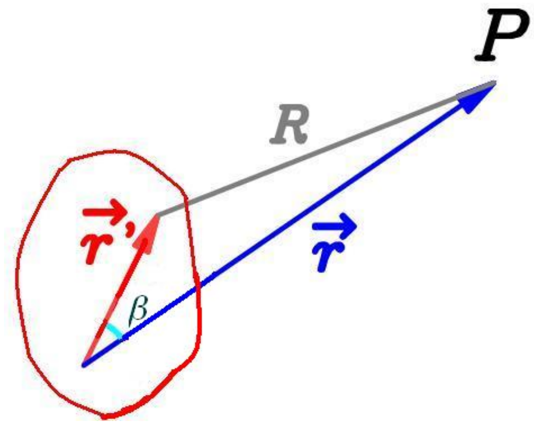

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二、例题

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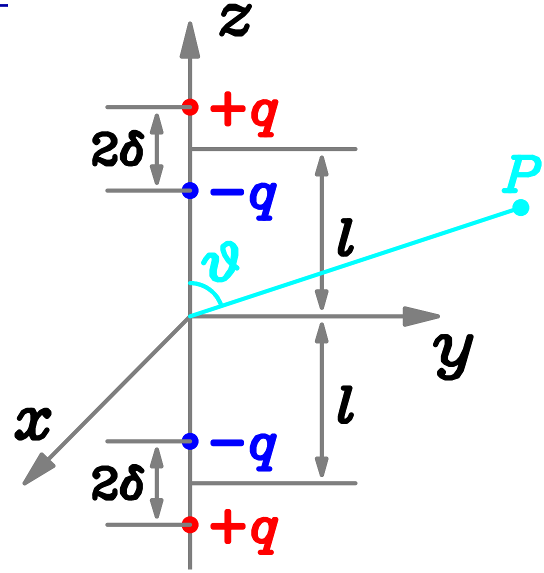
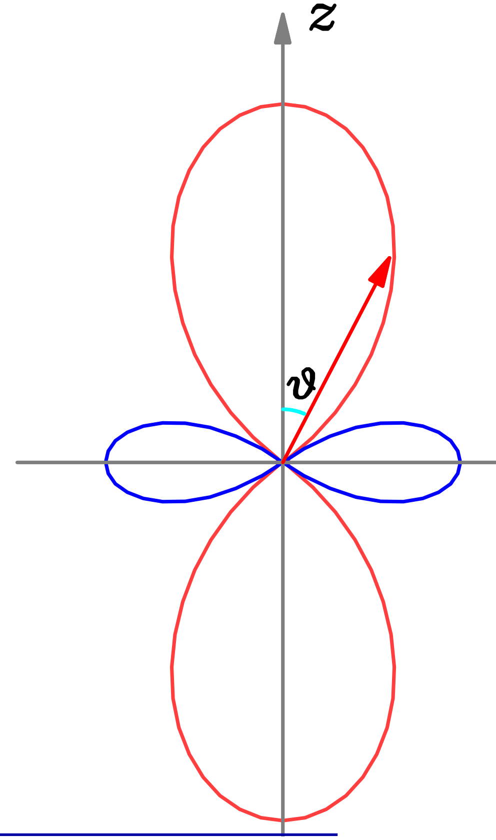
例 1：如图所示电荷分布的电四极子，并求其电势





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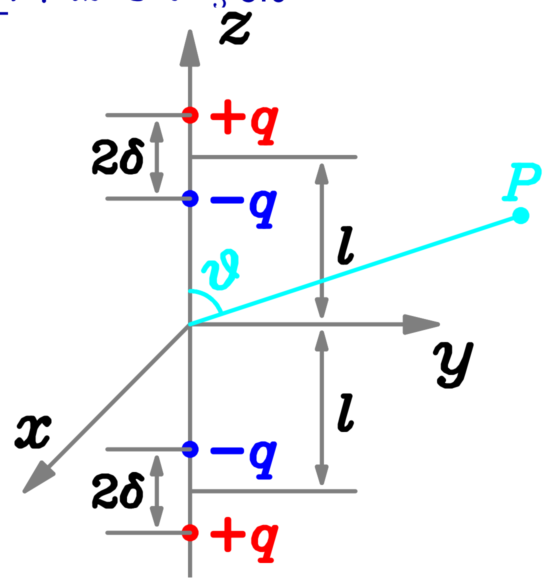
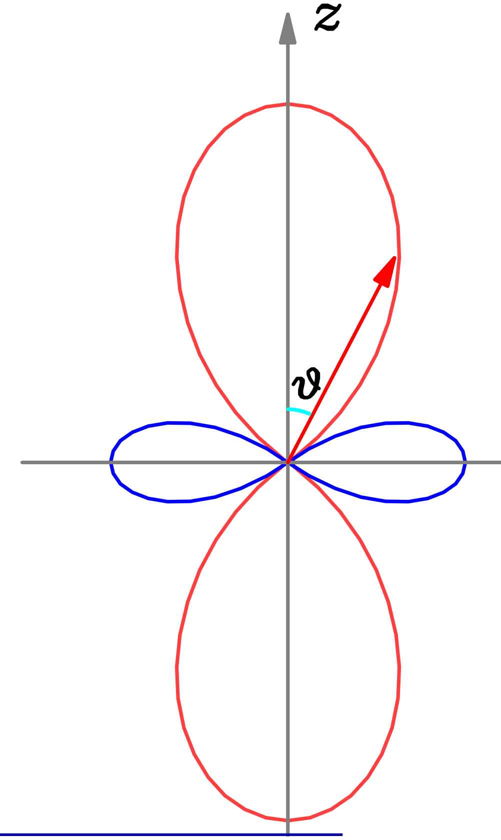


$$Q = \sum_i q_i = 0, \quad \vec{\boldsymbol{p}} = \sum_i \vec{\boldsymbol{r}}_i q_i = 0, \quad \overleftrightarrow{\boldsymbol{D}} = \sum_i q_i (3\vec{\boldsymbol{r}}_i\vec{\boldsymbol{r}}_i - r_i^2 \overleftrightarrow{\boldsymbol{I}})$$



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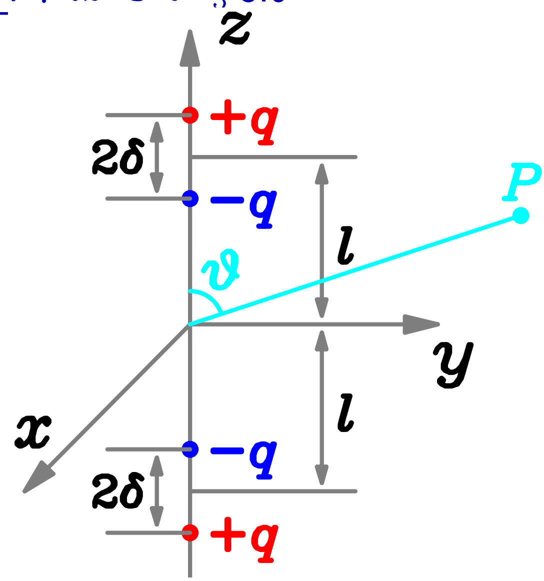

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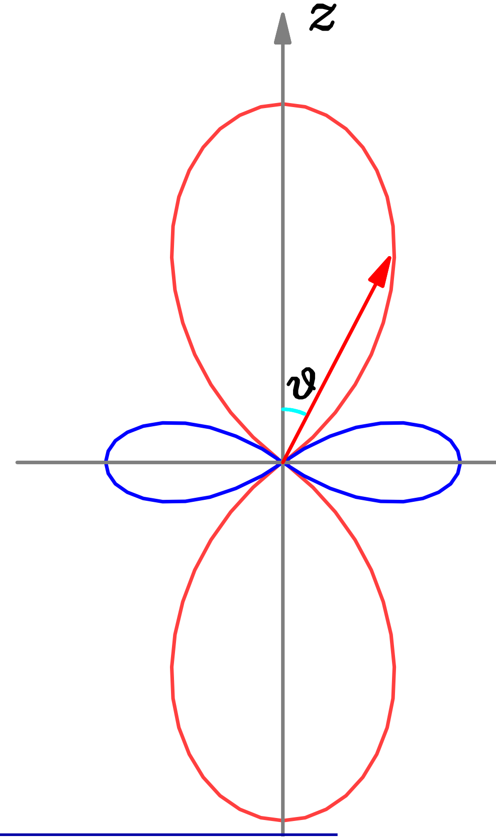
$$D_{33}=\sum_i q_i(3z_iz_i-r_i^2)=16ql\delta$$ （其中 $r_i=|z_i|$）



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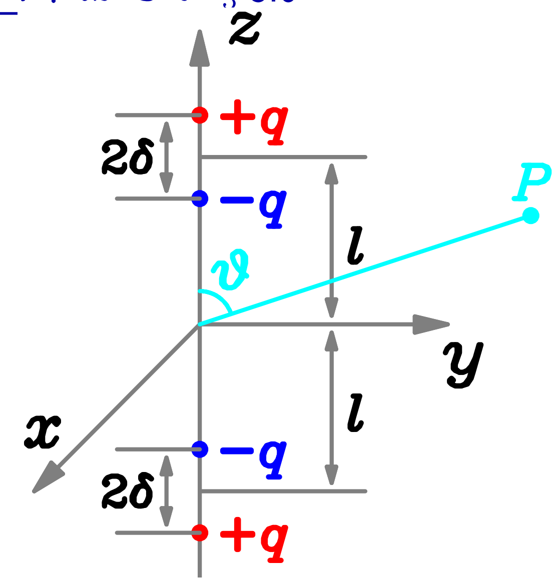

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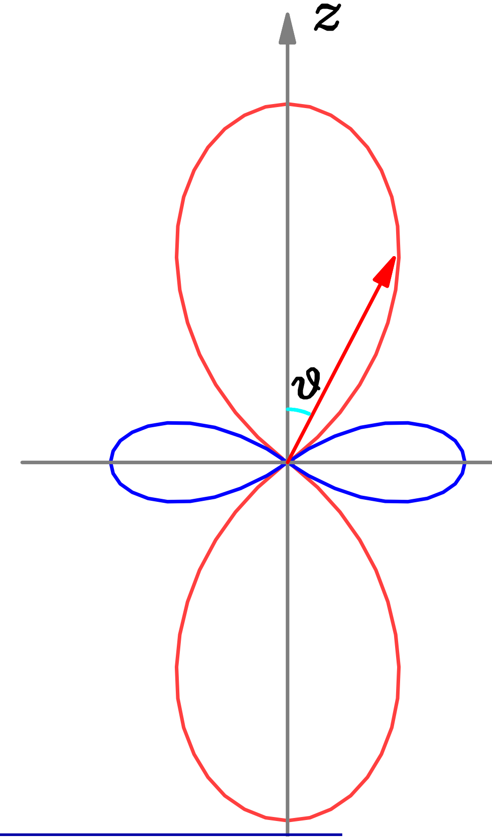
$$D_{33} = \sum_i q_i (3 z_i z_i - r_i^2) = 16 q l \delta$$ (其中 $r_i = |z_i|$)

$$D_{11} + D_{22} + D_{33} = 0 \quad \Longrightarrow \quad D_{11} = D_{22} = -8 q l \delta$$



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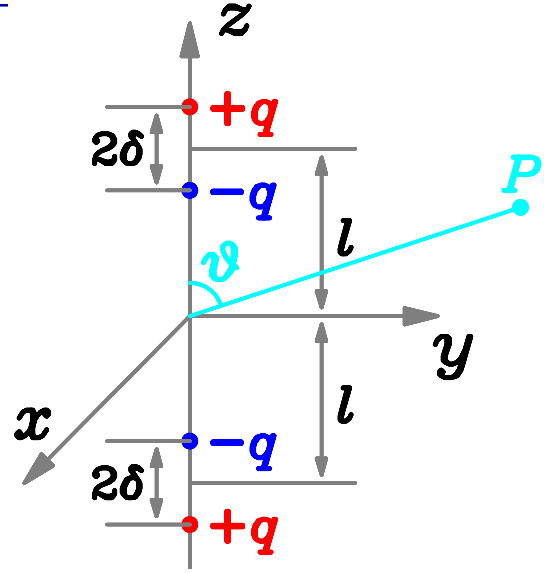

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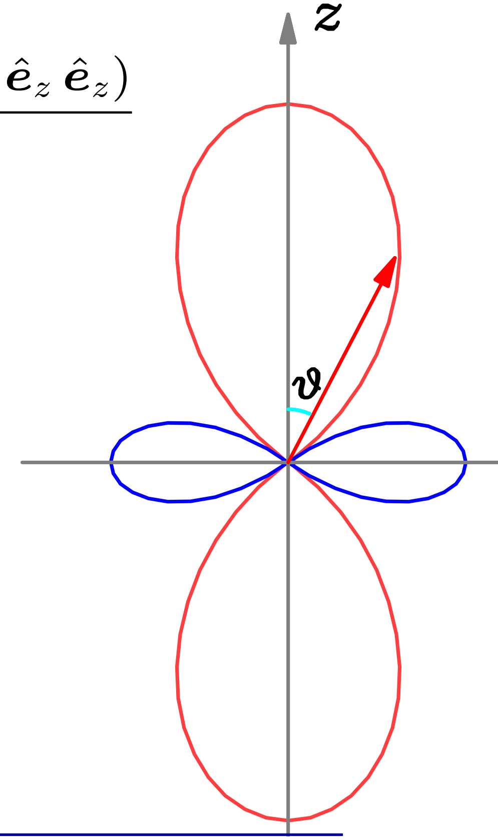
$$\varphi_2 = \frac{1}{8\pi\epsilon_0} \frac{\hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_r : \overleftrightarrow{\boldsymbol{D}}}{r^3} = \frac{1}{8\pi\epsilon_0} \frac{\hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_r : (D_{11} \hat{\boldsymbol{e}}_x \hat{\boldsymbol{e}}_x + D_{22} \hat{\boldsymbol{e}}_y \hat{\boldsymbol{e}}_y + D_{33} \hat{\boldsymbol{e}}_z \hat{\boldsymbol{e}}_z)}{r^3}$$



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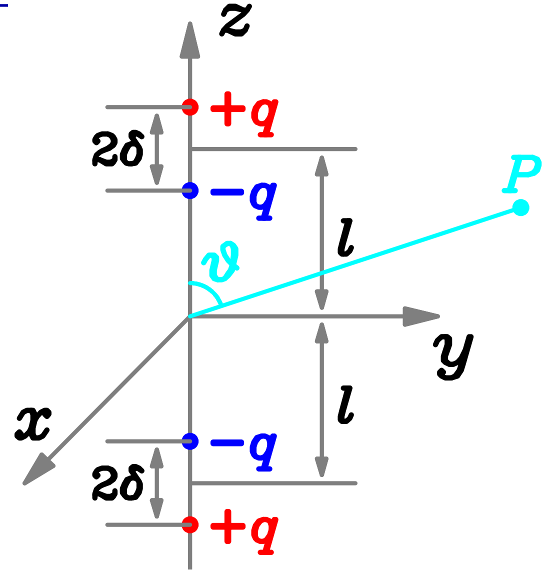

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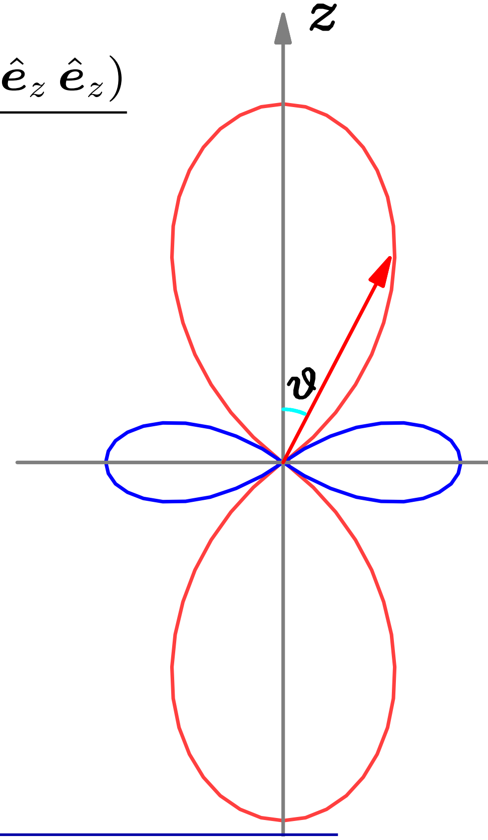
$$=\frac{1}{8\pi\epsilon_0}\frac{8ql\delta(-\sin^2\theta\cos^2\phi-\sin^2\theta\sin^2\phi)+16ql\delta\cos^2\theta}{r^3}$$



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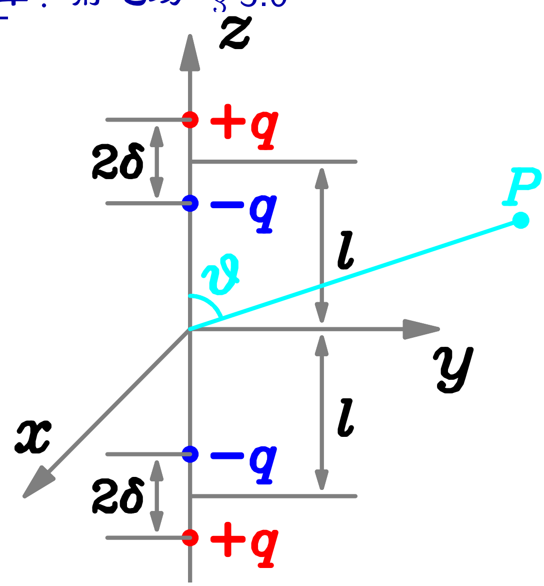

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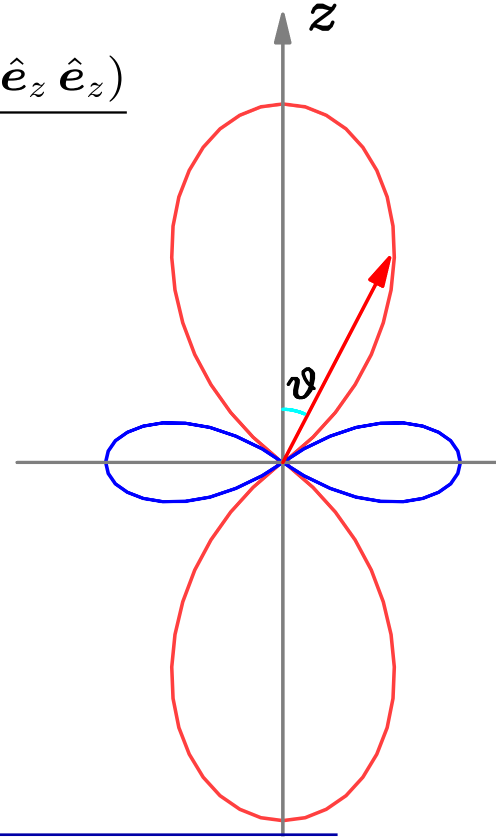
$$\varphi_2=\frac{1}{8\pi\epsilon_0}\frac{\hat{\boldsymbol{e}}_r\,\hat{\boldsymbol{e}}_r:\overleftrightarrow{\boldsymbol{D}}}{r^3}=\frac{1}{8\pi\epsilon_0}\frac{\hat{\boldsymbol{e}}_r\,\hat{\boldsymbol{e}}_r:(D_{11}\,\hat{\boldsymbol{e}}_x\,\hat{\boldsymbol{e}}_x+D_{22}\,\hat{\boldsymbol{e}}_y\,\hat{\boldsymbol{e}}_y+D_{33}\,\hat{\boldsymbol{e}}_z\,\hat{\boldsymbol{e}}_z)}{r^3}$$

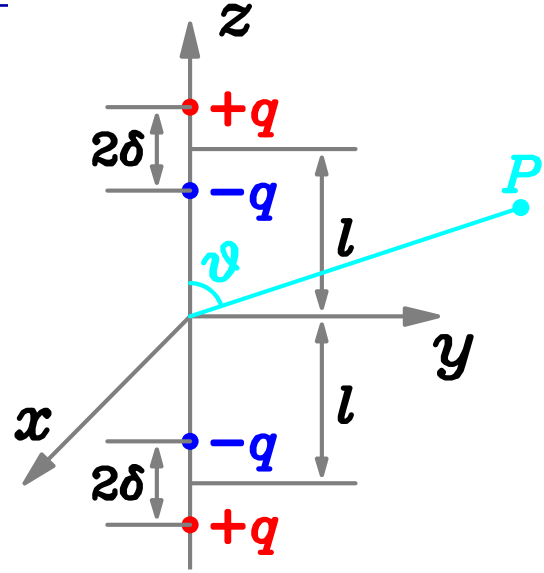
$$=\frac{1}{8\pi\epsilon_0}\frac{8ql\delta(-\sin^2\theta\cos^2\phi-\sin^2\theta\sin^2\phi)+16ql\delta\cos^2\theta}{r^3}$$

$$=\frac{ql\delta}{\pi\epsilon_0}\frac{3\cos^2\theta-1}{r^3}\sim\left(\frac{d}{r}\right)^2\frac{1}{r}$$



二、例题

例 1：如图所示电荷分布的电四极子，并求其电势



$$Q = \sum_i q_i = 0, \quad \vec{p} = \sum_i \vec{r}_i q_i = 0, \quad \overleftrightarrow{D} = \sum_i q_i (3\vec{r}_i\vec{r}_i - r_i^2 \overleftrightarrow{I})$$

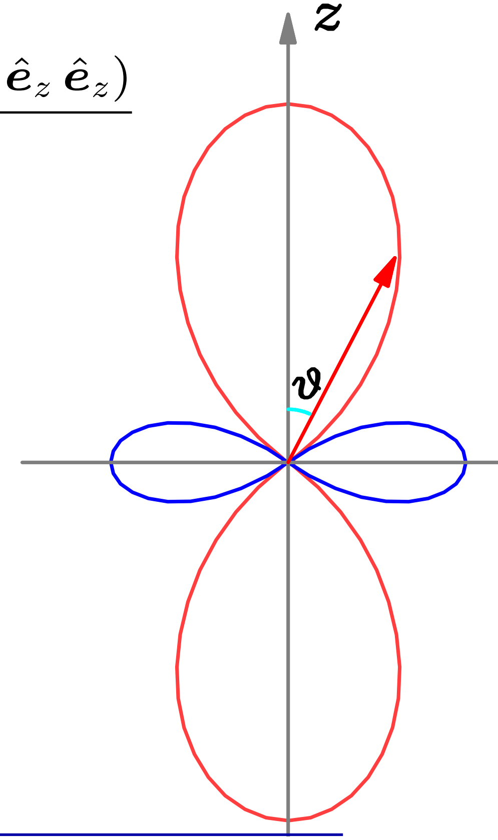
$$D_{33} = \sum_i q_i (3z_i z_i - r_i^2) = 16ql\delta \qquad (\text{其中 } r_i = |z_i|)$$

$$D_{11} + D_{22} + D_{33} = 0 \implies D_{11} = D_{22} = -8ql\delta$$

$$\varphi_2 = \frac{1}{8\pi\epsilon_0} \frac{\hat{e}_r\,\hat{e}_r : \overleftrightarrow{D}}{r^3} = \frac{1}{8\pi\epsilon_0} \frac{\hat{e}_r\,\hat{e}_r : (D_{11}\,\hat{e}_x\,\hat{e}_x + D_{22}\,\hat{e}_y\,\hat{e}_y + D_{33}\,\hat{e}_z\,\hat{e}_z)}{r^3}$$

$$= \frac{1}{8\pi\epsilon_0} \frac{8ql\delta(-\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi) + 16ql\delta\cos^2\theta}{r^3}$$

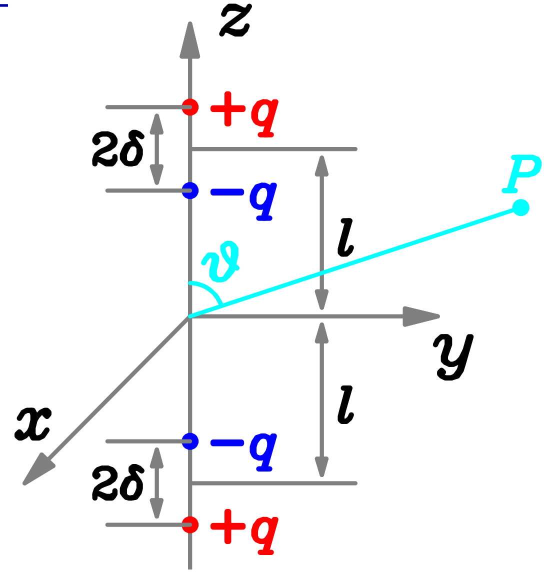
$$= \frac{ql\delta}{\pi\epsilon_0} \frac{3\cos^2\theta - 1}{r^3} \sim \left(\frac{d}{r}\right)^2 \frac{1}{r}$$



电势极坐标图如右图，兰色区 $(\theta_0 < \theta < \pi - \theta_0)$ 电势为负

二、例题

例 1：如图所示电荷分布的电四极子，并求其电势



$$Q=\sum_i q_i=0,\quad \vec{\boldsymbol{p}}=\sum_i \vec{\boldsymbol{r}}_i q_i=0,\quad \overleftrightarrow{\boldsymbol{D}}=\sum_i q_i(3\vec{\boldsymbol{r}}_i\vec{\boldsymbol{r}}_i-r_i^2\overleftrightarrow{\boldsymbol{I}})$$

$$D_{33}=\sum_i q_i(3z_iz_i-r_i^2)=16ql\delta \qquad (\text{其中 } r_i=|z_i|)$$

$$D_{11}+D_{22}+D_{33}=0 \implies D_{11}=D_{22}=-8ql\delta$$

$$\varphi_2=\frac{1}{8\pi\epsilon_0}\frac{\hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r:\overleftrightarrow{\boldsymbol{D}}}{r^3}=\frac{1}{8\pi\epsilon_0}\frac{\hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r:(D_{11}\hat{\boldsymbol{e}}_x\hat{\boldsymbol{e}}_x+D_{22}\hat{\boldsymbol{e}}_y\hat{\boldsymbol{e}}_y+D_{33}\hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_z)}{r^3}$$

$$=\frac{1}{8\pi\epsilon_0}\frac{8ql\delta(-\sin^2\theta\cos^2\phi-\sin^2\theta\sin^2\phi)+16ql\delta\cos^2\theta}{r^3}$$

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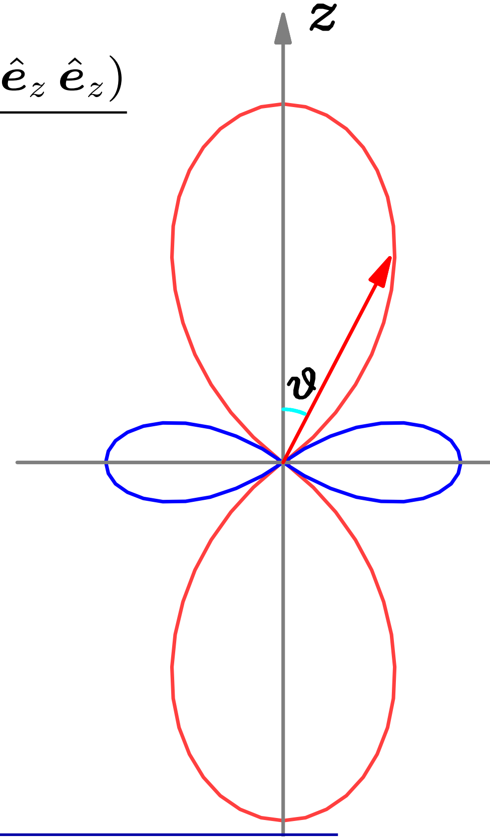
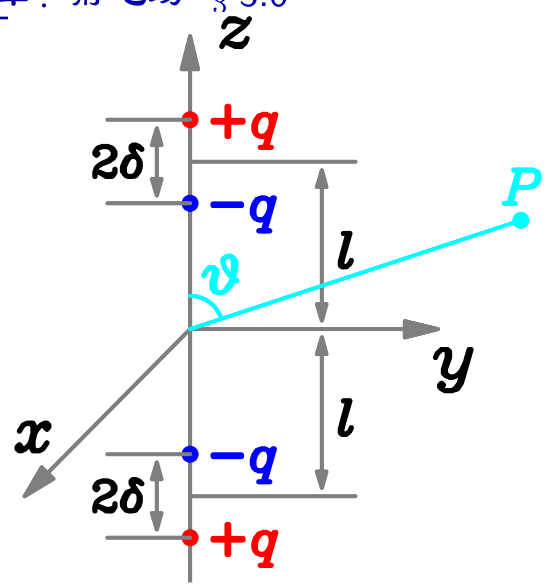


电势极坐标图如右图，兰色区 $(\theta_0<\theta<\pi-\theta_0)$ 电势为负

红色区 $(\theta<\theta_0$ 和 $\pi-\theta_0<\theta<2\pi)$ 为正



二、例题

例 1：如图所示电荷分布的电四极子，并求其电势

$$Q=\sum_i q_i=0,\quad \vec{\boldsymbol{p}}=\sum_i \vec{\boldsymbol{r}}_i q_i=0,\quad \overleftrightarrow{\boldsymbol{D}}=\sum_i q_i(3\vec{\boldsymbol{r}}_i\vec{\boldsymbol{r}}_i-r_i^2\overleftrightarrow{\boldsymbol{I}})$$

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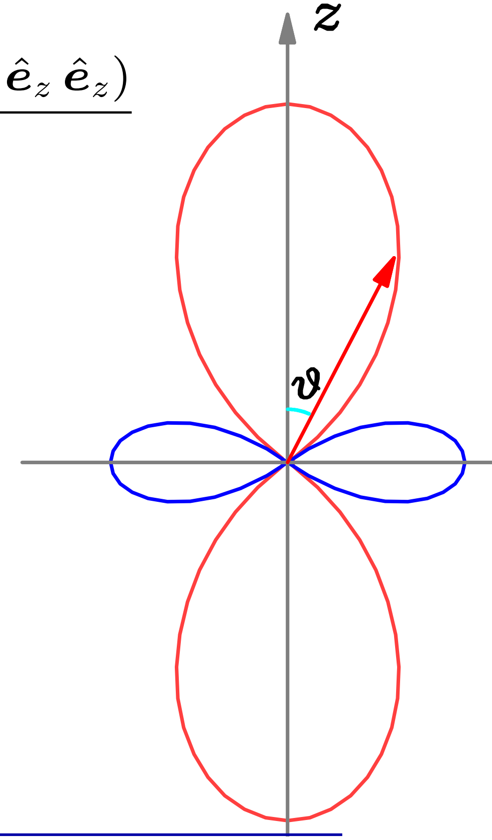
$$=\frac{1}{8\pi\epsilon_0}\frac{8ql\delta(-\sin^2\theta\cos^2\phi-\sin^2\theta\sin^2\phi)+16ql\delta\cos^2\theta}{r^3}$$

$$=\frac{ql\delta}{\pi\epsilon_0}\frac{3\cos^2\theta-1}{r^3}\sim\left(\frac{d}{r}\right)^2\frac{1}{r}$$



电势极坐标图如右图，兰色区 $(\theta_0<\theta<\pi-\theta_0)$ 电势为负

红色区 $(\theta<\theta_0$ 和 $\pi-\theta_0<\theta<2\pi)$ 为正

$$\theta_0=\arccos\sqrt{1/3}=54.7^\circ$$

例 2：试证明球内电荷分布在球内产生的电场之平均值为：$\vec{E}_{\text{ave}} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}}{a^3}$，其中 a 为球半径，$\vec{p}$ 为球内电荷的电偶极矩。另，球外电荷在球内产生的电场之平均值等于球外电荷在球心产生的电场。

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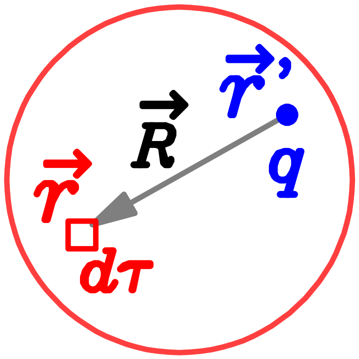
取球心为坐标原点，设在球内 $\vec{r}'$ 处有一点电荷 q



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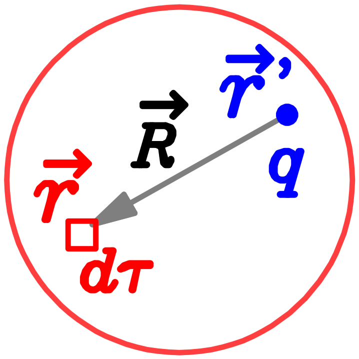

q 在球内 $\vec{r}$ 处产生的电场为：$\vec{E}_q^{\,\vec{r}\,'}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$

q 产生的电场在球内的平均值为：$\vec{E}_q^{\,\vec{r}\,'}(\text{ave}) = \dfrac{1}{\frac{4\pi}{3}a^3}\displaystyle\int_{\text{球}} \vec{E}_q^{\,\vec{r}\,'}(\vec{r})\,\mathrm{d}\tau$

例 2：试证明球内电荷分布在球内产生的电场之平均值为：$\vec{E}_{\text{ave}} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}}{a^3}$，其中 a 为球半径，$\vec{p}$ 为球内电荷的电偶极矩。另，球外电荷在球内产生的电场之平均值等于球外电荷在球心产生的电场。

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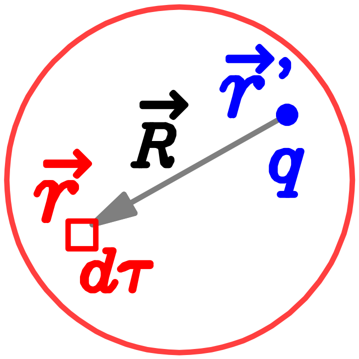

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故：$\vec{E}_q^{\,\vec{r}\,'}(\text{ave}) = \dfrac{1}{\frac{4\pi}{3}a^3}\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{球}} \dfrac{q\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\mathrm{d}\tau$

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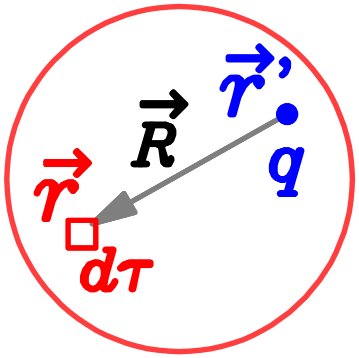

取球心为坐标原点，设在球内 $\vec{r}'$ 处有一点电荷 q

q 在球内 $\vec{r}$ 处产生的电场为：$\vec{E}_q^{\vec{r}'}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

q 产生的电场在球内的平均值为：$\vec{E}_q^{\vec{r}'}(\text{ave}) = \frac{1}{\frac{4\pi}{3}a^3}\int_{\text{球}} \vec{E}_q^{\vec{r}'}(\vec{r})\,\mathrm{d}\tau$

故：$\vec{E}_q^{\vec{r}'}(\text{ave}) = \frac{1}{\frac{4\pi}{3}a^3}\frac{1}{4\pi\epsilon_0}\int_{\text{球}} \frac{q\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,\mathrm{d}\tau$

现让 q 均匀分布于球上，再在球内 $\vec{r}$ 处取一小体积元 $\mathrm{d}\tau$，该小体积元的电量为 $\frac{q}{\frac{4\pi}{3}a^3}\,\mathrm{d}\tau$

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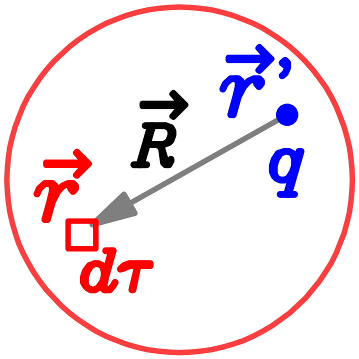

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该小体积元 $\mathrm{d}\tau$ 在球内 $\vec{r}\,'$ 处的电场为：$\mathrm{d}\vec{E}_{\mathrm{d}\tau}(\vec{r}\,') = \dfrac{1}{4\pi\epsilon_0}\dfrac{(\vec{r}\,'-\vec{r})}{|\vec{r}\,'-\vec{r}|^3}\dfrac{q}{\frac{4\pi}{3}a^3}\,\mathrm{d}\tau$

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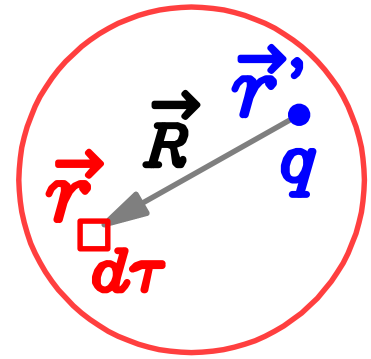

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故：$\vec{E}_q^{\vec{r}'}(\text{ave}) = \dfrac{1}{\frac{4\pi}{3}a^3}\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{球} \dfrac{q\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,\mathrm{d}\tau$

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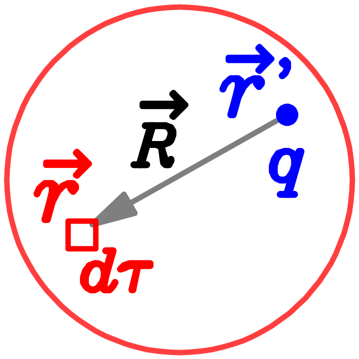
该小体积元 $\mathrm{d}\tau$ 在球内 $\vec{r}'$ 处的电场为：$\mathrm{d}\vec{E}_{\mathrm{d}\tau}(\vec{r}') = \dfrac{1}{4\pi\epsilon_0}\dfrac{(\vec{r}'-\vec{r})}{|\vec{r}'-\vec{r}|^3}\,\dfrac{q}{\frac{4\pi}{3}a^3}\,\mathrm{d}\tau$

整个均匀带电球在球内 $\vec{r}'$ 处的电场为：

例 2：试证明球内电荷分布在球内产生的电场之平均值为：$\vec{E}_{\text{ave}} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}}{a^3}$，其中 a 为球半径，$\vec{p}$ 为球内电荷的电偶极矩。另，球外电荷在球内产生的电场之平均值等于球外电荷在球心产生的电场。



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故：$\vec{E}_q^{\vec{r}'}(\text{ave}) = \dfrac{1}{\frac{4\pi}{3}a^3}\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{球}} \dfrac{q\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,\mathrm{d}\tau$

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整个均匀带电球在球内 $\vec{r}'$ 处的电场为：$\vec{E}_q^{\text{球}}(\vec{r}') = \displaystyle\int \mathrm{d}\vec{E}_{\mathrm{d}\tau}(\vec{r}') = \int \dfrac{1}{4\pi\epsilon_0}\dfrac{(\vec{r}'-\vec{r})}{|\vec{r}'-\vec{r}|^3}\dfrac{q}{\frac{4\pi}{3}a^3}\,\mathrm{d}\tau$

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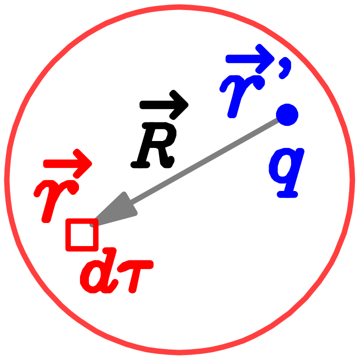

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q 在球内 $\vec{\boldsymbol{r}}$ 处产生的电场为：$\vec{\boldsymbol{E}}_q^{\vec{r}'}(\vec{\boldsymbol{r}}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q\,(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^3}$

q 产生的电场在球内的平均值为：$\vec{\boldsymbol{E}}_q^{\vec{r}'}(\text{ave}) = \dfrac{1}{\frac{4\pi}{3}a^3}\displaystyle\int_{\text{球}} \vec{\boldsymbol{E}}_q^{\vec{r}'}(\vec{\boldsymbol{r}})\,\mathrm{d}\tau$

故：$\vec{\boldsymbol{E}}_q^{\vec{r}'}(\text{ave}) = \dfrac{1}{\frac{4\pi}{3}a^3}\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{球}} \dfrac{q\,(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^3}\,\mathrm{d}\tau$

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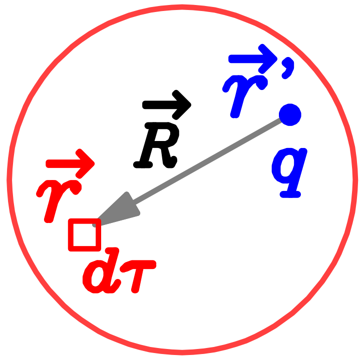

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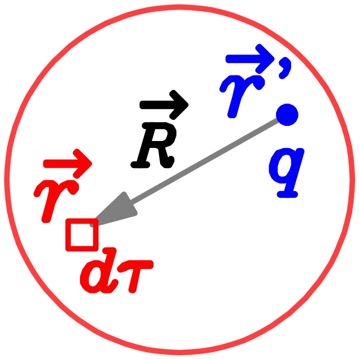

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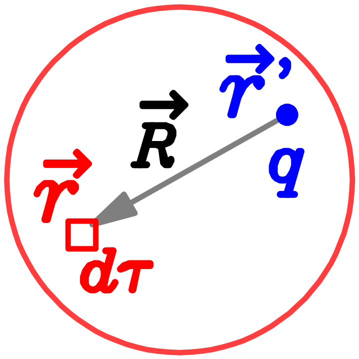

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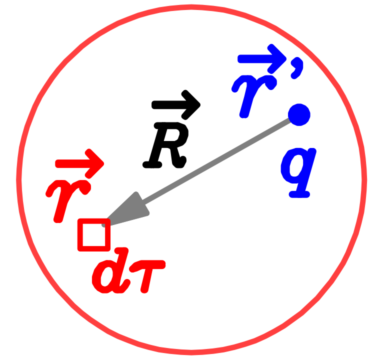

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Let there be light

$$\vec{\boldsymbol{E}}_q^{\vec{r}\,'}(\text{ave}) = -\frac{q\,\vec{\boldsymbol{r}}\,'}{4\pi\epsilon_0 a^3}$$

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而处于 $\vec{\boldsymbol{r}}\,'$ 的点电荷 q 在球心的电场 $\vec{\boldsymbol{E}}_q^{\vec{r}\,'}(0)$ 为：$\vec{\boldsymbol{E}}_q^{\vec{r}\,'}(0) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q\,(-\vec{\boldsymbol{r}}\,')}{r'^{\,3}} = -\vec{\boldsymbol{E}}_q^{\text{球}}(\vec{\boldsymbol{r}}\,')$

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$$\vec{E}_q^{\vec{r}\,'}(\text{ave}) = -\frac{q\,\vec{r}\,'}{4\pi\epsilon_0 a^3}$$

位于球内 $\vec{r}\,'$ 处的一点电荷 q 在球内产生的电场之平均值 $= -\dfrac{\text{该点电荷的电偶极矩}\ q\,\vec{r}\,'}{4\pi\epsilon_0 a^3}$

若球内有一电荷分布，由叠加原理知

该电荷分布在球内的电场之平均值为各小电荷元 $\mathrm{d}q$ 的球内电场平均值之和

从而有：$\vec{E}_{\text{ave}} = -\dfrac{1}{4\pi\epsilon_0 a^3}\displaystyle\int \vec{r}\,'\,\mathrm{d}q = -\dfrac{\vec{p}}{4\pi\epsilon_0 a^3}.$ qed

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即：球外任意处的点电荷元 q 在球内的电场之平均值等于 q 在球心的电场值

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由叠加原理易推广至球外任意电荷分布情况

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练习： 试对点电偶极子验证本例题的结论。