§1.4 并矢与张量

一、两个矢量的三种乘积运算

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点积（dot product, 也称标积）： (对重复下标求和)

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$$\vec{a}\cdot\vec{b}=a_ib_j\,\hat{e}_i\cdot\hat{e}_j=a_ib_j\delta_{ij}=a_ib_i$$ 结果为标量

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物理用例：功 $W=\vec{F}\cdot\vec{l}$

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$$\vec{a}\cdot\vec{b} = a_i b_j\,\hat{e}_i\cdot\hat{e}_j = a_i b_j \delta_{ij} = a_i b_i$$

结果为标量

物理用例：功 $W = \vec{F}\cdot\vec{l}$

叉积（cross product, 也称矢积）：　(对重复下标求和)

$$\vec{a}\times\vec{b} = a_i b_j\,\hat{e}_i\times\hat{e}_j = a_i b_j\,\varepsilon_{ijk}\,\hat{e}_k$$

结果为矢量

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物理用例：力矩 $\vec{L}=\vec{r}\times\vec{F}$

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结果为并矢 (dyad)，用 $\overset{\Rightarrow}{A}$ 表示

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$$\vec{a}\times\vec{b}=a_ib_j\,\hat{e}_i\times\,\hat{e}_j=a_ib_j\,\varepsilon_{ijk}\,\hat{e}_k$$

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并积（dyadic product）：　(对重复下标求和)

$$\vec{a}\,\vec{b}=a_ib_j\,\hat{e}_i\,\hat{e}_j$$

结果为并矢 (dyad)，用 $\overset{\Rightarrow}{A}$ 表示

物理用例：动量流 $\vec{v}\,\vec{p}$、自旋流 $\vec{v}\,\vec{s}$

并积 (dyadic product) 和并矢 (dyad) 的性质

并积 (dyadic product) 和并矢 (dyad) 的性质

1. 不满足交换律

$$\vec{a}\vec{b} \neq \vec{b}\vec{a}$$

并积 (dyadic product) 和并矢 (dyad) 的性质

1. 不满足交换律

$$\vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}} \neq \vec{\boldsymbol{b}}\,\vec{\boldsymbol{a}}$$

2. 矩阵表示，9 个分量，只有 6 个独立

$$\overset{\Rightarrow}{\boldsymbol{A}} = \vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}, \qquad \overset{\Rightarrow}{\boldsymbol{A}} = A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = a_ib_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j$$

并积 (dyadic product) 和并矢 (dyad) 的性质

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3. 并矢的转置 (transpose)　　类似于距阵之转置

$$(\overset{\Rightarrow}{A})^T = (\vec{a}\,\vec{b})^T = \vec{b}\,\vec{a} = \begin{bmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \end{bmatrix}, \quad (\overset{\Rightarrow}{A})^T = a_jb_i\,\hat{e}_i\,\hat{e}_j$$

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3. 并矢的转置 (transpose)　类似于距阵之转置

$$(\overset{\Rightarrow}{\boldsymbol{A}})^T = (\vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}})^T = \vec{\boldsymbol{b}}\,\vec{\boldsymbol{a}} = \begin{bmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \end{bmatrix}, \qquad (\overset{\Rightarrow}{\boldsymbol{A}})^T = a_jb_i\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j$$

4. 并矢之和：$\vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}\,\vec{\boldsymbol{d}} = (a_ib_j + c_id_j)\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j$　类似于距阵之和

并积 (dyadic product) 和并矢 (dyad) 的性质

1. 不满足交换律

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$$(\overset{\Rightarrow}{\boldsymbol{A}})^T = (\vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}})^T = \vec{\boldsymbol{b}}\,\vec{\boldsymbol{a}} = \begin{bmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \end{bmatrix}, \qquad (\overset{\Rightarrow}{\boldsymbol{A}})^T = a_jb_i\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j$$

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5. 并矢式 (dyadic)：两个或两个以上的并矢之和，9 个独立分量 A_{ij}

并积 (dyadic product) 和并矢 (dyad) 的性质

1. 不满足交换律

$$\vec{a}\,\vec{b} \neq \vec{b}\,\vec{a}$$

2. 矩阵表示，9 个分量，只有 6 个独立

$$\overset{\Rightarrow}{A} = \vec{a}\,\vec{b} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}, \qquad \overset{\Rightarrow}{A} = A_{ij}\,\hat{e}_i\,\hat{e}_j = a_ib_j\,\hat{e}_i\,\hat{e}_j$$

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4. 并矢之和：$\vec{a}\,\vec{b} + \vec{c}\,\vec{d} = (a_ib_j + c_id_j)\,\hat{e}_i\,\hat{e}_j$　　类似于距阵之和

5. 并矢式 (dyadic)：两个或两个以上的并矢之和，9 个独立分量 A_{ij}
也称张量 (tensor)，用 $\overset{\leftrightarrow}{A}$ 表示，$\overset{\leftrightarrow}{A} = A_{ij}\,\hat{e}_i\,\hat{e}_j$

二、张量代数

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1. 矢量与张量的点积：

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二、张量代数

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左点积： 相当于行矢量左乘距阵

二、张量代数

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左点积：　相当于行矢量左乘距阵

$$\vec{a}\cdot\overleftrightarrow{A} = (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)$$

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$$\vec{a}\cdot\overleftrightarrow{A} = (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k) = a_i A_{jk}\;\hat{e}_i\cdot\hat{e}_j\;\hat{e}_k$$

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$$
\begin{aligned}
\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k) \\
&= a_i A_{jk}\ \hat{e}_i\cdot\hat{e}_j\ \hat{e}_k \\
&= a_i A_{jk}\,\delta_{ij}\ \hat{e}_k
\end{aligned}
$$

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$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)\\ &= a_iA_{jk}\;\hat{e}_i\cdot\hat{e}_j\;\hat{e}_k\\ &= a_iA_{jk}\,\delta_{ij}\;\hat{e}_k = a_iA_{ik}\,\hat{e}_k\end{aligned}$$

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右点积： 相当于列矢量右乘距阵

$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)\\ &= a_iA_{jk}\;\hat{e}_i\cdot\hat{e}_j\;\hat{e}_k\\ &= a_iA_{jk}\,\delta_{ij}\;\hat{e}_k = a_iA_{ik}\,\hat{e}_k\end{aligned}$$

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$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)\\ &= a_iA_{jk}\,\hat{e}_i\cdot\hat{e}_j\,\hat{e}_k\\ &= a_iA_{jk}\,\delta_{ij}\,\hat{e}_k = a_iA_{ik}\,\hat{e}_k\end{aligned}$$

右点积：　相当于列矢量右乘距阵

$$\overleftrightarrow{A}\cdot\vec{a} = (A_{jk}\,\hat{e}_j\,\hat{e}_k)\cdot(a_i\,\hat{e}_i)$$

二、张量代数

1. 矢量与张量的点积： 注意：$\vec{a}\cdot\overleftrightarrow{A}\neq\overleftrightarrow{A}\cdot\vec{a}$

左点积： 相当于行矢量左乘距阵

$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)\\ &= a_iA_{jk}\,\hat{e}_i\cdot\hat{e}_j\,\hat{e}_k\\ &= a_iA_{jk}\,\delta_{ij}\,\hat{e}_k = a_iA_{ik}\,\hat{e}_k\end{aligned}$$

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2. 矢量与并矢的点积：

二、张量代数

1. 矢量与张量的点积： 注意：$\vec{a}\cdot\overleftrightarrow{A}\neq\overleftrightarrow{A}\cdot\vec{a}$

左点积： 相当于行矢量左乘距阵

$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k) \\ &= a_iA_{jk}\;\hat{e}_i\cdot\hat{e}_j\;\hat{e}_k \\ &= a_iA_{jk}\,\delta_{ij}\;\hat{e}_k = a_iA_{ik}\,\hat{e}_k\end{aligned}$$

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$$\begin{aligned}\overleftrightarrow{A}\cdot\vec{a} &= (A_{jk}\,\hat{e}_j\,\hat{e}_k)\cdot(a_i\,\hat{e}_i) \\ &= a_iA_{jk}\,\hat{e}_j\;\hat{e}_k\cdot\hat{e}_i \\ &= a_iA_{jk}\,\hat{e}_j\,\delta_{ik} = a_iA_{ji}\,\hat{e}_j\end{aligned}$$

2. 矢量与并矢的点积：并矢 $\overset{\rightrightarrows}{A}=\vec{b}\,\vec{c}=b_jc_k\,\hat{e}_j\,\hat{e}_k$,

二、张量代数

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$$\begin{aligned}\overleftrightarrow{A}\cdot\vec{a} &= (A_{jk}\,\hat{e}_j\,\hat{e}_k)\cdot(a_i\,\hat{e}_i)\\ &= a_iA_{jk}\,\hat{e}_j\;\hat{e}_k\cdot\hat{e}_i\\ &= a_iA_{jk}\,\hat{e}_j\,\delta_{ik} = a_iA_{ji}\,\hat{e}_j\end{aligned}$$

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二、张量代数

1. 矢量与张量的点积： 注意： $\vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{A}} \neq \overleftrightarrow{\boldsymbol{A}} \cdot \vec{\boldsymbol{a}}$

左点积： 相当于行矢量左乘距阵

$$\begin{aligned}\vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{A}} &= (a_i\, \hat{\boldsymbol{e}}_i) \cdot (A_{jk}\, \hat{\boldsymbol{e}}_j\, \hat{\boldsymbol{e}}_k) \\ &= a_i A_{jk}\; \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j\; \hat{\boldsymbol{e}}_k \\ &= a_i A_{jk}\, \delta_{ij}\; \hat{\boldsymbol{e}}_k = a_i A_{ik}\, \hat{\boldsymbol{e}}_k\end{aligned}$$

右点积： 相当于列矢量右乘距阵

$$\begin{aligned}\overleftrightarrow{\boldsymbol{A}} \cdot \vec{\boldsymbol{a}} &= (A_{jk}\, \hat{\boldsymbol{e}}_j\, \hat{\boldsymbol{e}}_k) \cdot (a_i\, \hat{\boldsymbol{e}}_i) \\ &= a_i A_{jk}\, \hat{\boldsymbol{e}}_j\; \hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}_i \\ &= a_i A_{jk}\, \hat{\boldsymbol{e}}_j\, \delta_{ik} = a_i A_{ji}\, \hat{\boldsymbol{e}}_j\end{aligned}$$

2. 矢量与并矢的点积：并矢 $\overrightarrow{\overrightarrow{\boldsymbol{A}}} = \vec{\boldsymbol{b}}\,\vec{\boldsymbol{c}} = b_j c_k\, \hat{\boldsymbol{e}}_j\, \hat{\boldsymbol{e}}_k$， 注意： $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}}\,\vec{\boldsymbol{c}}) \neq (\vec{\boldsymbol{b}}\,\vec{\boldsymbol{c}}) \cdot \vec{\boldsymbol{a}}$

左点积：

二、张量代数

1. 矢量与张量的点积：　注意：$\vec{a}\cdot\overleftrightarrow{A}\neq\overleftrightarrow{A}\cdot\vec{a}$

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$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\,\hat{e}_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)\\ &= a_iA_{jk}\;\hat{e}_i\cdot\hat{e}_j\;\hat{e}_k\\ &= a_iA_{jk}\,\delta_{ij}\;\hat{e}_k = a_iA_{ik}\,\hat{e}_k\end{aligned}$$

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5. 张量与张量的点积：相当于距阵相乘

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$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l$$

5. 张量与张量的点积：相当于距阵相乘

$$
\begin{aligned}
\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} &= (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l) \\
&= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l \\
&= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l
\end{aligned}
$$

5. 张量与张量的点积：相当于距阵相乘

$$
\begin{aligned}
\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} &= (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l) \\
&= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l \\
&= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l
\end{aligned}
$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l$$
$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\ \hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\ \hat{\boldsymbol{e}}_l$$
$$= A_{ij}B_{kl}\,\delta_{jk}\ \hat{\boldsymbol{e}}_i\ \hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\ \hat{\boldsymbol{e}}_i\ \hat{\boldsymbol{e}}_l \qquad \text{仍是张量}$$

6. 张量与张量的双点积：

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\,\cdot\,(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i)\cdot[(b_j\,\hat{\boldsymbol{e}}_j)\cdot(C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

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$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

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张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}$$
$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i) \cdot [(b_j\,\hat{\boldsymbol{e}}_j) \cdot (C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_l)]$$

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$$
\begin{aligned}
\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} &= (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l) \\
&= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l \\
&= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l \qquad \text{仍是张量}
\end{aligned}
$$

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$
\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}
$$

$$
\begin{aligned}
\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) &= (a_i\,\hat{\boldsymbol{e}}_i)\cdot[(b_j\,\hat{\boldsymbol{e}}_j)\cdot(C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)] \\
&= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_l)] \\
&= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) : (\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]
\end{aligned}
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5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}$$
$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i)\cdot[(b_j\,\hat{\boldsymbol{e}}_j)\cdot(C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_l)]$$
$$= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) : (\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) : \overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{a}}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l$$
$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

$$\overleftrightarrow{\boldsymbol{A}} : \overleftrightarrow{\boldsymbol{B}}$$

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}$$
$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i) \cdot [(b_j\,\hat{\boldsymbol{e}}_j) \cdot (C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_l)]$$
$$= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) : (\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) : \overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{a}}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}} : \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) : (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} = A_{11} + A_{22} + A_{33}$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i)\cdot[(b_j\,\hat{\boldsymbol{e}}_j)\cdot(C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= a_i b_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_l)]$$
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$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) : \overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{a}}$$

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$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) \cdot (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l$$
$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l \qquad \text{仍是张量}$$

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} = A_{11} + A_{22} + A_{33}$$

$$\overleftrightarrow{\boldsymbol{A}} : \overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) : (B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,[(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j) : (\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i) \cdot [(b_j\,\hat{\boldsymbol{e}}_j) \cdot (C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
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$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) : \overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{a}}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\cdot(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} \quad = A_{11} + A_{22} + A_{33}$$

$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i)\cdot[(b_j\,\hat{\boldsymbol{e}}_j)\cdot(C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)\,(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j):(\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

$$\overleftrightarrow{\boldsymbol{A}}:\overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j):(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}):\overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{a}}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\cdot(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} = A_{11} + A_{22} + A_{33}$$

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$$= a_ib_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\,\boldsymbol{:}\,(\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

$$\overleftrightarrow{\boldsymbol{A}}\,\boldsymbol{:}\,\overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\;\boldsymbol{:}\;(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,[(\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\,\boldsymbol{:}\,(\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= A_{ij}B_{kl}\,[(\,\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)\,(\,\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
$$= A_{ij}B_{kl}\,\delta_{jk}\delta_{il}$$

$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}})\,\boldsymbol{:}\,\overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{a}}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\cdot(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
$$= A_{ij}B_{kl}\,\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k\;\hat{\boldsymbol{e}}_l$$
$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} = A_{11}+A_{22}+A_{33}$$

$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}) = (a_i\,\hat{\boldsymbol{e}}_i)\cdot[(b_j\,\hat{\boldsymbol{e}}_j)\cdot(C_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j):(\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$

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$$= A_{ij}B_{kl}\,[(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j):(\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= A_{ij}B_{kl}\,[(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
$$= A_{ij}B_{kl}\,\delta_{jk}\delta_{il}$$
$$= A_{ij}B_{ji}$$

$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}):\overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{a}}$$

5. 张量与张量的点积：相当于距阵相乘

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{B}} = (A_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)\cdot(B_{kl}\,\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)$$
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$$= A_{ij}B_{kl}\,\delta_{jk}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l = A_{ij}B_{jl}\;\hat{\boldsymbol{e}}_i\;\hat{\boldsymbol{e}}_l$$

仍是张量

6. 张量与张量的双点积：相当于距阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{\boldsymbol{A}}_t = A_{ii} = A_{11}+A_{22}+A_{33}$$

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$$= a_ib_j\,C_{kl}\,[(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
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$$= A_{ij}B_{kl}\,[(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j):(\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l)]$$
$$= A_{ij}B_{kl}\,[(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)]$$
$$= A_{ij}B_{kl}\,\delta_{jk}\delta_{il}$$
$$= A_{ij}B_{ji}$$

退化为标量

$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}):\overleftrightarrow{\boldsymbol{C}} = \vec{\boldsymbol{b}}\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{a}}$$

$$(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j):(\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_l) \equiv (\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_l)$$

7. 单位张量：

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当张量的分量 $A_{ij}=\delta_{ij}$ 时，这时的张量称为单位张量

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$$\overleftrightarrow{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

7. 单位张量：

当张量的分量 $A_{ij} = \delta_{ij}$ 时，这时的张量称为单位张量

$$\overset{\leftrightarrow}{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}} \cdot \overset{\leftrightarrow}{\boldsymbol{I}} = a_i\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j = a_j\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{a}}$$

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当张量的分量 $A_{ij} = \delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{I}} = a_i\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j = a_j\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{a}} \quad \text{即：} \begin{cases} \vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{I}} = \vec{\boldsymbol{a}} \\ \text{同理可证} \\ \overleftrightarrow{\boldsymbol{I}} \cdot \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}} \end{cases}$$

7. 单位张量：

当张量的分量 $A_{ij} = \delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{I} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}}\cdot\overleftrightarrow{I} = a_i\,\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j = a_j\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{a}}$$

即：
$$\begin{cases} \vec{\boldsymbol{a}}\cdot\overleftrightarrow{I} = \vec{\boldsymbol{a}} \\ \text{同理可证} \\ \overleftrightarrow{I}\cdot\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}} \end{cases}$$

既然张量的点积同距阵的乘积，易证：

$$\overleftrightarrow{A}\cdot\overleftrightarrow{I} = \overleftrightarrow{I}\cdot\overleftrightarrow{A} = \overleftrightarrow{A}$$

7. 单位张量：

当张量的分量 $A_{ij} = \delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{I}} = a_i\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j = a_j\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{a}}$$

即：
$\vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{I}} = \vec{\boldsymbol{a}}$
同理可证
$\overleftrightarrow{\boldsymbol{I}} \cdot \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}$

既然张量的点积同距阵的乘积，易证：

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{I}} = \overleftrightarrow{\boldsymbol{I}} \cdot \overleftrightarrow{\boldsymbol{A}} = \overleftrightarrow{\boldsymbol{A}}$$

双点积： $\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i) : (A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k)$

7. 单位张量：

当张量的分量 $A_{ij} = \delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{I}} = a_i\,\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j = a_j\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{a}}$$

即：
$$\begin{cases} \vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{I}} = \vec{\boldsymbol{a}} \\ \text{同理可证} \\ \overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}} \end{cases}$$

既然张量的点积同矩阵的乘积，易证：

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{I}} = \overleftrightarrow{\boldsymbol{I}}\cdot\overleftrightarrow{\boldsymbol{A}} = \overleftrightarrow{\boldsymbol{A}}$$

双点积：
$$\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i) : (A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k) = A_{jk}(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k)$$

7. 单位张量：

当张量的分量 $A_{ij} = \delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{I}} = a_i\,\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j = a_j\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{a}}$$

即：
$$\begin{cases} \vec{\boldsymbol{a}} \cdot \overleftrightarrow{\boldsymbol{I}} = \vec{\boldsymbol{a}} \\ \text{同理可证} \\ \overleftrightarrow{\boldsymbol{I}} \cdot \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}} \end{cases}$$

既然张量的点积同距阵的乘积，易证：

$$\overleftrightarrow{\boldsymbol{A}} \cdot \overleftrightarrow{\boldsymbol{I}} = \overleftrightarrow{\boldsymbol{I}} \cdot \overleftrightarrow{\boldsymbol{A}} = \overleftrightarrow{\boldsymbol{A}}$$

双点积： $\overleftrightarrow{\boldsymbol{I}} : \overleftrightarrow{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i) : (A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k) = A_{jk}(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j)(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_k)$

$$= A_{jk}(\delta_{ij})(\delta_{ik})$$

7. 单位张量：

当张量的分量 $A_{ij}=\delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}}=\delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j=\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{I}}=a_i\,\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j=a_i\,\delta_{ij}\,\hat{\boldsymbol{e}}_j=a_j\,\hat{\boldsymbol{e}}_j=\vec{\boldsymbol{a}}$$

即：
$$\begin{cases}\vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{I}}=\vec{\boldsymbol{a}}\\ \text{同理可证}\\ \overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{a}}=\vec{\boldsymbol{a}}\end{cases}$$

既然张量的点积同距阵的乘积，易证：

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{I}}=\overleftrightarrow{\boldsymbol{I}}\cdot\overleftrightarrow{\boldsymbol{A}}=\overleftrightarrow{\boldsymbol{A}}$$

双点积：
$$\overleftrightarrow{\boldsymbol{I}}:\overleftrightarrow{\boldsymbol{A}}=(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i):(A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k)=A_{jk}(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k)$$

$$=A_{jk}(\delta_{ij})(\delta_{ik})=A_{ii}=$$

7. 单位张量：

当张量的分量 $A_{ij}=\delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{I}=\delta_{ij}\,\hat{e}_i\,\hat{e}_j=\hat{e}_i\,\hat{e}_i$$

$$\vec{a}\cdot\overleftrightarrow{I}=a_i\,\hat{e}_i\cdot\hat{e}_j\,\hat{e}_j=a_i\,\delta_{ij}\,\hat{e}_j=a_j\,\hat{e}_j=\vec{a}$$

即：$\vec{a}\cdot\overleftrightarrow{I}=\vec{a}$，同理可证 $\overleftrightarrow{I}\cdot\vec{a}=\vec{a}$

既然张量的点积同距阵的乘积，易证：

$$\overleftrightarrow{A}\cdot\overleftrightarrow{I}=\overleftrightarrow{I}\cdot\overleftrightarrow{A}=\overleftrightarrow{A}$$

双点积：

$$\overleftrightarrow{I}:\overleftrightarrow{A}=(\hat{e}_i\,\hat{e}_i):(A_{jk}\,\hat{e}_j\,\hat{e}_k)=A_{jk}(\hat{e}_i\cdot\hat{e}_j)(\hat{e}_i\cdot\hat{e}_k)$$

$$=A_{jk}(\delta_{ij})(\delta_{ik})=A_{ii}=\boxed{\overleftrightarrow{A}_t=\overleftrightarrow{I}:\overleftrightarrow{A}=\overleftrightarrow{A}:\overleftrightarrow{I}}$$

7. 单位张量：

当张量的分量 $A_{ij}=\delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}} = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i$$

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既然张量的点积同距阵的乘积，易证：

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{I}} = \overleftrightarrow{\boldsymbol{I}}\cdot\overleftrightarrow{\boldsymbol{A}} = \overleftrightarrow{\boldsymbol{A}}$$

双点积：
$$\overleftrightarrow{\boldsymbol{I}}:\overleftrightarrow{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i):(A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k) = A_{jk}(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k)$$

$$= A_{jk}(\delta_{ij})(\delta_{ik}) = A_{ii} = \boxed{\overleftrightarrow{\boldsymbol{A}}_t = \overleftrightarrow{\boldsymbol{I}}:\overleftrightarrow{\boldsymbol{A}} = \overleftrightarrow{\boldsymbol{A}}:\overleftrightarrow{\boldsymbol{I}}}$$

如 $\overleftrightarrow{\boldsymbol{A}}=\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}$, 则 $A_{ii}=a_ib_i$，从而：
$$\boxed{\overleftrightarrow{\boldsymbol{I}}:(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) = a_ib_i = \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}}$$

8. 若干常用恒等式

交换律：

$$\vec{\boldsymbol{u}} \cdot \overleftrightarrow{\boldsymbol{A}} = (\overleftrightarrow{\boldsymbol{A}})^T \cdot \vec{\boldsymbol{u}}$$

8. 若干常用恒等式

交换律：

$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

但是：

$$\vec{u}\times\overleftrightarrow{A}\neq(\overleftrightarrow{A})^T\times\vec{u}$$
$$\vec{u}\times\overleftrightarrow{A}\neq-(\overleftrightarrow{A})^T\times\vec{u}$$

8. 若干常用恒等式

交换律：

$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

$$\left[\overleftrightarrow{A}\times\vec{u}\right]^T=-\vec{u}\times(\overleftrightarrow{A})^T$$

但是：

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$$\vec{u}\cdot\overleftrightarrow{A} = (\overleftrightarrow{A})^T\cdot\vec{u}$$

但是：

$$\vec{u}\times\overleftrightarrow{A} \neq (\overleftrightarrow{A})^T\times\vec{u}$$

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$$\left[\overleftrightarrow{A}\times\vec{u}\right]^T = -\vec{u}\times(\overleftrightarrow{A})^T$$

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结合律：

8. 若干常用恒等式

交换律:

$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

但是:

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$$[\overleftrightarrow{A}\times\vec{u}]^T=-\vec{u}\times(\overleftrightarrow{A})^T$$

$$[\vec{u}\times\overleftrightarrow{A}]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律:

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

8. 若干常用恒等式

交换律:

$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

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$$\left[\overleftrightarrow{A}\times\vec{u}\right]^T=-\vec{u}\times(\overleftrightarrow{A})^T \qquad \left[\vec{u}\times\overleftrightarrow{A}\right]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律:

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v})=(\vec{u}\cdot\overleftrightarrow{A})\times\vec{v}$$

8. 若干常用恒等式

交换律：

$$\vec{u}\cdot\overleftrightarrow{A} = (\overleftrightarrow{A})^T\cdot\vec{u}$$

但是：

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$$[\overleftrightarrow{A}\times\vec{u}]^T = -\vec{u}\times(\overleftrightarrow{A})^T$$

$$[\vec{u}\times\overleftrightarrow{A}]^T = -(\overleftrightarrow{A})^T\times\vec{u}$$

结合律：

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c} = \vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c}) = \vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c} = \vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v}) = (\vec{u}\cdot\overleftrightarrow{A})\times\vec{v}$$

$$(\vec{u}\times\overleftrightarrow{A})\cdot\vec{v} = \vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$

8. 若干常用恒等式

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$$[\overleftrightarrow{A}\times\vec{u}]^T=-\vec{u}\times(\overleftrightarrow{A})^T$$

$$[\vec{u}\times\overleftrightarrow{A}]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律：

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v})=(\vec{u}\cdot\overleftrightarrow{A})\times\vec{v}$$

$$(\vec{u}\times\overleftrightarrow{A})\cdot\vec{v}=\vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$

$$\overleftrightarrow{B}\cdot(\overleftrightarrow{A}\times\vec{u})=(\overleftrightarrow{B}\cdot\overleftrightarrow{A})\times\vec{u}$$

8. 若干常用恒等式

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$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

但是:
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$$\left[\overleftrightarrow{A}\times\vec{u}\right]^T=-\vec{u}\times(\overleftrightarrow{A})^T \qquad \left[\vec{u}\times\overleftrightarrow{A}\right]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律:

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v})=(\vec{u}\cdot\overleftrightarrow{A})\times\vec{v} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\vec{v}=\vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$
$$\overleftrightarrow{B}\cdot(\overleftrightarrow{A}\times\vec{u})=(\overleftrightarrow{B}\cdot\overleftrightarrow{A})\times\vec{u} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\overleftrightarrow{B}=\vec{u}\times(\overleftrightarrow{A}\cdot\overleftrightarrow{B})$$

8. 若干常用恒等式

交换律:

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$$[\overleftrightarrow{A}\times\vec{u}]^T=-\vec{u}\times(\overleftrightarrow{A})^T \qquad [\vec{u}\times\overleftrightarrow{A}]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律:

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v})=(\vec{u}\cdot\overleftrightarrow{A})\times\vec{v} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\vec{v}=\vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$
$$\overleftrightarrow{B}\cdot(\overleftrightarrow{A}\times\vec{u})=(\overleftrightarrow{B}\cdot\overleftrightarrow{A})\times\vec{u} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\overleftrightarrow{B}=\vec{u}\times(\overleftrightarrow{A}\cdot\overleftrightarrow{B})$$

混合积:

类似于 $(\vec{w}\times\vec{u})\cdot\vec{v}=\vec{w}\cdot(\vec{u}\times\vec{v})$

8. 若干常用恒等式

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$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

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$$\vec{u}\times\overleftrightarrow{A}\neq(\overleftrightarrow{A})^T\times\vec{u}$$
$$\vec{u}\times\overleftrightarrow{A}\neq-(\overleftrightarrow{A})^T\times\vec{u}$$

$$[\overleftrightarrow{A}\times\vec{u}]^T=-\vec{u}\times(\overleftrightarrow{A})^T \qquad [\vec{u}\times\overleftrightarrow{A}]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律：

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v})=(\vec{u}\cdot\overleftrightarrow{A})\times\vec{v} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\vec{v}=\vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$
$$\overleftrightarrow{B}\cdot(\overleftrightarrow{A}\times\vec{u})=(\overleftrightarrow{B}\cdot\overleftrightarrow{A})\times\vec{u} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\overleftrightarrow{B}=\vec{u}\times(\overleftrightarrow{A}\cdot\overleftrightarrow{B})$$

混合积：

类似于 $(\vec{w}\times\vec{u})\cdot\vec{v}=\vec{w}\cdot(\vec{u}\times\vec{v})$ $\xrightarrow{\text{推广}}$ $\left\{ (\overleftrightarrow{A}\times\vec{u})\cdot\vec{v}=\overleftrightarrow{A}\cdot(\vec{u}\times\vec{v}) \right.$

8. 若干常用恒等式

交换律：

$$\vec{\boldsymbol{u}}\cdot\overleftrightarrow{\boldsymbol{A}} = (\overleftrightarrow{\boldsymbol{A}})^T\cdot\vec{\boldsymbol{u}}$$

但是：

$$\vec{\boldsymbol{u}}\times\overleftrightarrow{\boldsymbol{A}} \neq (\overleftrightarrow{\boldsymbol{A}})^T\times\vec{\boldsymbol{u}}$$

$$\vec{\boldsymbol{u}}\times\overleftrightarrow{\boldsymbol{A}} \neq -(\overleftrightarrow{\boldsymbol{A}})^T\times\vec{\boldsymbol{u}}$$

$$[\overleftrightarrow{\boldsymbol{A}}\times\vec{\boldsymbol{u}}]^T = -\vec{\boldsymbol{u}}\times(\overleftrightarrow{\boldsymbol{A}})^T \qquad [\vec{\boldsymbol{u}}\times\overleftrightarrow{\boldsymbol{A}}]^T = -(\overleftrightarrow{\boldsymbol{A}})^T\times\vec{\boldsymbol{u}}$$

结合律：

$$(\vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{B}})\cdot\vec{\boldsymbol{c}} = \vec{\boldsymbol{a}}\cdot(\overleftrightarrow{\boldsymbol{B}}\cdot\vec{\boldsymbol{c}}) = \vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{B}}\cdot\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}\cdot(\overleftrightarrow{\boldsymbol{B}})^T\cdot\vec{\boldsymbol{a}}$$

$$\vec{\boldsymbol{u}}\cdot(\overleftrightarrow{\boldsymbol{A}}\times\vec{\boldsymbol{v}}) = (\vec{\boldsymbol{u}}\cdot\overleftrightarrow{\boldsymbol{A}})\times\vec{\boldsymbol{v}} \qquad (\vec{\boldsymbol{u}}\times\overleftrightarrow{\boldsymbol{A}})\cdot\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}}\times(\overleftrightarrow{\boldsymbol{A}}\cdot\vec{\boldsymbol{v}})$$

$$\overleftrightarrow{\boldsymbol{B}}\cdot(\overleftrightarrow{\boldsymbol{A}}\times\vec{\boldsymbol{u}}) = (\overleftrightarrow{\boldsymbol{B}}\cdot\overleftrightarrow{\boldsymbol{A}})\times\vec{\boldsymbol{u}} \qquad (\vec{\boldsymbol{u}}\times\overleftrightarrow{\boldsymbol{A}})\cdot\overleftrightarrow{\boldsymbol{B}} = \vec{\boldsymbol{u}}\times(\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{B}})$$

混合积：

类似于 $(\vec{\boldsymbol{w}}\times\vec{\boldsymbol{u}})\cdot\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}\cdot(\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}})$ $\xrightarrow{\text{推广}}$ $\begin{cases}(\overleftrightarrow{\boldsymbol{A}}\times\vec{\boldsymbol{u}})\cdot\vec{\boldsymbol{v}} = \overleftrightarrow{\boldsymbol{A}}\cdot(\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}}) \\ (\overleftrightarrow{\boldsymbol{A}}\times\vec{\boldsymbol{u}})\cdot\overleftrightarrow{\boldsymbol{B}} = \overleftrightarrow{\boldsymbol{A}}\cdot(\vec{\boldsymbol{u}}\times\overleftrightarrow{\boldsymbol{B}})\end{cases}$

8. 若干常用恒等式

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$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

但是：
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$$[\overleftrightarrow{A}\times\vec{u}]^T=-\vec{u}\times(\overleftrightarrow{A})^T \qquad [\vec{u}\times\overleftrightarrow{A}]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律：
$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v})=(\vec{u}\cdot\overleftrightarrow{A})\times\vec{v} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\vec{v}=\vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$
$$\overleftrightarrow{B}\cdot(\overleftrightarrow{A}\times\vec{u})=(\overleftrightarrow{B}\cdot\overleftrightarrow{A})\times\vec{u} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\overleftrightarrow{B}=\vec{u}\times(\overleftrightarrow{A}\cdot\overleftrightarrow{B})$$

混合积：

类似于 $(\vec{w}\times\vec{u})\cdot\vec{v}=\vec{w}\cdot(\vec{u}\times\vec{v})$ $\xrightarrow{\text{推广}}$ $\begin{cases}(\overleftrightarrow{A}\times\vec{u})\cdot\vec{v}=\overleftrightarrow{A}\cdot(\vec{u}\times\vec{v})\\(\overleftrightarrow{A}\times\vec{u})\cdot\overleftrightarrow{B}=\overleftrightarrow{A}\cdot(\vec{u}\times\overleftrightarrow{B})\end{cases}$

$$\vec{u}\cdot(\vec{v}\times\vec{w})=(\vec{u}\times\vec{v})\cdot\vec{w}$$

8. 若干常用恒等式

交换律：

$$\vec{u}\cdot\overleftrightarrow{A}=(\overleftrightarrow{A})^T\cdot\vec{u}$$

但是：

$$\vec{u}\times\overleftrightarrow{A}\neq(\overleftrightarrow{A})^T\times\vec{u}$$
$$\vec{u}\times\overleftrightarrow{A}\neq-(\overleftrightarrow{A})^T\times\vec{u}$$

$$[\overleftrightarrow{A}\times\vec{u}]^T=-\vec{u}\times(\overleftrightarrow{A})^T \qquad [\vec{u}\times\overleftrightarrow{A}]^T=-(\overleftrightarrow{A})^T\times\vec{u}$$

结合律：

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c}=\vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c})=\vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c}=\vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

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混合积：

类似于 $(\vec{w}\times\vec{u})\cdot\vec{v}=\vec{w}\cdot(\vec{u}\times\vec{v}) \overset{\text{推广}}{\Longrightarrow} \begin{cases}(\overleftrightarrow{A}\times\vec{u})\cdot\vec{v}=\overleftrightarrow{A}\cdot(\vec{u}\times\vec{v})\\(\overleftrightarrow{A}\times\vec{u})\cdot\overleftrightarrow{B}=\overleftrightarrow{A}\cdot(\vec{u}\times\overleftrightarrow{B})\end{cases}$

$$\vec{u}\cdot(\vec{v}\times\vec{w})=(\vec{u}\times\vec{v})\cdot\vec{w} \overset{\text{推广}}{\Longrightarrow} \vec{u}\cdot(\vec{v}\times\overleftrightarrow{A})=(\vec{u}\times\vec{v})\cdot\overleftrightarrow{A}$$

连续叉积：

$$\vec{u} \times (\vec{v} \times \vec{w}) = \vec{v}\,(\vec{u} \cdot \vec{w}) - (\vec{u} \cdot \vec{v})\,\vec{w} = \vec{v}\,(\vec{w} \cdot \vec{u}) - (\vec{u} \cdot \vec{v})\,\vec{w}$$

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$\Downarrow$ 推广

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$$\Downarrow \text{推广}$$

$$\vec{u}\times(\vec{v}\times\overleftrightarrow{A}) = \vec{v}\,(\vec{u}\cdot\overleftrightarrow{A}) - (\vec{u}\cdot\vec{v})\overleftrightarrow{A}$$

连续叉积：

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$$\vec{u}\times(\vec{v}\times\overleftrightarrow{A}) = \vec{v}\,(\vec{u}\cdot\overleftrightarrow{A})-(\vec{u}\cdot\vec{v})\overleftrightarrow{A} \neq \vec{v}\,(\overleftrightarrow{A}\cdot\vec{u})-(\vec{u}\cdot\vec{v})\overleftrightarrow{A}$$

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$\Downarrow$ 推广

$$\vec{\boldsymbol{u}} \times (\vec{\boldsymbol{v}} \times \overleftrightarrow{\boldsymbol{A}}) = \vec{\boldsymbol{v}}\,(\vec{\boldsymbol{u}} \cdot \overleftrightarrow{\boldsymbol{A}}) - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\overleftrightarrow{\boldsymbol{A}} \neq \vec{\boldsymbol{v}}\,(\overleftrightarrow{\boldsymbol{A}} \cdot \vec{\boldsymbol{u}}) - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\overleftrightarrow{\boldsymbol{A}}$$

应保证 $\overleftrightarrow{\boldsymbol{A}}$ 在最右边

连续叉积：

$$\vec{u}\times(\vec{v}\times\vec{w}) = \vec{v}\,(\vec{u}\cdot\vec{w})-(\vec{u}\cdot\vec{v})\,\vec{w} = \vec{v}\,(\vec{w}\cdot\vec{u})-(\vec{u}\cdot\vec{v})\,\vec{w}$$

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以下公式在直角坐标下成立

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$$\nabla\vec{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\partial_i)(A_j\,\hat{\boldsymbol{e}}_j) = (\partial_i A_j)\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i A_j \equiv \frac{\partial A_j}{\partial x_i}$$

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$$\nabla\cdot(\vec{a}\vec{b}) = \partial_i(a_i b_k)\,\hat{\boldsymbol{e}}_k = (\partial_i a_i)b_k\,\hat{\boldsymbol{e}}_k + a_i\partial_i b_k\,\hat{\boldsymbol{e}}_k = (\nabla\cdot\vec{a})\vec{b} + \vec{a}\cdot(\nabla\vec{b})$$

三、张量分析（微分、积分）

以下公式在直角坐标下成立

矢量的梯度是张量：

$$\nabla \vec{A} = (\hat{\boldsymbol{e}}_i \partial_i)(A_j \hat{\boldsymbol{e}}_j) = (\partial_i A_j)\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i A_j \equiv \frac{\partial A_j}{\partial x_i}$$

张量的散度是矢量：

$$\nabla \cdot \overleftrightarrow{A} = (\hat{\boldsymbol{e}}_i \partial_i) \cdot (A_{jk} \hat{\boldsymbol{e}}_j \hat{\boldsymbol{e}}_k) = (\partial_i A_{jk})\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j \hat{\boldsymbol{e}}_k = (\partial_i A_{jk})\delta_{ij}\hat{\boldsymbol{e}}_k = \partial_i A_{ik}\hat{\boldsymbol{e}}_k$$

$$\nabla \cdot (\vec{a}\vec{b}) = \partial_i (a_i b_k)\hat{\boldsymbol{e}}_k = (\partial_i a_i) b_k \hat{\boldsymbol{e}}_k + a_i \partial_i b_k \hat{\boldsymbol{e}}_k = (\nabla \cdot \vec{a})\vec{b} + \vec{a} \cdot (\nabla \vec{b})$$

张量的旋度仍为张量：

三、张量分析（微分、积分）

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$$\nabla\vec{A} = (\hat{\boldsymbol{e}}_i\partial_i)(A_j\,\hat{\boldsymbol{e}}_j) = (\partial_i A_j)\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i A_j \equiv \frac{\partial A_j}{\partial x_i}$$

张量的散度是矢量：

$$\nabla\cdot\overleftrightarrow{A} = (\hat{\boldsymbol{e}}_i\partial_i)\cdot(A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k) = (\partial_i A_{jk})\,\hat{\boldsymbol{e}}_i\cdot\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k = (\partial_i A_{jk})\delta_{ij}\,\hat{\boldsymbol{e}}_k = \partial_i A_{ik}\,\hat{\boldsymbol{e}}_k$$

$$\nabla\cdot(\vec{a}\vec{b}) = \partial_i(a_i b_k)\,\hat{\boldsymbol{e}}_k = (\partial_i a_i)b_k\,\hat{\boldsymbol{e}}_k + a_i\partial_i b_k\,\hat{\boldsymbol{e}}_k = (\nabla\cdot\vec{a})\vec{b} + \vec{a}\cdot(\nabla\vec{b})$$

张量的旋度仍为张量：

$$\nabla\times\overleftrightarrow{A} = (\hat{\boldsymbol{e}}_i\partial_i)\times(A_{jk}\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k) = (\partial_i A_{jk})\,\hat{\boldsymbol{e}}_i\times\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k = (\partial_i A_{jk})\varepsilon_{ijl}\,\hat{\boldsymbol{e}}_l\,\hat{\boldsymbol{e}}_k$$

三、张量分析（微分、积分）

以下公式在直角坐标下成立

矢量的梯度是张量：

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张量的散度是矢量：

$$\nabla\cdot\overleftrightarrow{A}=(\hat{e}_i\partial_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k)=(\partial_iA_{jk})\,\hat{e}_i\cdot\,\hat{e}_j\,\hat{e}_k=(\partial_iA_{jk})\delta_{ij}\,\hat{e}_k=\partial_iA_{ik}\,\hat{e}_k$$

$$\nabla\cdot(\vec{a}\vec{b})=\partial_i(a_ib_k)\,\hat{e}_k=(\partial_ia_i)b_k\,\hat{e}_k+a_i\partial_ib_k\,\hat{e}_k=(\nabla\cdot\vec{a})\vec{b}+\vec{a}\cdot(\nabla\vec{b})$$

张量的旋度仍为张量：

$$\nabla\times\overleftrightarrow{A}=(\hat{e}_i\partial_i)\times(A_{jk}\,\hat{e}_j\,\hat{e}_k)=(\partial_iA_{jk})\,\hat{e}_i\times\,\hat{e}_j\,\hat{e}_k=(\partial_iA_{jk})\varepsilon_{ijl}\,\hat{e}_l\,\hat{e}_k$$

$$\nabla\times(\vec{a}\vec{b})=[\partial_i(a_jb_k)]\varepsilon_{ijl}\,\hat{e}_l\,\hat{e}_k=[\varepsilon_{ijl}\,(\partial_ia_j)\,\hat{e}_l]\,(b_k\,\hat{e}_k)-[\varepsilon_{jil}(a_j\partial_i)\,\hat{e}_l]\,(b_k\,\hat{e}_k)$$

三、张量分析（微分、积分）

以下公式在直角坐标下成立

矢量的梯度是张量：

$$\nabla\vec{A} = (\hat{e}_i\partial_i)(A_j\,\hat{e}_j) = (\partial_i A_j)\,\hat{e}_i\,\hat{e}_j, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i A_j \equiv \frac{\partial A_j}{\partial x_i}$$

张量的散度是矢量：

$$\nabla\cdot\overleftrightarrow{A} = (\hat{e}_i\partial_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k) = (\partial_i A_{jk})\,\hat{e}_i\cdot\,\hat{e}_j\,\hat{e}_k = (\partial_i A_{jk})\delta_{ij}\,\hat{e}_k = \partial_i A_{ik}\,\hat{e}_k$$

$$\nabla\cdot(\vec{a}\vec{b}) = \partial_i(a_i b_k)\,\hat{e}_k = (\partial_i a_i)b_k\,\hat{e}_k + a_i\partial_i b_k\,\hat{e}_k = (\nabla\cdot\vec{a})\vec{b} + \vec{a}\cdot(\nabla\vec{b})$$

张量的旋度仍为张量：

$$\nabla\times\overleftrightarrow{A} = (\hat{e}_i\partial_i)\times(A_{jk}\,\hat{e}_j\,\hat{e}_k) = (\partial_i A_{jk})\,\hat{e}_i\times\,\hat{e}_j\,\hat{e}_k = (\partial_i A_{jk})\varepsilon_{ijl}\,\hat{e}_l\,\hat{e}_k$$

$$\begin{aligned}\nabla\times(\vec{a}\vec{b}) = [\partial_i(a_j b_k)]\varepsilon_{ijl}\,\hat{e}_l\,\hat{e}_k &= [\varepsilon_{ijl}\,(\partial_i a_j)\,\hat{e}_l]\,(b_k\,\hat{e}_k) - [\varepsilon_{jil}(a_j\partial_i)\,\hat{e}_l]\,(b_k\,\hat{e}_k) \\ &= (\nabla\times\vec{a})\,\vec{b} - (\vec{a}\times\nabla)\,\vec{b}\end{aligned}$$

三、张量分析（微分、积分）

以下公式在直角坐标下成立

矢量的梯度是张量：

$$\nabla\vec{A} = (\hat{e}_i\partial_i)(A_j\,\hat{e}_j) = (\partial_i A_j)\,\hat{e}_i\,\hat{e}_j, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i A_j \equiv \frac{\partial A_j}{\partial x_i}$$

张量的散度是矢量：

$$\nabla\cdot\overleftrightarrow{A} = (\hat{e}_i\partial_i)\cdot(A_{jk}\,\hat{e}_j\,\hat{e}_k) = (\partial_i A_{jk})\,\hat{e}_i\cdot\,\hat{e}_j\,\hat{e}_k = (\partial_i A_{jk})\delta_{ij}\,\hat{e}_k = \partial_i A_{ik}\,\hat{e}_k$$

$$\nabla\cdot(\vec{a}\vec{b}) = \partial_i(a_i b_k)\,\hat{e}_k = (\partial_i a_i)b_k\,\hat{e}_k + a_i\partial_i b_k\,\hat{e}_k = (\nabla\cdot\vec{a})\vec{b} + \vec{a}\cdot(\nabla\vec{b})$$

张量的旋度仍为张量：

$$\nabla\times\overleftrightarrow{A} = (\hat{e}_i\partial_i)\times(A_{jk}\,\hat{e}_j\,\hat{e}_k) = (\partial_i A_{jk})\,\hat{e}_i\times\,\hat{e}_j\,\hat{e}_k = (\partial_i A_{jk})\varepsilon_{ijl}\,\hat{e}_l\,\hat{e}_k$$

$$\nabla\times(\vec{a}\vec{b}) = [\partial_i(a_j b_k)]\varepsilon_{ijl}\,\hat{e}_l\,\hat{e}_k = [\varepsilon_{ijl}\,(\partial_i a_j)\,\hat{e}_l]\,(b_k\,\hat{e}_k) - [\varepsilon_{jil}(a_j\partial_i)\,\hat{e}_l]\,(b_k\,\hat{e}_k)$$

$$= \nabla_a\times(\vec{a}\vec{b}) + \nabla_b\times(\vec{a}\vec{b}) = (\nabla\times\vec{a})\,\vec{b} - (\vec{a}\times\nabla)\,\vec{b}$$

例题

例题

$$\nabla\vec{r} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = (A_i\partial_i)B_j\,\hat{\boldsymbol{e}}_j$$

例题

$$\nabla \vec{r} = \partial_i x_j \, \hat{e}_i \, \hat{e}_j = \delta_{ij} \, \hat{e}_i \, \hat{e}_j = \overleftrightarrow{I}, \qquad \text{单位张量}$$

$$(\vec{A} \cdot \nabla)\vec{B} \quad = (A_i \partial_i) B_j \, \hat{e}_j = A_i (\partial_i B_j) \, \hat{e}_j$$

例题

$$\nabla\vec{r} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$\begin{aligned}(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} &= (A_i\partial_i)B_j\,\hat{\boldsymbol{e}}_j = A_i(\partial_i B_j)\,\hat{\boldsymbol{e}}_j \\ &= A_k\delta_{ik}\,\partial_i B_j\,\hat{\boldsymbol{e}}_j\end{aligned}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}},\qquad \text{单位张量}$$

$$\begin{aligned}(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} &= (A_i\partial_i)B_j\,\hat{\boldsymbol{e}}_j = A_i(\partial_i B_j)\,\hat{\boldsymbol{e}}_j \\ &= A_k\delta_{ik}\,\partial_i B_j\,\hat{\boldsymbol{e}}_j = A_k\,\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_i\,\partial_i B_j\,\hat{\boldsymbol{e}}_j\end{aligned}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$\begin{aligned}(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} &= (A_i\partial_i)B_j\,\hat{\boldsymbol{e}}_j = A_i(\partial_i B_j)\,\hat{\boldsymbol{e}}_j \\ &= A_k\delta_{ik}\,\partial_i B_j\,\hat{\boldsymbol{e}}_j = A_k\,\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_i\,\partial_i B_j\,\hat{\boldsymbol{e}}_j \\ &= A_k\,\hat{\boldsymbol{e}}_k\cdot\partial_i B_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j\end{aligned}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$\begin{aligned}
(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} &= (A_i\partial_i)B_j\,\hat{\boldsymbol{e}}_j = A_i(\partial_i B_j)\,\hat{\boldsymbol{e}}_j \\
&= A_k\delta_{ik}\,\partial_i B_j\,\hat{\boldsymbol{e}}_j = A_k\,\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_i\,\partial_i B_j\,\hat{\boldsymbol{e}}_j \\
&= A_k\,\hat{\boldsymbol{e}}_k\cdot\partial_i B_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})
\end{aligned}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overset{\leftrightarrow}{\boldsymbol{I}},\qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$$= \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}},$$ 单位张量

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}})$$

例题

$$\nabla \vec{r} = \partial_i x_j \, \hat{e}_i \, \hat{e}_j = \delta_{ij} \, \hat{e}_i \, \hat{e}_j = \overleftrightarrow{I}, \qquad \text{单位张量}$$

$$(\vec{A} \cdot \nabla)\vec{B} \quad = \vec{A} \cdot (\nabla \vec{B})$$

$\vec{a}$ 为常矢量

$$\nabla(\vec{a} \cdot \vec{A}) \quad = \hat{e}_i \, \partial_i \, (a_j A_j)$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}},\qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) \quad = \hat{\boldsymbol{e}}_i\,\partial_i\,(a_j A_j) = a_j\,(\partial_i A_j)\,\hat{\boldsymbol{e}}_i$$

例题

$$\nabla\vec{\boldsymbol{r}}=\partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j=\delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j=\overset{\leftrightarrow}{\boldsymbol{I}},\qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}}\quad=\vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}})\quad=\hat{\boldsymbol{e}}_i\,\partial_i\,(a_jA_j)=a_j\,(\partial_iA_j)\,\hat{\boldsymbol{e}}_i$$

$$=a_k\,(\,\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_j)\,(\partial_iA_j)\,\hat{\boldsymbol{e}}_i$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overset{\leftrightarrow}{\boldsymbol{I}},\qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\begin{aligned}\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) \quad &= \hat{\boldsymbol{e}}_i\,\partial_i\,(a_j A_j) = a_j\,(\partial_i A_j)\,\hat{\boldsymbol{e}}_i \\ &= a_k\,(\,\hat{\boldsymbol{e}}_k\cdot\,\hat{\boldsymbol{e}}_j)\,(\partial_i A_j)\,\hat{\boldsymbol{e}}_i = (\partial_i A_j)\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j\cdot a_k\,\hat{\boldsymbol{e}}_k\end{aligned}$$

例题

$$\nabla\vec{r} = \partial_i x_j\,\hat{e}_i\,\hat{e}_j = \delta_{ij}\,\hat{e}_i\,\hat{e}_j = \overleftrightarrow{I}, \qquad \text{单位张量}$$

$$(\vec{A}\cdot\nabla)\vec{B} = \vec{A}\cdot(\nabla\vec{B})$$

$\vec{a}$ 为常矢量

$$\nabla(\vec{a}\cdot\vec{A}) = \hat{e}_i\,\partial_i\,(a_jA_j) = a_j\,(\partial_iA_j)\,\hat{e}_i$$

$$= a_k\,(\,\hat{e}_k\cdot\,\hat{e}_j)\,(\partial_iA_j)\,\hat{e}_i = (\partial_iA_j)\,\hat{e}_i\,\hat{e}_j\cdot a_k\,\hat{e}_k = (\nabla\vec{A})\cdot\vec{a}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overset{\leftrightarrow}{\boldsymbol{I}},\qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

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$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) \quad = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{a}}$$

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例题

$$\nabla \vec{\boldsymbol{r}} = \partial_i x_j \,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) \quad = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{a}}$$

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}})$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\, \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j = \delta_{ij}\, \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j = \overset{\leftrightarrow}{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{a}}$$

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}\times(\nabla\times\vec{\boldsymbol{B}}) + (\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} + \vec{\boldsymbol{B}}\times(\nabla\times\vec{\boldsymbol{A}}) + (\vec{\boldsymbol{B}}\cdot\nabla)\vec{\boldsymbol{A}}$$

例题

$$\nabla \vec{\boldsymbol{r}} = \partial_i x_j \, \hat{\boldsymbol{e}}_i \, \hat{\boldsymbol{e}}_j = \delta_{ij} \, \hat{\boldsymbol{e}}_i \, \hat{\boldsymbol{e}}_j = \overset{\leftrightarrow}{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}} \cdot \nabla)\vec{\boldsymbol{B}} \quad = \vec{\boldsymbol{A}} \cdot (\nabla \vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{A}}) \quad = (\nabla \vec{\boldsymbol{A}}) \cdot \vec{\boldsymbol{a}}$$

$$\nabla(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}) \quad = \vec{\boldsymbol{A}} \times (\nabla \times \vec{\boldsymbol{B}}) + (\vec{\boldsymbol{A}} \cdot \nabla)\vec{\boldsymbol{B}} + \vec{\boldsymbol{B}} \times (\nabla \times \vec{\boldsymbol{A}}) + (\vec{\boldsymbol{B}} \cdot \nabla)\vec{\boldsymbol{A}}$$

$$\nabla(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}})$$

例题

$$\nabla\vec{r} = \partial_i x_j\,\hat{e}_i\,\hat{e}_j = \delta_{ij}\,\hat{e}_i\,\hat{e}_j = \overleftrightarrow{I}, \qquad \text{单位张量}$$

$$(\vec{A}\cdot\nabla)\vec{B} \quad = \vec{A}\cdot(\nabla\vec{B})$$

$\vec{a}$ 为常矢量

$$\nabla(\vec{a}\cdot\vec{A}) \quad = (\nabla\vec{A})\cdot\vec{a}$$

$$\nabla(\vec{A}\cdot\vec{B}) \quad = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

$$\nabla(\vec{A}\cdot\vec{B}) \quad = \nabla_A(\vec{A}\cdot\vec{B}) + \nabla_B(\vec{B}\cdot\vec{A})$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \delta_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{a}}$$

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}\times(\nabla\times\vec{\boldsymbol{B}}) + (\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} + \vec{\boldsymbol{B}}\times(\nabla\times\vec{\boldsymbol{A}}) + (\vec{\boldsymbol{B}}\cdot\nabla)\vec{\boldsymbol{A}}$$

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \nabla_A(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) + \nabla_B(\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{A}}) = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{B}} + (\nabla\vec{\boldsymbol{B}})\cdot\vec{\boldsymbol{A}}$$

例题

$$\nabla\vec{r} = \partial_i x_j\,\hat{e}_i\,\hat{e}_j = \delta_{ij}\,\hat{e}_i\,\hat{e}_j = \overleftrightarrow{I}, \qquad \text{单位张量}$$

$$(\vec{A}\cdot\nabla)\vec{B} \quad = \vec{A}\cdot(\nabla\vec{B})$$

$\vec{a}$ 为常矢量

$$\nabla(\vec{a}\cdot\vec{A}) \quad = (\nabla\vec{A})\cdot\vec{a}$$

$$\nabla(\vec{A}\cdot\vec{B}) \quad = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

$$\nabla(\vec{A}\cdot\vec{B}) \quad = \nabla_A(\vec{A}\cdot\vec{B}) + \nabla_B(\vec{B}\cdot\vec{A}) = (\nabla\vec{A})\cdot\vec{B} + (\nabla\vec{B})\cdot\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B}) = \vec{A}\times(\nabla_B\times\vec{B}) = \nabla_B(\vec{B}\cdot\vec{A}) - (\vec{A}\cdot\nabla_B)\vec{B} = (\nabla\vec{B})\cdot\vec{A} - \vec{A}\cdot(\nabla\vec{B})$$

例题

$$\nabla\vec{r}=\partial_i x_j\,\hat{e}_i\,\hat{e}_j=\delta_{ij}\,\hat{e}_i\,\hat{e}_j=\overleftrightarrow{I},\qquad \text{单位张量}$$

$$(\vec{A}\cdot\nabla)\vec{B}\quad=\vec{A}\cdot(\nabla\vec{B})$$

$\vec{a}$ 为常矢量

$$\nabla(\vec{a}\cdot\vec{A})\quad=(\nabla\vec{A})\cdot\vec{a}$$

$$\nabla(\vec{A}\cdot\vec{B})\quad=\vec{A}\times(\nabla\times\vec{B})+(\vec{A}\cdot\nabla)\vec{B}+\vec{B}\times(\nabla\times\vec{A})+(\vec{B}\cdot\nabla)\vec{A}$$

$$\nabla(\vec{A}\cdot\vec{B})\quad=\nabla_A(\vec{A}\cdot\vec{B})+\nabla_B(\vec{B}\cdot\vec{A})=(\nabla\vec{A})\cdot\vec{B}+(\nabla\vec{B})\cdot\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B})=\vec{A}\times(\nabla_B\times\vec{B})=\nabla_B(\vec{B}\cdot\vec{A})-(\vec{A}\cdot\nabla_B)\vec{B}=(\nabla\vec{B})\cdot\vec{A}-\vec{A}\cdot(\nabla\vec{B})$$

因此若 $\nabla\times\vec{B}=0$，则有

例题

$$\nabla\vec{r} = \partial_i x_j\,\hat{e}_i\,\hat{e}_j = \delta_{ij}\,\hat{e}_i\,\hat{e}_j = \overset{\leftrightarrow}{I}, \qquad \text{单位张量}$$

$$(\vec{A}\cdot\nabla)\vec{B} \quad = \vec{A}\cdot(\nabla\vec{B})$$

$\vec{a}$ 为常矢量

$$\nabla(\vec{a}\cdot\vec{A}) \quad = (\nabla\vec{A})\cdot\vec{a}$$

$$\nabla(\vec{A}\cdot\vec{B}) \quad = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

$$\nabla(\vec{A}\cdot\vec{B}) \quad = \nabla_A(\vec{A}\cdot\vec{B}) + \nabla_B(\vec{B}\cdot\vec{A}) = (\nabla\vec{A})\cdot\vec{B} + (\nabla\vec{B})\cdot\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B}) = \vec{A}\times(\nabla_B\times\vec{B}) = \nabla_B(\vec{B}\cdot\vec{A}) - (\vec{A}\cdot\nabla_B)\vec{B} = (\nabla\vec{B})\cdot\vec{A} - \vec{A}\cdot(\nabla\vec{B})$$

因此若 $\nabla\times\vec{B} = 0$，则有

$$(\nabla\vec{B})\cdot\vec{A} = \vec{A}\cdot(\nabla\vec{B}) \qquad \text{if} \quad \nabla\times\vec{B} = 0$$

$$(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}}$$

$$(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} \quad = (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}}$$

Let there be light

$$(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} \quad = (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}}$$

利用 $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \times \vec{\boldsymbol{c}} = \vec{\boldsymbol{b}}(\vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{b}}) - \vec{\boldsymbol{a}}(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{c}})$

$$(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} = (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}}$$

利用 $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \times \vec{\boldsymbol{c}} = \vec{\boldsymbol{b}}(\vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{b}}) - \vec{\boldsymbol{a}}(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{c}})$

$$= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}})$$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \quad \text{利用 } \nabla(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{a}}) = (\nabla \vec{\boldsymbol{A}}) \cdot \vec{\boldsymbol{a}} \text{ 和 } \nabla \cdot \vec{\boldsymbol{r}} = 3
\end{aligned}
$$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \quad \text{利用 } \nabla(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{a}}) = (\nabla \vec{\boldsymbol{A}}) \cdot \vec{\boldsymbol{a}} \text{ 和 } \nabla \cdot \vec{\boldsymbol{r}} = 3 \\
&= (\nabla_r \vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r \vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}}
\end{aligned}
$$

利用 $\nabla \vec{\boldsymbol{r}} = \overleftrightarrow{\boldsymbol{I}}$ 和 $\overleftrightarrow{\boldsymbol{I}} \cdot \vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}$

$$\begin{aligned}(\vec{\boldsymbol{A}}\times\nabla)\times\vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}}\times\nabla_r)\times\vec{\boldsymbol{r}} \\ &= \nabla_r\,(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}\,(\nabla\cdot\vec{\boldsymbol{r}}) \\ &= (\nabla_r\,\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\ &= -2\vec{\boldsymbol{A}}\end{aligned}$$

利用 $\nabla\vec{\boldsymbol{r}} = \overleftrightarrow{\boldsymbol{I}}$ 和 $\overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r \vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\nabla(\vec{\boldsymbol{r}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) = \nabla(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2)
$$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r \vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\nabla(\vec{\boldsymbol{r}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) = \nabla(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2)
$$

分别作用于两个 $\vec{\boldsymbol{r}}$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r\,(\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}\,(\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r\,\vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{\boldsymbol{r}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) &= \nabla(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) \\
&= \nabla_1(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) + \nabla_2(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2)
\end{aligned}
$$

分别作用于两个 $\vec{\boldsymbol{r}}$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r\,(\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}\,(\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r\,\vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{\boldsymbol{r}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) &= \nabla(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) \\
&= \nabla_1(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) + \nabla_2(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2)
\end{aligned}
$$

利用 $\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2 = \vec{\boldsymbol{r}}_2 \cdot (\overleftrightarrow{\boldsymbol{C}})^T \cdot \vec{\boldsymbol{r}}_1$

$$
\begin{aligned}
(\vec{A}\times\nabla)\times\vec{r} &= (\vec{A}\times\nabla_r)\times\vec{r}\\
&= \nabla_r(\vec{r}\cdot\vec{A}) - \vec{A}(\nabla\cdot\vec{r})\\
&= (\nabla_r\vec{r})\cdot\vec{A} - 3\vec{A}\\
&= -2\vec{A}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{r}\cdot\overleftrightarrow{C}\cdot\vec{r}) &= \nabla(\vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2)\\
&= \nabla_1(\vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2) + \nabla_2(\vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2) \quad \text{利用 } \vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2 = \vec{r}_2\cdot(\overleftrightarrow{C})^T\cdot\vec{r}_1\\
&= \nabla_1\Big[\vec{r}_1\cdot\underbrace{(\overleftrightarrow{C}\cdot\vec{r}_2)}_{\text{常矢量}}\Big] + \nabla_2\Big\{\vec{r}_2\cdot\underbrace{[(\overleftrightarrow{C})^T\cdot\vec{r}_1]}_{\text{常矢量}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r \vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{\boldsymbol{r}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) &= \nabla(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) \\
&= \nabla_1(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) + \nabla_2(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) \\
&= \nabla_1\Big[\vec{\boldsymbol{r}}_1 \cdot \underbrace{(\overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2)}_{\text{常矢量}}\Big] + \nabla_2\Big\{\vec{\boldsymbol{r}}_2 \cdot \underbrace{[(\overleftrightarrow{\boldsymbol{C}})^T \cdot \vec{\boldsymbol{r}}_1]}_{\text{常矢量}}\Big\}
\end{aligned}
$$

利用 $\nabla(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}) = (\nabla \vec{\boldsymbol{a}}) \cdot \vec{\boldsymbol{c}}$

$$
\begin{aligned}
(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}} \times \nabla_r) \times \vec{\boldsymbol{r}} \\
&= \nabla_r (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} (\nabla \cdot \vec{\boldsymbol{r}}) \\
&= (\nabla_r \vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{\boldsymbol{r}} \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) &= \nabla(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) \\
&= \nabla_1(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) + \nabla_2(\vec{\boldsymbol{r}}_1 \cdot \overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2) \\
&= \nabla_1\Big[\vec{\boldsymbol{r}}_1 \cdot \underbrace{(\overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}_2)}_{\text{常矢量}}\Big] + \nabla_2\Big\{\vec{\boldsymbol{r}}_2 \cdot \underbrace{[(\overleftrightarrow{\boldsymbol{C}})^T \cdot \vec{\boldsymbol{r}}_1]}_{\text{常矢量}}\Big\} \\
&= (\nabla\vec{\boldsymbol{r}}) \cdot (\overleftrightarrow{\boldsymbol{C}} \cdot \vec{\boldsymbol{r}}) + \nabla\vec{\boldsymbol{r}} \cdot [(\overleftrightarrow{\boldsymbol{C}})^T \cdot \vec{\boldsymbol{r}}]
\end{aligned}
$$

利用 $\nabla(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}) = (\nabla\vec{\boldsymbol{a}}) \cdot \vec{\boldsymbol{c}}$

$$
\begin{aligned}
(\vec{\boldsymbol{A}}\times\nabla)\times\vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}}\times\nabla_r)\times\vec{\boldsymbol{r}}\\
&= \nabla_r\,(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}\,(\nabla\cdot\vec{\boldsymbol{r}})\\
&= (\nabla_r\,\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}}\\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{\boldsymbol{r}}\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) &= \nabla(\vec{\boldsymbol{r}}_1\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2)\\
&= \nabla_1(\vec{\boldsymbol{r}}_1\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2) + \nabla_2(\vec{\boldsymbol{r}}_1\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2)\\
&= \nabla_1\Big[\,\vec{\boldsymbol{r}}_1\cdot\underbrace{(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2)}_{\text{常矢量}}\Big] + \nabla_2\Big\{\,\vec{\boldsymbol{r}}_2\cdot\underbrace{\big[(\overleftrightarrow{\boldsymbol{C}})^T\cdot\vec{\boldsymbol{r}}_1\big]}_{\text{常矢量}}\Big\}\\
&= (\nabla\vec{\boldsymbol{r}})\cdot(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) + \nabla\vec{\boldsymbol{r}}\cdot\big[(\overleftrightarrow{\boldsymbol{C}})^T\cdot\vec{\boldsymbol{r}}\big] \qquad \text{利用 } \nabla\vec{\boldsymbol{r}} = \overleftrightarrow{\boldsymbol{I}}
\end{aligned}
$$

$$(\vec{A}\times\nabla)\times\vec{r} = (\vec{A}\times\nabla_r)\times\vec{r}$$

$$= \nabla_r(\vec{r}\cdot\vec{A}) - \vec{A}(\nabla\cdot\vec{r})$$

$$= (\nabla_r\vec{r})\cdot\vec{A} - 3\vec{A}$$

$$= -2\vec{A}$$

$$\nabla(\vec{r}\cdot\overleftrightarrow{C}\cdot\vec{r}) = \nabla(\vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2)$$

$$= \nabla_1(\vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2) + \nabla_2(\vec{r}_1\cdot\overleftrightarrow{C}\cdot\vec{r}_2)$$

$$= \nabla_1\Big[\vec{r}_1\cdot\underbrace{(\overleftrightarrow{C}\cdot\vec{r}_2)}_{\text{常矢量}}\Big] + \nabla_2\Big\{\vec{r}_2\cdot\underbrace{[(\overleftrightarrow{C})^T\cdot\vec{r}_1]}_{\text{常矢量}}\Big\}$$

$$= (\nabla\vec{r})\cdot(\overleftrightarrow{C}\cdot\vec{r}) + \nabla\vec{r}\cdot[(\overleftrightarrow{C})^T\cdot\vec{r}] \qquad \text{利用 } \nabla\vec{r} = \overleftrightarrow{I}$$

$$= [\overleftrightarrow{C} + (\overleftrightarrow{C})^T]\cdot\vec{r}$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\vec{\boldsymbol{c}}\cdot\vec{\boldsymbol{r}})=\nabla(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{c}})=(\nabla\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{c}}=\overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{c}}=\vec{\boldsymbol{c}}$$

$$\nabla(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}})=\nabla\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right]=(\nabla\vec{\boldsymbol{r}})\cdot(\overleftrightarrow{\boldsymbol{C}})^T=\overleftrightarrow{\boldsymbol{I}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T=(\overleftrightarrow{\boldsymbol{C}})^T$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] =$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] =$$

$\Uparrow$

利用 $\vec{a}\cdot(\vec{b}\cdot\overleftrightarrow{C}) = (\vec{a}\vec{b}):\overleftrightarrow{C}$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r}):(\overleftrightarrow{C})^T$$

$\Uparrow$

利用 $\vec{a}\cdot(\vec{b}\cdot\overleftrightarrow{C}) = (\vec{a}\vec{b}):\overleftrightarrow{C}$

$$\nabla(\vec{\boldsymbol{c}}\cdot\vec{\boldsymbol{r}}) = \nabla(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{c}}) = (\nabla\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{c}} = \overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}$$

$$\nabla(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) = \nabla\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right] = (\nabla\vec{\boldsymbol{r}})\cdot(\overleftrightarrow{\boldsymbol{C}})^T = \overleftrightarrow{\boldsymbol{I}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T = (\overleftrightarrow{\boldsymbol{C}})^T$$

$$\nabla\cdot(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) = \nabla\cdot\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right] = (\nabla\vec{\boldsymbol{r}}):(\overleftrightarrow{\boldsymbol{C}})^T = \overleftrightarrow{\boldsymbol{I}}:(\overleftrightarrow{\boldsymbol{C}})^T = (\overleftrightarrow{\boldsymbol{C}})^T_t = \overleftrightarrow{\boldsymbol{C}}_t$$

$$\nabla(\vec{\boldsymbol{c}}\cdot\vec{\boldsymbol{r}}) = \nabla(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{c}}) = (\nabla\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{c}} = \overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}$$

$$\nabla(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) = \nabla\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right] = (\nabla\vec{\boldsymbol{r}})\cdot(\overleftrightarrow{\boldsymbol{C}})^T = \overleftrightarrow{\boldsymbol{I}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T = (\overleftrightarrow{\boldsymbol{C}})^T$$

$$\nabla\cdot(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) = \nabla\cdot\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right] = (\nabla\vec{\boldsymbol{r}}) : (\overleftrightarrow{\boldsymbol{C}})^T = \overleftrightarrow{\boldsymbol{I}} : (\overleftrightarrow{\boldsymbol{C}})^T = (\overleftrightarrow{\boldsymbol{C}})^T_t = \overleftrightarrow{\boldsymbol{C}}_t$$

矢量 $\nabla\dfrac{1}{r} = -\dfrac{\vec{\boldsymbol{r}}}{r^3}$ 的梯度：

$$\nabla(\vec{\boldsymbol{c}}\cdot\vec{\boldsymbol{r}}) = \nabla(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{c}}) = (\nabla\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{c}} = \overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}$$

$$\nabla(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) = \nabla\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right] = (\nabla\vec{\boldsymbol{r}})\cdot(\overleftrightarrow{\boldsymbol{C}})^T = \overleftrightarrow{\boldsymbol{I}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T = (\overleftrightarrow{\boldsymbol{C}})^T$$

$$\nabla\cdot(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) = \nabla\cdot\left[\vec{\boldsymbol{r}}\cdot(\overleftrightarrow{\boldsymbol{C}})^T\right] = (\nabla\vec{\boldsymbol{r}}):(\overleftrightarrow{\boldsymbol{C}})^T = \overleftrightarrow{\boldsymbol{I}}:(\overleftrightarrow{\boldsymbol{C}})^T = (\overleftrightarrow{\boldsymbol{C}})^T_t = \overleftrightarrow{\boldsymbol{C}}_t$$

矢量 $\nabla\frac{1}{r} = -\frac{\vec{\boldsymbol{r}}}{r^3}$ 的梯度：

$$\nabla\nabla\frac{1}{r} = \nabla(-\frac{\vec{\boldsymbol{r}}}{r^3}) \quad = -\frac{(\nabla\vec{\boldsymbol{r}})}{r^3} - \nabla\left(\frac{1}{r^3}\right)\vec{\boldsymbol{r}}$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r}):(\overleftrightarrow{C})^T = \overleftrightarrow{I}:(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T_t = \overleftrightarrow{C}_t$$

矢量 $\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}$ 的梯度：

$$\nabla\nabla\frac{1}{r} = \nabla(-\frac{\vec{r}}{r^3}) \quad = -\frac{(\nabla\vec{r})}{r^3} - \nabla\left(\frac{1}{r^3}\right)\vec{r} = -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\nabla r\right]\vec{r}$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r}):(\overleftrightarrow{C})^T = \overleftrightarrow{I}:(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T_t = \overleftrightarrow{C}_t$$

矢量 $\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}$ 的梯度：

$$\nabla\nabla\frac{1}{r} = \nabla(-\frac{\vec{r}}{r^3}) = -\frac{(\nabla\vec{r})}{r^3} - \nabla\left(\frac{1}{r^3}\right)\vec{r} = -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\nabla r\right]\vec{r}$$

$$= -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\frac{\vec{r}}{r}\right]\vec{r}$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r}):(\overleftrightarrow{C})^T = \overleftrightarrow{I}:(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T_t = \overleftrightarrow{C}_t$$

矢量 $\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}$ 的梯度：

$$\nabla\nabla\frac{1}{r} = \nabla(-\frac{\vec{r}}{r^3}) \quad = -\frac{(\nabla\vec{r})}{r^3} - \nabla\left(\frac{1}{r^3}\right)\vec{r} = -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\nabla r\right]\vec{r}$$

$$= -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\frac{\vec{r}}{r}\right]\vec{r} = \frac{1}{r^3}\left(\frac{3\vec{r}\vec{r}}{r^2} - \overleftrightarrow{I}\right)$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

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$$= \nabla^2\frac{1}{r} = 0 \qquad \text{if } r\neq 0$$

例：如果 $\overleftrightarrow{\boldsymbol{T}}$ 为对称张量，即 $(\overleftrightarrow{\boldsymbol{T}})^T = \overleftrightarrow{\boldsymbol{T}}$，$T_{ij} = T_{ji}$，试证明：

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利用分量形式

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$$\oint_{\mathcal{S}} \mathrm{d}\sigma\, \vec{\boldsymbol{n}}\, \vec{\boldsymbol{A}} = \int_{\mathcal{V}} \mathrm{d}\tau\, \nabla\, \vec{\boldsymbol{A}}$$

对于张量的 Gauss 定理和 Stokes 定理，应特别注意点积、叉积的次序:

Gauss 定理：
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Stokes 定理：
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例：如果在区域 $\mathcal{V}$ 内矢量 $\vec{\boldsymbol{j}}$ 满足 $\nabla\cdot\vec{\boldsymbol{j}}=0$，在 $\mathcal{V}$ 边界 $\mathcal{S}$ 上 $j_n=\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{j}}=0$，试证：$\displaystyle\int_{\mathcal{V}}\vec{\boldsymbol{j}}\,\mathrm{d}\tau=0$。

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由于在 $\mathcal{V}$ 内 $\nabla\cdot\vec{j}=0$，故

$$\vec{j}\quad=\vec{j}\cdot\overleftrightarrow{I}$$

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\vec{j} &= \vec{j}\cdot\overleftrightarrow{I} && \text{利用 } \nabla\vec{r}=\overleftrightarrow{I} \\
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\end{aligned}
$$

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&= (\vec{\boldsymbol{j}}\cdot\nabla)\vec{\boldsymbol{r}} && \text{利用 } \nabla\cdot(\vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}})=(\nabla\cdot\vec{\boldsymbol{a}})\,\vec{\boldsymbol{b}}+(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{b}}\\
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\end{aligned}$$

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$$\int_{\mathcal{V}}\vec{j}\,\mathrm{d}\tau=\int_{\mathcal{V}}\nabla\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\tau\qquad\text{利用了上式 }\vec{j}=\nabla\cdot(\vec{j}\,\vec{r})\ \text{ if }\nabla\cdot\vec{j}=0$$

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$$\int_{\mathcal{V}}\vec{j}\,\mathrm{d}\tau=\int_{\mathcal{V}}\nabla\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\tau$$ 利用了上式 $\vec{j}=\nabla\cdot(\vec{j}\vec{r})$ if $\nabla\cdot\vec{j}=0$

$$=\oint_{\mathcal{S}}\vec{n}\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\sigma$$

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$$\int_{\mathcal{V}}\vec{j}\,\mathrm{d}\tau=\int_{\mathcal{V}}\nabla\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\tau\qquad\text{利用了上式 }\vec{j}=\nabla\cdot(\vec{j}\vec{r})\ \text{ if }\nabla\cdot\vec{j}=0$$

$$=\oint_{\mathcal{S}}\vec{n}\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\sigma\qquad\text{利用了高斯定理 }\oint_{\mathcal{S}}\vec{n}\cdot\overleftrightarrow{T}\,\mathrm{d}\sigma=\int_{\mathcal{V}}(\nabla\cdot\overleftrightarrow{T})\,\mathrm{d}\tau$$

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由于在 $\mathcal{V}$ 内 $\nabla\cdot\vec{j}=0$，故

$$\begin{aligned}\vec{j}\ &=\vec{j}\cdot\overleftrightarrow{I} && \text{利用 } \nabla\vec{r}=\overleftrightarrow{I}\\ &=\vec{j}\cdot(\nabla\vec{r}) && \text{利用 } (\vec{a}\cdot\nabla)\vec{b}=\vec{a}\cdot(\nabla\vec{b})\\ &=(\vec{j}\cdot\nabla)\vec{r} && \text{利用 } \nabla\cdot(\vec{a}\,\vec{b})=(\nabla\cdot\vec{a})\,\vec{b}+(\vec{a}\cdot\nabla)\vec{b}\\ &=\nabla\cdot(\vec{j}\,\vec{r})-\underbrace{(\nabla\cdot\vec{j})}_{0}\,\vec{r}\ =\nabla\cdot(\vec{j}\,\vec{r})\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{V}}\vec{j}\,\mathrm{d}\tau &=\int_{\mathcal{V}}\nabla\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\tau && \text{利用了上式 } \vec{j}=\nabla\cdot(\vec{j}\vec{r})\ \text{ if } \nabla\cdot\vec{j}=0\\ &=\oint_{\mathcal{S}}\vec{n}\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\sigma && \text{利用了高斯定理 } \oint_{\mathcal{S}}\vec{n}\cdot\overleftrightarrow{T}\,\mathrm{d}\sigma=\int_{\mathcal{V}}(\nabla\cdot\overleftrightarrow{T})\,\mathrm{d}\tau\\ &=\int_{\mathcal{S}}(\vec{n}\cdot\vec{j})\,\vec{r}\,\mathrm{d}\sigma\end{aligned}$$

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$$=\oint_{\mathcal{S}}\vec{n}\cdot(\vec{j}\,\vec{r})\,\mathrm{d}\sigma\qquad\text{利用了高斯定理 }\oint_{\mathcal{S}}\vec{n}\cdot\overleftrightarrow{T}\,\mathrm{d}\sigma=\int_{\mathcal{V}}(\nabla\cdot\overleftrightarrow{T})\,\mathrm{d}\tau$$

$$=\int_{\mathcal{S}}(\vec{n}\cdot\vec{j})\,\vec{r}\,\mathrm{d}\sigma\qquad\text{利用了 }\vec{n}\cdot(\vec{a}\vec{b})=(\vec{n}\cdot\vec{a})\vec{b}$$

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\vec{\boldsymbol{j}} &= \vec{\boldsymbol{j}}\cdot\overleftrightarrow{\boldsymbol{I}} && \text{利用 } \nabla\vec{\boldsymbol{r}}=\overleftrightarrow{\boldsymbol{I}}\\
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\end{aligned}$$

$$\begin{aligned}
\int_{\mathcal{V}}\vec{\boldsymbol{j}}\,\mathrm{d}\tau &= \int_{\mathcal{V}}\nabla\cdot(\vec{\boldsymbol{j}}\,\vec{\boldsymbol{r}})\,\mathrm{d}\tau && \text{利用了上式 } \vec{\boldsymbol{j}}=\nabla\cdot(\vec{\boldsymbol{j}}\vec{\boldsymbol{r}}) \text{ if } \nabla\cdot\vec{\boldsymbol{j}}=0\\
&= \oint_{\mathcal{S}}\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{j}}\,\vec{\boldsymbol{r}})\,\mathrm{d}\sigma && \text{利用了高斯定理 } \oint_{\mathcal{S}}\vec{\boldsymbol{n}}\cdot\overleftrightarrow{\boldsymbol{T}}\,\mathrm{d}\sigma=\int_{\mathcal{V}}(\nabla\cdot\overleftrightarrow{\boldsymbol{T}})\,\mathrm{d}\tau\\
&= \int_{\mathcal{S}}(\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{j}})\,\vec{\boldsymbol{r}}\,\mathrm{d}\sigma && \text{利用了 } \vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}})=(\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{a}})\vec{\boldsymbol{b}}\\
&= 0 && \text{因为在边界 } \mathcal{S} \text{ 上 } \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{j}}=0
\end{aligned}$$

$$\vec{\boldsymbol{F}} = \oint_{\mathcal{P}} I \mathrm{d}\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}}$$

$$\vec{\boldsymbol{F}} = \oint_{\mathcal{P}} I\mathrm{d}\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}} \qquad = \int_{\mathcal{S}} I\mathrm{d}\sigma(\vec{\boldsymbol{n}} \times \nabla) \times B$$

$$\vec{\boldsymbol{F}} = \oint_{\mathcal{P}} I \mathrm{d}\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}} \qquad = \int_{\mathcal{S}} I \mathrm{d}\sigma (\vec{\boldsymbol{n}} \times \nabla) \times B$$

$$= \int_{\mathcal{S}} I \mathrm{d}\sigma [\nabla_B (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}) - \vec{\boldsymbol{n}} (\nabla \cdot \vec{\boldsymbol{B}})]$$

Let there be light

$$\vec{\boldsymbol{F}} = \oint_{\mathcal{P}} I \mathrm{d}\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}} = \int_{\mathcal{S}} I \mathrm{d}\sigma(\vec{\boldsymbol{n}} \times \nabla) \times B$$

$$= \int_{\mathcal{S}} I \mathrm{d}\sigma[\nabla_B(\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}) - \vec{\boldsymbol{n}}(\nabla \cdot \vec{\boldsymbol{B}})]$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0 \qquad = \int_{\mathcal{S}} I \mathrm{d}\sigma(\nabla \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{n}}$$

$$\vec{F} = \oint_{\mathcal{P}} I \mathrm{d}\vec{l} \times \vec{B} = \int_{\mathcal{S}} I \mathrm{d}\sigma (\vec{n} \times \nabla) \times B$$

$$= \int_{\mathcal{S}} I \mathrm{d}\sigma [\nabla_B (\vec{B} \cdot \vec{n}) - \vec{n}(\nabla \cdot \vec{B})]$$

$\nabla \cdot \vec{B} = 0$

$$= \int_{\mathcal{S}} I \mathrm{d}\sigma (\nabla \vec{B}) \cdot \vec{n}$$

if $\nabla \times \vec{B} = 0$

$(\nabla \vec{B}) \cdot \vec{n} = \vec{n} \cdot (\nabla \vec{B})$

$$= \int_{\mathcal{S}} I \mathrm{d}\sigma \vec{n} \cdot (\nabla \vec{B})$$

$$\vec{F} = \oint_{\mathcal{P}} I\mathrm{d}\vec{l} \times \vec{B} \qquad = \int_{\mathcal{S}} I\mathrm{d}\sigma(\vec{n} \times \nabla) \times B$$

$$= \int_{\mathcal{S}} I\mathrm{d}\sigma[\nabla_B(\vec{B}\cdot\vec{n}) - \vec{n}(\nabla\cdot\vec{B})]$$

$$\nabla\cdot\vec{B} = 0 \qquad = \int_{\mathcal{S}} I\mathrm{d}\sigma(\nabla\vec{B})\cdot\vec{n}$$

if $\nabla\times\vec{B} = 0$
$(\nabla\vec{B})\cdot\vec{n} = \vec{n}\cdot(\nabla\vec{B})$

$$= \int_{\mathcal{S}} I\mathrm{d}\sigma\vec{n}\cdot(\nabla\vec{B})$$

$$\mathrm{d}\vec{\mu} = I\vec{n}\,\mathrm{d}\sigma \qquad = \int_{\mathcal{S}} \mathrm{d}\vec{\mu}\cdot(\nabla\vec{B})$$