§1.2 微分

一、梯度

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梯度是个矢量

$$\nabla T = \frac{\partial T}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial T}{\partial y}\hat{\boldsymbol{e}}_y + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

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全微分

$$\mathrm{d}\,T = \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial y}\mathrm{d}y + \frac{\partial T}{\partial z}\mathrm{d}z$$

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$$\mathrm{d}\,T = \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial y}\mathrm{d}y + \frac{\partial T}{\partial z}\mathrm{d}z \qquad \text{而 } \mathrm{d}\vec{\boldsymbol{l}} = \hat{\boldsymbol{e}}_x\mathrm{d}x + \hat{\boldsymbol{e}}_y\mathrm{d}y + \hat{\boldsymbol{e}}_z\mathrm{d}z$$

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全微分

$$\begin{aligned} \mathrm{d}\,T &= \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial y}\mathrm{d}y + \frac{\partial T}{\partial z}\mathrm{d}z \qquad \text{而 } \mathrm{d}\vec{\boldsymbol{l}} = \hat{\boldsymbol{e}}_x\mathrm{d}x + \hat{\boldsymbol{e}}_y\mathrm{d}y + \hat{\boldsymbol{e}}_z\mathrm{d}z \\ &= (\nabla\,T)\cdot(\mathrm{d}\,\vec{\boldsymbol{l}}) = |\nabla T|\,|\mathrm{d}\,\vec{\boldsymbol{l}}|\cos\theta \end{aligned}$$

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几何意义：

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全微分

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几何意义：

梯度 ∇T 的方向指向函数 T 的最大变化率（方向导数）方向，其大小即为函数 T 的最大变化率（方向导数）。

二、算符 ∇

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梯度写成

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$$\nabla T = \frac{\partial T}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial T}{\partial y}\hat{\boldsymbol{e}}_y + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z = \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right) T$$

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$$\nabla=\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}$$

类比：

$$\vec{\boldsymbol{A}}\, a \qquad \Longrightarrow \qquad \nabla T$$ 梯度 (gradient)

类比：

$$\vec{\boldsymbol{A}}\,a \quad \Longrightarrow \quad \nabla\, T \qquad \text{梯度 (gradient)}$$

$$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} \quad \Longrightarrow \quad \nabla\cdot\vec{\boldsymbol{v}} \qquad \text{散度 (divergence)}$$

类比：

$$\vec{\boldsymbol{A}}\, a \quad \Longrightarrow \quad \nabla T \quad \text{梯度 (gradient)}$$

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$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \quad \Longrightarrow \quad \nabla \times \vec{\boldsymbol{v}} \quad \text{旋度 (curl)}$$

类比：

$\vec{\boldsymbol{A}}\, a$	$\Longrightarrow$	$\nabla\, T$	梯度 (gradient)
$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}$	$\Longrightarrow$	$\nabla \cdot \vec{\boldsymbol{v}}$	散度 (divergence)
$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$	$\Longrightarrow$	$\nabla \times \vec{\boldsymbol{v}}$	旋度 (curl)

三、散度

类比：

$$\vec{A}\,a \quad \Longrightarrow \quad \nabla T \quad \text{梯度 (gradient)}$$

$$\vec{A}\cdot\vec{B} \quad \Longrightarrow \quad \nabla\cdot\vec{v} \quad \text{散度 (divergence)}$$

$$\vec{A}\times\vec{B} \quad \Longrightarrow \quad \nabla\times\vec{v} \quad \text{旋度 (curl)}$$

三、散度

$$\begin{aligned}\nabla\cdot\vec{v} &= \left(\hat{e}_x\frac{\partial}{\partial x}+\hat{e}_y\frac{\partial}{\partial y}+\hat{e}_z\frac{\partial}{\partial z}\right)\cdot(v_x\,\hat{e}_x+v_y\,\hat{e}_y+v_z\,\hat{e}_z)\\ &= \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\end{aligned}$$

四、旋度

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$$\nabla \times \vec{\boldsymbol{v}} = \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right) \times \left(v_x\,\hat{\boldsymbol{e}}_x + v_y\,\hat{\boldsymbol{e}}_y + v_z\,\hat{\boldsymbol{e}}_z\right)$$

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$$= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

四、旋度

$$
\begin{aligned}
\nabla \times \vec{\boldsymbol{v}} &= \left(\hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}\right) \times \left(v_x\, \hat{\boldsymbol{e}}_x + v_y\, \hat{\boldsymbol{e}}_y + v_z\, \hat{\boldsymbol{e}}_z\right) \\
&= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
&= \hat{\boldsymbol{e}}_x \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\boldsymbol{e}}_y \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\boldsymbol{e}}_z \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)
\end{aligned}
$$

四、旋度

$$
\begin{aligned}
\nabla\times\vec{\boldsymbol{v}} &= \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\times\left(v_x\,\hat{\boldsymbol{e}}_x+v_y\,\hat{\boldsymbol{e}}_y+v_z\,\hat{\boldsymbol{e}}_z\right)\\
&= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}\\
&= \hat{\boldsymbol{e}}_x\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{\boldsymbol{e}}_y\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{\boldsymbol{e}}_z\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)
\end{aligned}
$$

五、例题

四、旋度

$$\nabla\times\vec{\boldsymbol{v}} = \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\times\left(v_x\,\hat{\boldsymbol{e}}_x+v_y\,\hat{\boldsymbol{e}}_y+v_z\,\hat{\boldsymbol{e}}_z\right)$$

$$= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\boldsymbol{e}}_x\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{\boldsymbol{e}}_y\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{\boldsymbol{e}}_z\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

五、例题

$$\nabla r = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}\,\hat{\boldsymbol{e}}_x+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y}\,\hat{\boldsymbol{e}}_y+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z}\,\hat{\boldsymbol{e}}_z$$

四、旋度

$$
\begin{aligned}
\nabla \times \vec{\boldsymbol{v}} &= \left(\hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}\right) \times (v_x \hat{\boldsymbol{e}}_x + v_y \hat{\boldsymbol{e}}_y + v_z \hat{\boldsymbol{e}}_z) \\
&= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
&= \hat{\boldsymbol{e}}_x \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\boldsymbol{e}}_y \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\boldsymbol{e}}_z \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)
\end{aligned}
$$

五、例题

$$
\begin{aligned}
\nabla r &= \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial x} \hat{\boldsymbol{e}}_x + \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial y} \hat{\boldsymbol{e}}_y + \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial z} \hat{\boldsymbol{e}}_z \\
\nabla r &= \frac{x}{r} \hat{\boldsymbol{e}}_x + \frac{y}{r} \hat{\boldsymbol{e}}_y + \frac{z}{r} \hat{\boldsymbol{e}}_z
\end{aligned}
$$

四、旋度

$$\nabla\times\vec{\boldsymbol{v}} = \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\times\left(v_x\,\hat{\boldsymbol{e}}_x+v_y\,\hat{\boldsymbol{e}}_y+v_z\,\hat{\boldsymbol{e}}_z\right)$$

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$$= \hat{\boldsymbol{e}}_x\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{\boldsymbol{e}}_y\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{\boldsymbol{e}}_z\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

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$$\nabla r = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}\hat{\boldsymbol{e}}_x+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y}\hat{\boldsymbol{e}}_y+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z}\hat{\boldsymbol{e}}_z$$

$$\nabla r = \frac{x}{r}\hat{\boldsymbol{e}}_x+\frac{y}{r}\hat{\boldsymbol{e}}_y+\frac{z}{r}\hat{\boldsymbol{e}}_z=\frac{\vec{\boldsymbol{r}}}{r}=\hat{\boldsymbol{e}}_r$$

$$\nabla r = \frac{\vec{\boldsymbol{r}}}{r} = \hat{\boldsymbol{e}}_r,$$

$$\nabla r = \frac{\vec{\boldsymbol{r}}}{r} = \hat{\boldsymbol{e}}_r, \qquad \nabla f(u) = \frac{\mathsf{d}f}{\mathsf{d}u}\nabla u$$

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$$\nabla r^2 = 2r\nabla r = 2\vec{\boldsymbol{r}}$$

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$$\nabla \cdot \vec{\boldsymbol{r}} = \left(\hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}\right) \cdot (x\,\hat{\boldsymbol{e}}_x + y\,\hat{\boldsymbol{e}}_y + z\,\hat{\boldsymbol{e}}_z) = 3$$

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$$\nabla \times \vec{r} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

六、乘积的梯度、散度、旋度

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∇ 既是矢量又是线性算符

六、乘积的梯度、散度、旋度

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分配率：

六、乘积的梯度、散度、旋度

∇ 既是矢量又是线性算符

分配率：

$$\nabla(f+g)=\nabla f+\nabla g$$

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如果 k 是常数

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如果 k 是常数

$$\nabla (kf) = k \nabla f$$

$$\nabla \cdot (k\vec{\boldsymbol{A}}) = k \nabla \cdot \vec{\boldsymbol{A}}$$

$$\nabla \times (k\vec{\boldsymbol{A}}) = k \nabla \times \vec{\boldsymbol{A}}$$

乘积形式：$fg, \quad \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}, \quad f\vec{\boldsymbol{A}}, \quad \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}$

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$\nabla(fg)$

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$\nabla(fg)$

先利用 ∇ 的算符（求导）特性，乘积的求导分成两项；

1. ∇ 对 f 作用（求导），g 视为常量，记为 ∇_f
2. ∇ 对 g 作用（求导），f 视为常量，记为 ∇_g

乘积形式：fg，$\vec{A}\cdot\vec{B}$，$f\vec{A}$，$\vec{A}\times\vec{B}$

$$\nabla(fg) = \nabla_f(fg)+\nabla_g(fg)$$

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再利用 ∇ 的矢量运算特性；$\vec{\boldsymbol{C}}(fg) = g\vec{\boldsymbol{C}}f = f\vec{\boldsymbol{C}}g$

将受 ∇ 作用的函数移到 ∇ 的右边，

不受 ∇ 作用的函数移到 ∇ 的左边

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$$= g\nabla_f f + f\nabla_g g$$

$$\nabla(fg) = g\nabla f + f\nabla g$$

乘积形式：$fg,\quad \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}},\quad f\vec{\boldsymbol{A}},\quad \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}$

$$\nabla(fg) = \nabla_f(fg) + \nabla_g(fg)$$

先利用 ∇ 的算符（求导）特性，乘积的求导分成两项：

1. ∇ 对 f 作用（求导），g 视为常量，记为 ∇_f
2. ∇ 对 g 作用（求导），f 视为常量，记为 ∇_g

再利用 ∇ 的矢量运算特性：$\vec{\boldsymbol{C}}(fg) = g\vec{\boldsymbol{C}}f = f\vec{\boldsymbol{C}}g$

将受 ∇ 作用的函数移到 ∇ 的右边，

不受 ∇ 作用的函数移到 ∇ 的左边

$$= g\nabla_f f + f\nabla_g g$$

$$\nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla\cdot(f\vec{\boldsymbol{A}})$$

乘积形式：fg, $\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}$, $f\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}$

$$\nabla(fg) = \nabla_f(fg) + \nabla_g(fg)$$

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1. ∇ 对 f 求导，$\vec{\boldsymbol{A}}$ 视为常量，记为 ∇_f
2. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，f 视为常量，记为 ∇_A

乘积形式：fg, $\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}$, $f\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}$

$$\nabla(fg) = \nabla_f(fg)+\nabla_g(fg)$$

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2. ∇ 对 g 作用（求导），f 视为常量，记为 ∇_g

$$= g\nabla_f f + f\nabla_g g$$

再利用 ∇ 的矢量运算特性：$\vec{\boldsymbol{C}}(fg) = g\vec{\boldsymbol{C}}f = f\vec{\boldsymbol{C}}g$

将受 ∇ 作用的函数移到 ∇ 的右边，

不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla\cdot(f\vec{\boldsymbol{A}}) = \nabla_f\cdot(f\vec{\boldsymbol{A}})+\nabla_A\cdot(f\vec{\boldsymbol{A}})$$

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2. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，f 视为常量，记为 ∇_A

乘积形式：fg, $\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}$, $f\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}$

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再利用 ∇ 的矢量运算特性：

$$\vec{\boldsymbol{C}}\cdot(f\vec{\boldsymbol{A}}) = \vec{\boldsymbol{A}}\cdot(\vec{\boldsymbol{C}}f) = f(\vec{\boldsymbol{C}}\cdot\vec{\boldsymbol{A}})$$

将受 ∇ 作用的函数移到 ∇ 的右边，
不受 ∇ 作用的函数移到 ∇ 的左边

乘积形式：fg, $\vec{A}\cdot\vec{B}$, $f\vec{A}$, $\vec{A}\times\vec{B}$

$$\nabla(fg) = \nabla_f(fg) + \nabla_g(fg)$$

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2. ∇ 对 $\vec{A}$ 求导，f 视为常量，记为 ∇_A

$$= \vec{A}\cdot(\nabla_f f) + f(\nabla_A\cdot\vec{A})$$

再利用 ∇ 的矢量运算特性；

$$\vec{C}\cdot(f\vec{A}) = \vec{A}\cdot(\vec{C}f) = f(\vec{C}\cdot\vec{A})$$

将受 ∇ 作用的函数移到 ∇ 的右边，

不受 ∇ 作用的函数移到 ∇ 的左边

乘积形式：$fg,\quad \vec{A}\cdot\vec{B},\quad f\vec{A},\quad \vec{A}\times\vec{B}$

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$$\nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla\cdot(f\vec{A}) = \nabla_f\cdot(f\vec{A}) + \nabla_A\cdot(f\vec{A})$$

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再利用 ∇ 的矢量运算特性：

$$\vec{C}\cdot(f\vec{A}) = \vec{A}\cdot(\vec{C}f) = f(\vec{C}\cdot\vec{A})$$

将受 ∇ 作用的函数移到 ∇ 的右边，

不受 ∇ 作用的函数移到 ∇ 的左边

$$= \vec{A}\cdot(\nabla_f f) + f(\nabla_A\cdot\vec{A})$$

$$\nabla\cdot(f\vec{A}) = \vec{A}\cdot\nabla f + f\nabla\cdot\vec{A}$$

$$\nabla \times (f\vec{\boldsymbol{A}})$$

$\nabla \times (f\vec{\boldsymbol{A}})$

先利用 ∇ 的求导特性，求导分成两项；
1. ∇ 对 f 求导，$\vec{\boldsymbol{A}}$ 视为常量，即 ∇_f
2. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，f 视为常量，即 ∇_A

$$\nabla\times(f\vec{A}) \quad = \nabla_f\times(f\vec{A})+\nabla_A\times(f\vec{A})$$

先利用 ∇ 的求导特性，求导分成两项；
1. ∇ 对 f 求导，$\vec{A}$ 视为常量，即 ∇_f
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$$\nabla \times (f\vec{A}) = \nabla_f \times (f\vec{A}) + \nabla_A \times (f\vec{A})$$

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2. ∇ 对 $\vec{A}$ 求导，f 视为常量，即 ∇_A

再利用 ∇ 的矢量运算特性；

$$\vec{C} \times (f\vec{A}) = -\vec{A} \times (\vec{C} f) = f(\vec{C} \times \vec{A})$$

将受 ∇ 作用的函数移到 ∇ 的右边，
不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla \times (f\vec{\boldsymbol{A}}) = \nabla_f \times (f\vec{\boldsymbol{A}}) + \nabla_A \times (f\vec{\boldsymbol{A}})$$

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1. ∇ 对 f 求导，$\vec{\boldsymbol{A}}$ 视为常量，即 ∇_f
2. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，f 视为常量，即 ∇_A

$$= -\vec{\boldsymbol{A}} \times (\nabla_f f) + f(\nabla_A \times \vec{\boldsymbol{A}})$$

再利用 ∇ 的矢量运算特性；

$$\vec{\boldsymbol{C}} \times (f\vec{\boldsymbol{A}}) = -\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{C}} f) = f(\vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}})$$

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不受 ∇ 作用的函数移到 ∇ 的左边

$$= -\vec{A} \times (\nabla_f f) + f(\nabla_A \times \vec{A})$$

$$\nabla \times (f\vec{A}) = -\vec{A} \times \nabla f + f\nabla \times \vec{A}$$

Let there be light

$$\nabla \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$

$$\nabla \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$

先利用 ∇ 的求导特性，求导分成两项；

1. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，$\vec{\boldsymbol{B}}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{\boldsymbol{B}}$ 求导，$\vec{\boldsymbol{A}}$ 视为常量，记为 ∇_B

$$\nabla \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) \quad = \nabla_A \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) + \nabla_B \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$

先利用 ∇ 的求导特性，求导分成两项：

1. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，$\vec{\boldsymbol{B}}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{\boldsymbol{B}}$ 求导，$\vec{\boldsymbol{A}}$ 视为常量，记为 ∇_B

$$\nabla\cdot(\vec{A}\times\vec{B}) \quad = \nabla_A\cdot(\vec{A}\times\vec{B}) + \nabla_B\cdot(\vec{A}\times\vec{B})$$

先利用 ∇ 的求导特性，求导分成两项；

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

再利用 ∇ 的矢量运算特性；

$$\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$$

将受 ∇ 作用的函数移到 ∇ 的右边，

不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla\cdot(\vec{A}\times\vec{B}) = \nabla_A\cdot(\vec{A}\times\vec{B})+\nabla_B\cdot(\vec{A}\times\vec{B})$$

先利用 ∇ 的求导特性，求导分成两项；

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= \vec{B}\cdot(\nabla_A\times\vec{A})-\vec{A}\cdot(\nabla_B\times\vec{B})$$

再利用 ∇ 的矢量运算特性；

$\vec{C}\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{C}\times\vec{A})=-\vec{A}\cdot(\vec{C}\times\vec{B})$

将受 ∇ 作用的函数移到 ∇ 的右边，

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$$\nabla\cdot(\vec{A}\times\vec{B}) = \nabla_A\cdot(\vec{A}\times\vec{B}) + \nabla_B\cdot(\vec{A}\times\vec{B})$$

先利用 ∇ 的求导特性，求导分成两项；
1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= \vec{B}\cdot(\nabla_A\times\vec{A}) - \vec{A}\cdot(\nabla_B\times\vec{B})$$

再利用 ∇ 的矢量运算特性；
$\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$
将受 ∇ 作用的函数移到 ∇ 的右边，
不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$

$\nabla \times (\vec{A} \times \vec{B})$

先利用 ∇ 的求导特性，求导分成两项：

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$\nabla \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = \nabla_A \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) + \nabla_B \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$

先利用 ∇ 的求导特性，求导分成两项：

1. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，$\vec{\boldsymbol{B}}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{\boldsymbol{B}}$ 求导，$\vec{\boldsymbol{A}}$ 视为常量，记为 ∇_B

$$\nabla\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) \quad = \nabla_A\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) + \nabla_B\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}})$$

先利用 ∇ 的求导特性，求导分成两项：

1. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，$\vec{\boldsymbol{B}}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{\boldsymbol{B}}$ 求导，$\vec{\boldsymbol{A}}$ 视为常量，记为 ∇_B

再利用 ∇ 的矢量运算特性：

$$\begin{aligned}\vec{\boldsymbol{C}}\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) &= (\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{C}})\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}(\vec{\boldsymbol{C}}\cdot\vec{\boldsymbol{A}}) \\ &= \vec{\boldsymbol{A}}(\vec{\boldsymbol{C}}\cdot\vec{\boldsymbol{B}}) - (\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{C}})\vec{\boldsymbol{B}}\end{aligned}$$

将受 ∇ 作用的函数移到 ∇ 的右边，
不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla \times (\vec{A} \times \vec{B}) = \nabla_A \times (\vec{A} \times \vec{B}) + \nabla_B \times (\vec{A} \times \vec{B})$$

先利用 ∇ 的求导特性，求导分成两项：
1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= (\vec{B} \cdot \nabla_A)\vec{A} - \vec{B}(\nabla_A \cdot \vec{A}) + \vec{A}(\nabla_B \cdot \vec{B}) - (\vec{A} \cdot \nabla_B)\vec{B}$$

再利用 ∇ 的矢量运算特性：

$$\vec{C} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{C})\vec{A} - \vec{B}(\vec{C} \cdot \vec{A})$$
$$= \vec{A}(\vec{C} \cdot \vec{B}) - (\vec{A} \cdot \vec{C})\vec{B}$$

将受 ∇ 作用的函数移到 ∇ 的右边，
不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla\times(\vec{A}\times\vec{B}) = \nabla_A\times(\vec{A}\times\vec{B})+\nabla_B\times(\vec{A}\times\vec{B})$$

先利用 ∇ 的求导特性，求导分成两项：

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= (\vec{B}\cdot\nabla_A)\vec{A}-\vec{B}(\nabla_A\cdot\vec{A})+\vec{A}(\nabla_B\cdot\vec{B})-(\vec{A}\cdot\nabla_B)\vec{B}$$

再利用 ∇ 的矢量运算特性；

$$\vec{C}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{C})\vec{A}-\vec{B}(\vec{C}\cdot\vec{A})$$
$$= \vec{A}(\vec{C}\cdot\vec{B})-(\vec{A}\cdot\vec{C})\vec{B}$$

将受 ∇ 作用的函数移到 ∇ 的右边，
不受 ∇ 作用的函数移到 ∇ 的左边

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A}-\vec{B}(\nabla\cdot\vec{A})-(\vec{A}\cdot\nabla)\vec{B}+\vec{A}(\nabla\cdot\vec{B})$$

其中

$$(\vec{\boldsymbol{B}} \cdot \nabla) = \left[(B_x\,\hat{\boldsymbol{e}}_x + B_y\,\hat{\boldsymbol{e}}_y + B_z\,\hat{\boldsymbol{e}}_z) \cdot \left(\hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z} \right) \right]$$

其中

$$
\begin{aligned}
(\vec{\boldsymbol{B}}\cdot\nabla) &= \left[(B_x\,\hat{\boldsymbol{e}}_x + B_y\,\hat{\boldsymbol{e}}_y + B_z\,\hat{\boldsymbol{e}}_z)\cdot\left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\right] \\
&= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}
\end{aligned}
$$

其中

$$(\vec{\boldsymbol{B}}\cdot\nabla) = \left[(B_x\,\hat{\boldsymbol{e}}_x + B_y\,\hat{\boldsymbol{e}}_y + B_z\,\hat{\boldsymbol{e}}_z)\cdot\left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\right]$$

$$= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z} \quad\Longrightarrow\quad \text{标量算符}$$

其中

$$(\vec{\boldsymbol{B}}\cdot\nabla) = \left[(B_x\,\hat{\boldsymbol{e}}_x + B_y\,\hat{\boldsymbol{e}}_y + B_z\,\hat{\boldsymbol{e}}_z)\cdot\left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\right]$$

$$= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z} \implies \text{标量算符}$$

从而

$$(\vec{\boldsymbol{B}}\cdot\nabla)\vec{\boldsymbol{A}} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(A_x\,\hat{\boldsymbol{e}}_x + A_y\,\hat{\boldsymbol{e}}_y + A_z\,\hat{\boldsymbol{e}}_z)$$

其中

$$(\vec{B}\cdot\nabla) = \left[(B_x\,\hat{e}_x + B_y\,\hat{e}_y + B_z\,\hat{e}_z)\cdot\left(\hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}\right)\right]$$

$$= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z} \quad\Longrightarrow\quad \text{标量算符}$$

从而

$$(\vec{B}\cdot\nabla)\vec{A} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(A_x\,\hat{e}_x + A_y\,\hat{e}_y + A_z\,\hat{e}_z)$$

$$= B_x\frac{\partial A_x}{\partial x}\,\hat{e}_x + B_y\frac{\partial A_x}{\partial y}\,\hat{e}_x + B_z\frac{\partial A_x}{\partial z}\,\hat{e}_x$$

$$+ B_x\frac{\partial A_y}{\partial x}\,\hat{e}_y + B_y\frac{\partial A_y}{\partial y}\,\hat{e}_y + B_z\frac{\partial A_y}{\partial z}\,\hat{e}_y$$

$$+ B_x\frac{\partial A_z}{\partial x}\,\hat{e}_z + B_y\frac{\partial A_z}{\partial y}\,\hat{e}_z + B_z\frac{\partial A_z}{\partial z}\,\hat{e}_z$$

$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}})$

$\nabla(\vec{A} \cdot \vec{B})$

先利用 ∇ 的求导特性，求导分成两项：

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) \quad = \nabla_A(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) + \nabla_B(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}})$$

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再利用 ∇ 的矢量运算特性：

$$\vec{B}\times(\vec{C}\times\vec{A}) = \vec{C}(\vec{A}\cdot\vec{B}) - (\vec{B}\cdot\vec{C})\vec{A}$$

$$\Downarrow$$

$$\vec{C}(\vec{A}\cdot\vec{B}) = \vec{B}\times(\vec{C}\times\vec{A}) + (\vec{B}\cdot\vec{C})\vec{A}$$

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将受 ∇ 作用的函数移到 ∇ 的右边，
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七、二重算符作用

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可能的情况：

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可能的情况：

梯度的散度 $\nabla \cdot (\nabla f)$

梯度的旋度 $\nabla \times (\nabla f)$

七、二重算符作用

可能的情况：

梯度的散度	$\nabla\cdot(\nabla f)$	起始于标量函数
梯度的旋度	$\nabla\times(\nabla f)$	

七、二重算符作用

可能的情况：

梯度的散度	$\nabla \cdot (\nabla f)$	起始于标量函数
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$$\nabla\cdot(\nabla f) = \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{e}}_x\frac{\partial f}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial f}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \equiv \nabla^2 f$$

标量函数 f 的 Laplacian，也是个标量

注意 $\hat{\boldsymbol{e}}_x$，$\hat{\boldsymbol{e}}_y$，$\hat{\boldsymbol{e}}_z$ 均为常矢量

标量算符 Laplacian：∇^2

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$$

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标量的Laplacian仍为标量

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标量的Laplacian仍为标量

$$\nabla^2 \vec{\boldsymbol{A}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_x \, \hat{\boldsymbol{e}}_x + A_y \, \hat{\boldsymbol{e}}_y + A_z \, \hat{\boldsymbol{e}}_z)$$

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$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$$

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标量的Laplacian仍为标量

$$\begin{aligned}\nabla^2 \vec{\boldsymbol{A}} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_x\,\hat{\boldsymbol{e}}_x + A_y\,\hat{\boldsymbol{e}}_y + A_z\,\hat{\boldsymbol{e}}_z)\\ &= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{\boldsymbol{e}}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{\boldsymbol{e}}_y\\ &\quad + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{\boldsymbol{e}}_z\end{aligned}$$

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标量的Laplacian仍为标量

$$\begin{aligned}\nabla^2 \vec{A} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z) \\ &= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{e}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{e}_y \\ &\quad + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{e}_z\end{aligned}$$

矢量的Laplacian仍为矢量

标量算符 Laplacian：∇^2

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标量的Laplacian仍为标量

$$\begin{aligned}
\nabla^2 \vec{\boldsymbol{A}} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_x\,\hat{\boldsymbol{e}}_x + A_y\,\hat{\boldsymbol{e}}_y + A_z\,\hat{\boldsymbol{e}}_z) \\
&= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{\boldsymbol{e}}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{\boldsymbol{e}}_y \\
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\end{aligned}$$

矢量的Laplacian仍为矢量

$$\nabla^2 \vec{\boldsymbol{A}} = (\nabla^2 A_x)\,\hat{\boldsymbol{e}}_x + (\nabla^2 A_y)\,\hat{\boldsymbol{e}}_y + (\nabla^2 A_z)\,\hat{\boldsymbol{e}}_z$$

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矢量的Laplacian仍为矢量

$$\nabla^2 \vec{\boldsymbol{A}} = (\nabla^2 A_x)\,\hat{\boldsymbol{e}}_x + (\nabla^2 A_y)\,\hat{\boldsymbol{e}}_y + (\nabla^2 A_z)\,\hat{\boldsymbol{e}}_z$$

$\hat{\boldsymbol{e}}_x$，$\hat{\boldsymbol{e}}_y$，$\hat{\boldsymbol{e}}_z$ 为常矢量

梯度的旋度恒为零：

$$\nabla \times (\nabla f) = (\nabla \times \nabla) f = 0$$

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试从分量形式加以证明

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试从分量形式加以证明

注意：

$$(\vec{C} g) \times (\vec{C} f) = (\vec{C} \times \vec{C}) g f = 0,$$

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$$(\nabla g) \times (\nabla f) = (\nabla_g\, g) \times (\nabla_f\, f) = (\nabla_g \times \nabla_f)\, g f \neq 0$$

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利用了：$\vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{B} \times \vec{C}) \cdot \vec{A}$

旋度的旋度常用于定义矢量的 Laplacian：

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一般矢量的 Laplacian：

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$$

旋度的旋度常用于定义矢量的 Laplacian：

$$\nabla\times(\nabla\times\vec{\boldsymbol{A}})=\nabla(\nabla\cdot\vec{\boldsymbol{A}})-(\nabla\cdot\nabla)\vec{\boldsymbol{A}}=\nabla(\nabla\cdot\vec{\boldsymbol{A}})-\nabla^2\vec{\boldsymbol{A}}$$

利用了：$\vec{\boldsymbol{B}}\times(\vec{\boldsymbol{C}}\times\vec{\boldsymbol{A}})=\vec{\boldsymbol{C}}(\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{A}})-(\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{C}})\vec{\boldsymbol{A}}$

一般矢量的 Laplacian：

$$\nabla^2\vec{\boldsymbol{A}}=\nabla(\nabla\cdot\vec{\boldsymbol{A}})-\nabla\times(\nabla\times\vec{\boldsymbol{A}})$$

$\nabla^2\vec{\boldsymbol{A}}=(\nabla^2A_x)\,\hat{\boldsymbol{e}}_x+(\nabla^2A_y)\,\hat{\boldsymbol{e}}_y+(\nabla^2A_z)\,\hat{\boldsymbol{e}}_z$ 仅对直角坐标成立。

旋度的旋度常用于定义矢量的 Laplacian：

$$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-(\nabla\cdot\nabla)\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

利用了：$\vec{B}\times(\vec{C}\times\vec{A})=\vec{C}(\vec{B}\cdot\vec{A})-(\vec{B}\cdot\vec{C})\vec{A}$

一般矢量的 Laplacian：

$$\nabla^2\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla\times(\nabla\times\vec{A})$$

$\nabla^2\vec{A}=(\nabla^2A_x)\,\hat{e}_x+(\nabla^2A_y)\,\hat{e}_y+(\nabla^2A_z)\,\hat{e}_z$ 仅对直角坐标成立。

如果 $\begin{cases}\vec{R}=\vec{r}-\vec{r}\,' \\ \vec{r}=x\,\hat{e}_x+y\,\hat{e}_y+z\,\hat{e}_z, \\ \vec{r}\,'=x'\,\hat{e}_x+y'\,\hat{e}_y+z'\,\hat{e}_z, \\ \nabla=\hat{e}_i\dfrac{\partial}{\partial x_i},\quad \nabla'=\hat{e}_i\dfrac{\partial}{\partial x_i'}\end{cases}\implies \nabla'\,[g(\vec{R})]=-\nabla\,[g(\vec{R})]$

八、例题

八、例题

以下 $\vec{A}$ 为任意矢量，$\vec{a}$ 为常矢量。

$$(\vec{A}\cdot\nabla)\vec{r} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z) = \vec{A}$$

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$$\begin{aligned}(\vec{\boldsymbol{A}}\cdot\nabla)r \quad &= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\sqrt{x^2+y^2+z^2} \\ &= (A_x x + A_y y + A_z z)/r = \vec{\boldsymbol{A}}\cdot\frac{\vec{\boldsymbol{r}}}{r}\end{aligned}$$

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$$\begin{aligned}(\vec{A}\cdot\nabla)r \quad &= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\sqrt{x^2+y^2+z^2} \\ &= (A_x x + A_y y + A_z z)/r = \vec{A}\cdot\frac{\vec{r}}{r} = \vec{A}\cdot(\nabla r)\end{aligned}$$

八、例题

以下 $\vec{A}$ 为任意矢量，$\vec{a}$ 为常矢量。

$$(\vec{A}\cdot\nabla)\vec{r} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z) = \vec{A}$$

$$\begin{aligned}(\vec{A}\cdot\nabla)r &= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\sqrt{x^2+y^2+z^2} \\ &= (A_x x + A_y y + A_z z)/r = \vec{A}\cdot\frac{\vec{r}}{r} = \vec{A}\cdot(\nabla r)\end{aligned}$$

$$\nabla(\vec{a}\cdot\vec{r})$$

八、例题

以下 $\vec{A}$ 为任意矢量，$\vec{a}$ 为常矢量。

$$(\vec{A}\cdot\nabla)\vec{r}=(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z})(x\,\hat{e}_x+y\,\hat{e}_y+z\,\hat{e}_z)=\vec{A}$$

$$\begin{aligned}(\vec{A}\cdot\nabla)r &= (A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z})\sqrt{x^2+y^2+z^2}\\ &= (A_x x+A_y y+A_z z)/r=\vec{A}\cdot\frac{\vec{r}}{r}=\vec{A}\cdot(\nabla r)\end{aligned}$$

$$\nabla(\vec{a}\cdot\vec{r})=\nabla(a_x x+a_y y+a_z z)$$

八、例题

以下 $\vec{\boldsymbol{A}}$ 为任意矢量，$\vec{\boldsymbol{a}}$ 为常矢量。

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{r}} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\,\hat{\boldsymbol{e}}_x + y\,\hat{\boldsymbol{e}}_y + z\,\hat{\boldsymbol{e}}_z) = \vec{\boldsymbol{A}}$$

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$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}) = \nabla(a_x x + a_y y + a_z z) = a_x\,\hat{\boldsymbol{e}}_x + a_y\,\hat{\boldsymbol{e}}_y + a_z\,\hat{\boldsymbol{e}}_z$$

八、例题

以下 $\vec{A}$ 为任意矢量，$\vec{a}$ 为常矢量。

$$(\vec{A}\cdot\nabla)\vec{r} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z) = \vec{A}$$

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$$\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})$$

八、例题

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$$\nabla\cdot(\vec{a}\times\vec{r}) = \nabla_r\cdot(\vec{a}\times\vec{r})$$

利用 $\vec{b}\cdot(\vec{a}\times\vec{c}) = -\vec{a}\cdot(\vec{b}\times\vec{c})$

八、例题

以下 $\vec{\boldsymbol{A}}$ 为任意矢量，$\vec{\boldsymbol{a}}$ 为常矢量。

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{r}} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\,\hat{\boldsymbol{e}}_x + y\,\hat{\boldsymbol{e}}_y + z\,\hat{\boldsymbol{e}}_z) = \vec{\boldsymbol{A}}$$

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$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}) = \nabla(a_x x + a_y y + a_z z) = a_x\,\hat{\boldsymbol{e}}_x + a_y\,\hat{\boldsymbol{e}}_y + a_z\,\hat{\boldsymbol{e}}_z = \vec{\boldsymbol{a}}$$

$$\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \nabla_r\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{a}}\cdot(\nabla\times\vec{\boldsymbol{r}})$$

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$$\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \nabla_r\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{a}}\cdot(\nabla\times\vec{\boldsymbol{r}}) = 0$$

八、例题

以下 $\vec{\boldsymbol{A}}$ 为任意矢量，$\vec{\boldsymbol{a}}$ 为常矢量。

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$$\nabla(\vec{a}\cdot\vec{r})=\nabla(a_x x+a_y y+a_z z)=a_x\,\hat{e}_x+a_y\,\hat{e}_y+a_z\,\hat{e}_z=\vec{a}$$

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$$\nabla\times(\vec{a}\times\vec{r})=\nabla_r\times(\vec{a}\times\vec{r})$$

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$$\nabla\times(\vec{a}\times\vec{r}) = \nabla_r\times(\vec{a}\times\vec{r})$$

利用 $\vec{b}\times(\vec{a}\times\vec{c}) = \vec{a}(\vec{b}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\vec{c}$

八、例题

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$$\nabla\cdot(\vec{a}\times\vec{r}) = \nabla_r\cdot(\vec{a}\times\vec{r}) = -\vec{a}\cdot(\nabla\times\vec{r}) = 0$$

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利用 $\vec{b}\times(\vec{a}\times\vec{c}) = \vec{a}(\vec{b}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\vec{c}$

八、例题

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$$\nabla(\vec{a}\cdot\vec{r}) = \nabla(a_x x + a_y y + a_z z) = a_x\,\hat{e}_x + a_y\,\hat{e}_y + a_z\,\hat{e}_z = \vec{a}$$

$$\nabla\cdot(\vec{a}\times\vec{r}) = \nabla_r\cdot(\vec{a}\times\vec{r}) = -\vec{a}\cdot(\nabla\times\vec{r}) = 0$$

$$\nabla\times(\vec{a}\times\vec{r}) = \nabla_r\times(\vec{a}\times\vec{r}) = \vec{a}(\nabla\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r}$$

利用 $\nabla\cdot\vec{r} = 3$ 和 $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$

八、例题

以下 $\vec{A}$ 为任意矢量，$\vec{a}$ 为常矢量。

$$(\vec{A}\cdot\nabla)\vec{r} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z) = \vec{A}$$

$$\begin{aligned}(\vec{A}\cdot\nabla)r &= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\sqrt{x^2+y^2+z^2} \\ &= (A_x x + A_y y + A_z z)/r = \vec{A}\cdot\frac{\vec{r}}{r} = \vec{A}\cdot(\nabla r)\end{aligned}$$

$$\nabla(\vec{a}\cdot\vec{r}) = \nabla(a_x x + a_y y + a_z z) = a_x\,\hat{e}_x + a_y\,\hat{e}_y + a_z\,\hat{e}_z = \vec{a}$$

$$\nabla\cdot(\vec{a}\times\vec{r}) = \nabla_r\cdot(\vec{a}\times\vec{r}) = -\vec{a}\cdot(\nabla\times\vec{r}) = 0$$

$$\nabla\times(\vec{a}\times\vec{r}) = \nabla_r\times(\vec{a}\times\vec{r}) = \vec{a}(\nabla\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r} = 2\vec{a}$$

利用 $\nabla\cdot\vec{r} = 3$ 和 $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$

$$\nabla \cdot [(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{r}}]$$

$$\nabla \cdot [(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{r}}]$$

利用 $\nabla \cdot (f\vec{\boldsymbol{A}}) = (\nabla f) \cdot \vec{\boldsymbol{A}} + f\nabla \cdot \vec{\boldsymbol{A}}$

$$\nabla \cdot [(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] = [\nabla(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})] \cdot \vec{\boldsymbol{r}} + (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\nabla \cdot \vec{\boldsymbol{r}}$$

利用 $\nabla \cdot (f\vec{\boldsymbol{A}}) = (\nabla f) \cdot \vec{\boldsymbol{A}} + f\nabla \cdot \vec{\boldsymbol{A}}$

$$\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] = [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}$$

利用 $\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})=\vec{\boldsymbol{a}}$ 及 $\nabla\cdot\vec{\boldsymbol{r}}=3$

$$\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] = [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r} + (\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}$$
$$= \vec{a}\cdot\vec{r} + 3\vec{a}\cdot\vec{r} = 4\vec{a}\cdot\vec{r}$$

利用 $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ 及 $\nabla\cdot\vec{r} = 3$

$$
\begin{aligned}
\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}\\
&= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}} = 4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}
\end{aligned}
$$

$$\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] = [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r} + (\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}$$

$$= \vec{a}\cdot\vec{r} + 3\vec{a}\cdot\vec{r} = 4\vec{a}\cdot\vec{r}$$

$$\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}]$$

利用 $\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f\nabla\times\vec{A}$

$$\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] = [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r} + (\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}$$

$$= \vec{a}\cdot\vec{r} + 3\vec{a}\cdot\vec{r} = 4\vec{a}\cdot\vec{r}$$

$$\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}] = [\nabla(\vec{a}\cdot\vec{r})]\times\vec{r} + (\vec{a}\cdot\vec{r})\nabla\times\vec{r}$$

利用 $\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f\nabla\times\vec{A}$

$$
\begin{aligned}
\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}\\
&= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}
\end{aligned}
$$

$$
\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] = [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}}
$$

利用 $\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})=\vec{\boldsymbol{a}}$ 及 $\nabla\times\vec{\boldsymbol{r}}=0$

$$\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] = [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r} + (\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}$$

$$= \vec{a}\cdot\vec{r} + 3\vec{a}\cdot\vec{r} = 4\vec{a}\cdot\vec{r}$$

$$\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}] = [\nabla(\vec{a}\cdot\vec{r})]\times\vec{r} + (\vec{a}\cdot\vec{r})\nabla\times\vec{r}$$

$$= \vec{a}\times\vec{r}$$

利用 $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ 及 $\nabla\times\vec{r} = 0$

$$\begin{aligned}\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}\\ &= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}\end{aligned}$$

$$\begin{aligned}\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}}\\ &= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}\end{aligned}$$

$$\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]$$

$$
\begin{aligned}
\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}\\
&= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}\\
\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}}\\
&= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}
\end{aligned}
$$

$$
\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] = \nabla_r\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]+\nabla_A\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]
$$

$$
\begin{aligned}
\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r}+(\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}\\
&= \vec{a}\cdot\vec{r}+3\vec{a}\cdot\vec{r}=4\vec{a}\cdot\vec{r}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\times\vec{r}+(\vec{a}\cdot\vec{r})\nabla\times\vec{r}\\
&= \vec{a}\times\vec{r}
\end{aligned}
$$

$$
\nabla\cdot[\vec{r}\times\vec{A}(r)] = \nabla_r\cdot[\vec{r}\times\vec{A}(r)]+\nabla_A\cdot[\vec{r}\times\vec{A}(r)]
$$

利用 $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}=-\vec{b}\cdot(\vec{a}\times\vec{c})$

$$\begin{aligned}\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r}+(\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}\\ &= \vec{a}\cdot\vec{r}+3\vec{a}\cdot\vec{r}=4\vec{a}\cdot\vec{r}\end{aligned}$$

$$\begin{aligned}\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\times\vec{r}+(\vec{a}\cdot\vec{r})\nabla\times\vec{r}\\ &= \vec{a}\times\vec{r}\end{aligned}$$

$$\begin{aligned}\nabla\cdot[\vec{r}\times\vec{A}(r)] &= \nabla_r\cdot[\vec{r}\times\vec{A}(r)]+\nabla_A\cdot[\vec{r}\times\vec{A}(r)]\\ &= (\nabla\times\vec{r})\cdot\vec{A}(r)-\vec{r}\cdot[\nabla\times\vec{A}(r)]\end{aligned}$$

利用 $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}=-\vec{b}\cdot(\vec{a}\times\vec{c})$

$$\begin{aligned}\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r}+(\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}\\ &= \vec{a}\cdot\vec{r}+3\vec{a}\cdot\vec{r}=4\vec{a}\cdot\vec{r}\end{aligned}$$

$$\begin{aligned}\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\times\vec{r}+(\vec{a}\cdot\vec{r})\nabla\times\vec{r}\\ &= \vec{a}\times\vec{r}\end{aligned}$$

$$\begin{aligned}\nabla\cdot[\vec{r}\times\vec{A}(r)] &= \nabla_r\cdot[\vec{r}\times\vec{A}(r)]+\nabla_A\cdot[\vec{r}\times\vec{A}(r)]\\ &= (\nabla\times\vec{r})\cdot\vec{A}(r)-\vec{r}\cdot[\nabla\times\vec{A}(r)]\end{aligned}$$

利用 $\nabla\times\vec{r}=0$ 及 $\nabla\times\vec{A}(r)=-\vec{A}'(r)\times\nabla r$

$$\begin{aligned}\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}\\ &= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}\end{aligned}$$

$$\begin{aligned}\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}}\\ &= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}\end{aligned}$$

$$\begin{aligned}\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] &= \nabla_r\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]+\nabla_A\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]\\ &= (\nabla\times\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}}(r)-\vec{\boldsymbol{r}}\cdot[\nabla\times\vec{\boldsymbol{A}}(r)]\\ &= \vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\nabla r]\end{aligned}$$

利用 $\nabla\times\vec{\boldsymbol{r}}=0$ 及 $\nabla\times\vec{\boldsymbol{A}}(r)=-\vec{\boldsymbol{A}}'(r)\times\nabla r$

$$
\begin{aligned}
\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}} \\
&= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}} \\
\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}} \\
&= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] &= \nabla_r\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]+\nabla_A\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] \\
&= (\nabla\times\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}}(r)-\vec{\boldsymbol{r}}\cdot[\nabla\times\vec{\boldsymbol{A}}(r)] \\
&= \vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\nabla r]
\end{aligned}
$$

利用 $\nabla r=\vec{\boldsymbol{r}}/r$

$$\begin{aligned}\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\cdot\vec{r}+(\vec{a}\cdot\vec{r})\nabla\cdot\vec{r}\\ &= \vec{a}\cdot\vec{r}+3\vec{a}\cdot\vec{r}=4\vec{a}\cdot\vec{r}\end{aligned}$$

$$\begin{aligned}\nabla\times[(\vec{a}\cdot\vec{r})\vec{r}] &= [\nabla(\vec{a}\cdot\vec{r})]\times\vec{r}+(\vec{a}\cdot\vec{r})\nabla\times\vec{r}\\ &= \vec{a}\times\vec{r}\end{aligned}$$

$$\begin{aligned}\nabla\cdot[\vec{r}\times\vec{A}(r)] &= \nabla_r\cdot[\vec{r}\times\vec{A}(r)]+\nabla_A\cdot[\vec{r}\times\vec{A}(r)]\\ &= (\nabla\times\vec{r})\cdot\vec{A}(r)-\vec{r}\cdot[\nabla\times\vec{A}(r)]\\ &= \vec{r}\cdot[\vec{A}'(r)\times\nabla r]=\vec{r}\cdot[\vec{A}'(r)\times\vec{r}/r]\end{aligned}$$

利用 $\nabla r=\vec{r}/r$

$$\begin{aligned}\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}\\ &= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}\end{aligned}$$

$$\begin{aligned}\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}}\\ &= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}\end{aligned}$$

$$\begin{aligned}\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] &= \nabla_r\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]+\nabla_A\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]\\ &= (\nabla\times\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}}(r)-\vec{\boldsymbol{r}}\cdot[\nabla\times\vec{\boldsymbol{A}}(r)]\\ &= \vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\nabla r]=\vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\vec{\boldsymbol{r}}/r]\end{aligned}$$

利用 $[\vec{\boldsymbol{A}}'(r)\times\vec{\boldsymbol{r}}]\perp\vec{\boldsymbol{r}}$

$$\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] = [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}}$$
$$= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}} = 4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}$$

$$\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] = [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}}$$
$$= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}$$

$$\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] = \nabla_r\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]+\nabla_A\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]$$
$$= (\nabla\times\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}}(r)-\vec{\boldsymbol{r}}\cdot[\nabla\times\vec{\boldsymbol{A}}(r)]$$
$$= \vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\nabla r] = \vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\vec{\boldsymbol{r}}/r] = 0$$

利用 $[\vec{\boldsymbol{A}}'(r)\times\vec{\boldsymbol{r}}]\perp\vec{\boldsymbol{r}}$

$$\nabla \cdot [\phi(r)(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})]$$

$$\nabla \cdot [\phi(r)(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})]$$

利用 $\nabla \cdot (f\vec{\boldsymbol{A}}) = (\nabla f) \cdot \vec{\boldsymbol{A}} + f\nabla \cdot \vec{\boldsymbol{A}}$

$$\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] \quad = [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})$$

利用 $\nabla\cdot(f\vec{\boldsymbol{A}})=(\nabla f)\cdot\vec{\boldsymbol{A}}+f\nabla\cdot\vec{\boldsymbol{A}}$

$$\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] \quad = [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) + \phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})$$

利用 $\nabla f(u) = f'(u)\nabla u$ 和 $\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = 0$

$$
\begin{aligned}
\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) \\
&= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})
\end{aligned}
$$

利用 $\nabla f(u) = f'(u)\nabla u$ 和 $\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = 0$

$$\begin{aligned}\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) \\ &= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\end{aligned}$$

利用 $\nabla r = \dfrac{\vec{\boldsymbol{r}}}{r}$

$$
\begin{aligned}
\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) \\
&= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})
\end{aligned}
$$

利用 $\nabla r = \dfrac{\vec{\boldsymbol{r}}}{r}$

$$\begin{aligned}\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\\ &= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\end{aligned}$$

利用 $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}\perp\vec{\boldsymbol{r}}$

$$\begin{aligned}\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\\ &= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = 0\end{aligned}$$

利用 $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}\perp\vec{\boldsymbol{r}}$

$$\begin{aligned}\nabla \cdot [\phi(r)(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})] &= [\nabla \phi(r)] \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) + \phi(r) \nabla \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) \\ &= [\phi'(r) \nabla r] \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r} \vec{\boldsymbol{r}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) = 0\end{aligned}$$

$$\nabla \times [\vec{\boldsymbol{E}}_0 \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)]$$

$$\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] = [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})$$
$$= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = 0$$

$$\nabla\times[\vec{\boldsymbol{E}}_0\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)]$$

利用 $\nabla\times(\vec{\boldsymbol{A}}f) = (\nabla\times\vec{\boldsymbol{A}})f-\vec{\boldsymbol{A}}\times(\nabla f)$

$$\nabla \cdot [\phi(r)(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})] = [\nabla \phi(r)] \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) + \phi(r) \nabla \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})$$
$$= [\phi'(r) \nabla r] \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r} \vec{\boldsymbol{r}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) = 0$$

$$\nabla \times [\vec{\boldsymbol{E}}_0 \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)] = \underbrace{(\nabla \times \vec{\boldsymbol{E}}_0)}_{0} \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t) - \vec{\boldsymbol{E}}_0 \times [\nabla \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)]$$

利用 $\nabla \times (\vec{\boldsymbol{A}} f) = (\nabla \times \vec{\boldsymbol{A}}) f - \vec{\boldsymbol{A}} \times (\nabla f)$

$$\begin{aligned}\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\\ &= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})=\frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})=0\end{aligned}$$

$$\nabla\times[\vec{\boldsymbol{E}}_0\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)] = \underbrace{(\nabla\times\vec{\boldsymbol{E}}_0)}_{0}\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)-\vec{\boldsymbol{E}}_0\times[\nabla\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)]$$

利用 $\nabla f(u)=\dfrac{df}{du}\nabla u$

$$
\begin{aligned}
\nabla \cdot [\phi(r)(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})] &= [\nabla \phi(r)] \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) + \phi(r) \nabla \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) \\
&= [\phi'(r) \nabla r] \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r} \vec{\boldsymbol{r}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times [\vec{\boldsymbol{E}}_0 \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)] &= \underbrace{(\nabla \times \vec{\boldsymbol{E}}_0)}_{0} \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t) - \vec{\boldsymbol{E}}_0 \times [\nabla \sin(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)] \\
&= -\vec{\boldsymbol{E}}_0 \times [\cos(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)] \nabla (\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}})
\end{aligned}
$$

利用 $\nabla f(u) = \dfrac{df}{du} \nabla u$

$$\begin{aligned}\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\\ &= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = 0\end{aligned}$$

$$\begin{aligned}\nabla\times[\vec{\boldsymbol{E}}_0\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)] &= \underbrace{(\nabla\times\vec{\boldsymbol{E}}_0)}_{0}\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)-\vec{\boldsymbol{E}}_0\times[\nabla\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)]\\ &= -\vec{\boldsymbol{E}}_0\times[\cos(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)]\nabla(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}})\end{aligned}$$

利用 $\nabla(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}})=\vec{\boldsymbol{k}}$

$$\begin{aligned}\nabla\cdot[\phi(r)(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})] &= [\nabla\phi(r)]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})+\phi(r)\nabla\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})\\ &= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol{r}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}) = 0\end{aligned}$$

$$\begin{aligned}\nabla\times[\vec{\boldsymbol{E}}_0\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)] &= \underbrace{(\nabla\times\vec{\boldsymbol{E}}_0)}_{0}\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)-\vec{\boldsymbol{E}}_0\times[\nabla\sin(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)]\\ &= -\vec{\boldsymbol{E}}_0\times[\cos(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)]\nabla(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}})\\ &= \vec{\boldsymbol{k}}\times\vec{\boldsymbol{E}}_0\cos(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)\end{aligned}$$

利用 $\nabla(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}})=\vec{\boldsymbol{k}}$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用当 $\vec{\boldsymbol{a}}$ 为常矢量时：$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) = (\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{A}} + \vec{\boldsymbol{a}}\times(\nabla\times\vec{\boldsymbol{A}})$

$$\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^3}\right) = (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)+\vec{a}\times\left(\nabla\times\frac{\vec{r}}{r^3}\right)$$

利用当 $\vec{a}$ 为常矢量时：$\nabla(\vec{a}\cdot\vec{A}) = (\vec{a}\cdot\nabla)\vec{A}+\vec{a}\times(\nabla\times\vec{A})$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用 $\nabla\times\dfrac{\vec{\boldsymbol{r}}}{r^n}=0$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用 $\nabla\times\frac{\vec{\boldsymbol{r}}}{r^n}=0$

$$\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^3}\right) = (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right) + \vec{a}\times\left(\nabla\times\frac{\vec{r}}{r^3}\right)$$

$$= (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)$$

标量算符 $(\vec{a}\cdot\nabla)$ 分别作用于 $\vec{r}$ 与 $\frac{1}{r^3}$

$$
\begin{aligned}
\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) &= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)\\
&= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)=\frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]
\end{aligned}
$$

标量算符 $(\vec{\boldsymbol{a}}\cdot\nabla)$ 分别作用于 $\vec{\boldsymbol{r}}$ 与 $\frac{1}{r^3}$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[\,(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

利用 $(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}=\vec{\boldsymbol{a}}$ 和 $(\vec{\boldsymbol{a}}\cdot\nabla)f(u)=f'(u)[(a\cdot\nabla)u]$

$$\begin{aligned}
\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) &= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)\\
&= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)=\frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]\\
&= \frac{\vec{\boldsymbol{a}}}{r^3}+\vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right]
\end{aligned}$$

利用 $(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}=\vec{\boldsymbol{a}}$ 和 $(\vec{\boldsymbol{a}}\cdot\nabla)f(u)=f'(u)[(a\cdot\nabla)u]$

$$
\begin{aligned}
\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) &= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)\\
&= {\color{red}(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)} = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]\\
&= \frac{\vec{\boldsymbol{a}}}{r^3}+\vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right]
\end{aligned}
$$

利用 $(\vec{\boldsymbol{a}}\cdot\nabla)r=\dfrac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$

$$
\begin{aligned}
\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) &= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)\\
&= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]\\
&= \frac{\vec{\boldsymbol{a}}}{r^3}+\vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3}-\left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}
\end{aligned}
$$

利用 $(\vec{\boldsymbol{a}}\cdot\nabla)r = \dfrac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3}+\vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3}-\left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3}-\frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3}+\vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3}-\left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3}-\frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\times\left(\vec{\boldsymbol{a}}\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) + \vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right] + \vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} + \vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3} - \left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} - \frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\times\left(\vec{\boldsymbol{a}}\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用当 $\vec{\boldsymbol{a}}$ 为常矢量时：$\nabla\times(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{A}}) = \vec{\boldsymbol{a}}(\nabla\cdot\vec{\boldsymbol{A}}) - (\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{A}}$

$$
\begin{aligned}
\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^3}\right) &= (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)+\vec{a}\times\left(\nabla\times\frac{\vec{r}}{r^3}\right)\\
&= (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{a}\cdot\nabla)\vec{r}\right]+\vec{r}\left[(\vec{a}\cdot\nabla)\frac{1}{r^3}\right]\\
&= \frac{\vec{a}}{r^3}+\vec{r}\left(-\frac{3}{r^4}\right)\left[(\vec{a}\cdot\nabla)r\right] = \frac{\vec{a}}{r^3}-\left(\frac{3\vec{r}}{r^4}\right)\frac{(\vec{a}\cdot\vec{r})}{r}\\
&= \frac{\vec{a}}{r^3}-\frac{3(\vec{a}\cdot\vec{r})\vec{r}}{r^5}
\end{aligned}
$$

$$
\nabla\times\left(\vec{a}\times\frac{\vec{r}}{r^3}\right) = \vec{a}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)-(\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)
$$

利用当 $\vec{a}$ 为常矢量时：$\nabla\times(\vec{a}\times\vec{A}) = \vec{a}(\nabla\cdot\vec{A})-(\vec{a}\cdot\nabla)\vec{A}$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) + \vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right] + \vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} + \vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3} - \left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} - \frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\times\left(\vec{\boldsymbol{a}}\times\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \vec{\boldsymbol{a}}\left(\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) - (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用 $\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3} = \frac{\nabla\cdot\vec{\boldsymbol{r}}}{r^3} + \left(\nabla\frac{1}{r^3}\right)\cdot\vec{\boldsymbol{r}} = \frac{3}{r^3} + \left(-\frac{3}{r^4}\,\frac{\vec{\boldsymbol{r}}}{r}\right)\cdot\vec{\boldsymbol{r}} = 0$

$$\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^3}\right) = (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)+\vec{a}\times\left(\nabla\times\frac{\vec{r}}{r^3}\right)$$

$$= (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{a}\cdot\nabla)\vec{r}\right]+\vec{r}\left[(\vec{a}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{a}}{r^3}+\vec{r}\left(-\frac{3}{r^4}\right)\left[(\vec{a}\cdot\nabla)r\right] = \frac{\vec{a}}{r^3}-\left(\frac{3\vec{r}}{r^4}\right)\frac{(\vec{a}\cdot\vec{r})}{r}$$

$$= \frac{\vec{a}}{r^3}-\frac{3(\vec{a}\cdot\vec{r})\vec{r}}{r^5}$$

$$\nabla\times\left(\vec{a}\times\frac{\vec{r}}{r^3}\right) = \vec{a}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)-(\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right) = -(\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)$$

利用 $\nabla\cdot\frac{\vec{r}}{r^3} = \frac{\nabla\cdot\vec{r}}{r^3}+\left(\nabla\frac{1}{r^3}\right)\cdot\vec{r} = \frac{3}{r^3}+\left(-\frac{3}{r^4}\frac{\vec{r}}{r}\right)\cdot\vec{r} = 0$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) + \vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right] + \vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} + \vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3} - \left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} - \frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\times\left(\vec{\boldsymbol{a}}\times\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \vec{\boldsymbol{a}}\left(\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) - (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = -(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用 $(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{\vec{\boldsymbol{a}}}{r^3} - \frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$ （上面红色部分）

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)+\vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right]+\vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3}+\vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3}-\left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3}-\frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\times\left(\vec{\boldsymbol{a}}\times\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \vec{\boldsymbol{a}}\left(\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right)-(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = -(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

利用 $(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{\vec{\boldsymbol{a}}}{r^3}-\frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$ （上面红色部分）

$$= -\frac{\vec{\boldsymbol{a}}}{r^3}+\frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\left(\vec{\boldsymbol{a}}\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) = (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) + \vec{\boldsymbol{a}}\times\left(\nabla\times\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{r}}\right] + \vec{\boldsymbol{r}}\left[(\vec{\boldsymbol{a}}\cdot\nabla)\frac{1}{r^3}\right]$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} + \vec{\boldsymbol{r}}\left(-\frac{3}{r^4}\right)\left[(\vec{\boldsymbol{a}}\cdot\nabla)r\right] = \frac{\vec{\boldsymbol{a}}}{r^3} - \left(\frac{3\vec{\boldsymbol{r}}}{r^4}\right)\frac{(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})}{r}$$

$$= \frac{\vec{\boldsymbol{a}}}{r^3} - \frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

$$\nabla\times\left(\vec{\boldsymbol{a}}\times\frac{\vec{\boldsymbol{r}}}{r^3}\right) = \vec{\boldsymbol{a}}\left(\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3}\right) - (\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) = -(\vec{\boldsymbol{a}}\cdot\nabla)\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right)$$

$$= -\frac{\vec{\boldsymbol{a}}}{r^3} + \frac{3(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}}{r^5}$$

Let there be light

$$(\vec{A} \times \nabla) \times \vec{r}$$

$$(\vec{A}\times\nabla)\times\vec{r} \quad = \underbrace{\Big[\ \overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$(\vec{\boldsymbol{A}} \times \nabla) \times \vec{\boldsymbol{r}} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i \partial_j)}^{\vec{\boldsymbol{A}} \times \nabla \text{ 的 } k \text{ 分量}} \hat{\boldsymbol{e}}_k \Big]}_{\text{对重复下标求和}} \times (x_l \, \hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i \partial_j x_l)(\hat{\boldsymbol{e}}_k \times \hat{\boldsymbol{e}}_l)$$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m)$$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m)$$

有 δ_{jl} 项，$\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$(\vec{\boldsymbol{A}}\times\nabla)\times\vec{\boldsymbol{r}} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{\boldsymbol{A}}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l = \varepsilon_{klm}\,\hat{\boldsymbol{e}}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{\boldsymbol{e}}_m)$$

有 δ_{jl} 项，$\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{\boldsymbol{e}}_m$$

$$
\begin{aligned}
(\vec{A}\times\nabla)\times\vec{r} &= \Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\hat{e}_k\Big]_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l) \\
&= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l) \qquad \text{利用}\begin{cases}\partial_j x_l=\delta_{jl}\\ \hat{e}_k\times\hat{e}_l=\varepsilon_{klm}\,\hat{e}_m\end{cases} \\
&= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m) \qquad \text{有 }\delta_{jl}\text{ 项，}\begin{cases}\text{对 }l\text{ 先求和}\\ \text{只留 }l=j\text{ 项}\end{cases} \\
&= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{e}_m \qquad \text{连续利用 }\varepsilon_{ijk}=-\varepsilon_{ikj}
\end{aligned}
$$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\ \overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l = \varepsilon_{klm}\,\hat{\boldsymbol{e}}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{\boldsymbol{e}}_m)$$

有 δ_{jl} 项，$\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{\boldsymbol{e}}_m$$

连续利用 $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$= -[\varepsilon_{ijk}\varepsilon_{mjk}]A_i\,\hat{\boldsymbol{e}}_m$$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m)$$

有 δ_{jl} 项，$\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{e}_m$$

连续利用 $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$= -\big[\varepsilon_{ijk}\varepsilon_{mjk}\big]A_i\,\hat{e}_m$$

[ ] 内为Levi-Civita 符号的两重求和

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m)$$

有 δ_{jl} 项，$\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{e}_m$$

连续利用 $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$= -[\varepsilon_{ijk}\varepsilon_{mjk}]A_i\,\hat{e}_m$$

[ ] 内为Levi-Civita 符号的两重求和

$$= -2\delta_{im}A_i\,\hat{e}_m$$

满足 $\varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im}$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m)$$

有 δ_{jl} 项， $\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{e}_m$$

连续利用 $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$= -[\varepsilon_{ijk}\varepsilon_{mjk}]A_i\,\hat{e}_m$$

[ ] 内为Levi-Civita 符号的两重求和

$$= -2\delta_{im}A_i\,\hat{e}_m$$

满足 $\varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im}$

$$= -2A_i\,\hat{e}_i = -2\vec{A}$$

$$(\vec{A}\times\nabla)\times\vec{r} = \Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\hat{e}_k\Big]\times(x_l\,\hat{e}_l)$$

对重复下标求和

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\,\hat{e}_m)$$

有 δ_{jl} 项，$\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{e}_m$$

连续利用 $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$= -[\varepsilon_{ijk}\varepsilon_{mjk}]A_i\,\hat{e}_m$$

[ ] 内为Levi-Civita 符号的两重求和

$$= -2\delta_{im}A_i\,\hat{e}_m$$

满足 $\varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im}$

$$= -2A_i\,\hat{e}_i = -2\vec{A}$$

$$(\vec{A}\times\nabla)\times\vec{r} = -2\vec{A}$$

$$(\vec{A}\times\nabla)\times\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\times\hat{e}_l = \varepsilon_{klm}\,\hat{e}_m\end{cases}$

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有 δ_{jl} 项， $\begin{cases}\text{对 } l \text{ 先求和}\\ \text{只留 } l=j \text{ 项}\end{cases}$

$$= \varepsilon_{ijk}A_i\varepsilon_{kjm}\,\hat{e}_m$$

连续利用 $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$= -[\varepsilon_{ijk}\varepsilon_{mjk}]A_i\,\hat{e}_m$$

[ ] 内为Levi-Civita 符号的两重求和

$$= -2\delta_{im}A_i\,\hat{e}_m$$

满足 $\varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im}$

$$= -2A_i\,\hat{e}_i = -2\vec{A}$$

$$(\vec{A}\times\nabla)\times\vec{r} = -2\vec{A}$$

$\vec{A}$ 可以为任意矢量

$$(\vec{\boldsymbol{A}} \times \nabla) \cdot \vec{\boldsymbol{r}}$$

$$(\vec{A}\times\nabla)\cdot\vec{r} \;=\; \underbrace{\Big[\;\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\;\;\hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{\boldsymbol{e}}_l)$$

$$(\vec{\boldsymbol{A}} \times \nabla) \cdot \vec{\boldsymbol{r}} = \underbrace{\Big[\ \overbrace{\varepsilon_{ijk}(A_i \partial_j)}^{\vec{\boldsymbol{A}} \times \nabla \text{ 的 } k \text{ 分量}}\ \hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}} \cdot (x_l\, \hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i \partial_j x_l)(\hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}_l)$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_l) \qquad \text{利用}\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_l = \delta_{kl}\end{cases}$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl}$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l) \qquad \text{利用}\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = 0$$

$$(\vec{\boldsymbol{A}} \times \nabla) \cdot \vec{\boldsymbol{r}} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{\boldsymbol{A}} \times \nabla \text{ 的 } k \text{ 分量}} \hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}} \cdot (x_l\, \hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i \partial_j x_l)(\hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}_l) \qquad \text{利用} \begin{cases} \partial_j x_l = \delta_{jl} \\ \hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}_l = \delta_{kl} \end{cases}$$

$$= \varepsilon_{ijk}(A_i \delta_{jl})\delta_{kl} = \varepsilon_{ill} A_i = 0$$

$$(\vec{\boldsymbol{A}} \times \nabla) \cdot \vec{\boldsymbol{r}} = 0 \qquad \vec{\boldsymbol{A}} \text{ 可以为任意矢量}$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\,\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{e}_l)$$

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$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = 0 \qquad \vec{A}\text{ 可以为任意矢量}$$

另证

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\ \overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = 0$$

$\vec{A}$ 可以为任意矢量

另证 $(\vec{A}\times\nabla)\cdot\vec{r} = \vec{A}\cdot(\nabla\times\vec{r}) = 0$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{\boldsymbol{e}}_k\Big]\cdot(x_l\,\hat{\boldsymbol{e}}_l)}_{\text{对重复下标求和}}$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_l) \qquad \text{利用}\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_l = \delta_{kl}\end{cases}$$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = 0$$

$\vec{A}$ 可以为任意矢量

另证 $(\vec{A}\times\nabla)\cdot\vec{r} = \vec{A}\cdot(\nabla\times\vec{r}) = 0$

$$(\vec{A}\times\nabla)\times\vec{r} = (\vec{A}\times\nabla_r)\times\vec{r}$$

$$(\vec{\boldsymbol{A}}\times\nabla)\cdot\vec{\boldsymbol{r}} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{\boldsymbol{A}}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{\boldsymbol{e}}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{\boldsymbol{e}}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_l) \qquad \text{利用}\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{\boldsymbol{e}}_k\cdot\hat{\boldsymbol{e}}_l = \delta_{kl}\end{cases}$$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

$$(\vec{\boldsymbol{A}}\times\nabla)\cdot\vec{\boldsymbol{r}} = 0$$

$\vec{\boldsymbol{A}}$ 可以为任意矢量

另证 $(\vec{\boldsymbol{A}}\times\nabla)\cdot\vec{\boldsymbol{r}} = \vec{\boldsymbol{A}}\cdot(\nabla\times\vec{\boldsymbol{r}}) = 0$

$$(\vec{\boldsymbol{A}}\times\nabla)\times\vec{\boldsymbol{r}} = (\vec{\boldsymbol{A}}\times\nabla_r)\times\vec{\boldsymbol{r}}$$

利用 $(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})\times\vec{\boldsymbol{c}} = \vec{\boldsymbol{b}}\,(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{c}}) - \vec{\boldsymbol{a}}\,(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{c}})$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l) \qquad \text{利用}\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$$

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$$(\vec{A}\times\nabla)\cdot\vec{r} = 0$$

$\vec{A}$ 可以为任意矢量

另证 $(\vec{A}\times\nabla)\cdot\vec{r} = \vec{A}\cdot(\nabla\times\vec{r}) = 0$

$$(\vec{A}\times\nabla)\times\vec{r} = (\vec{A}\times\nabla_r)\times\vec{r}$$

利用 $(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{a}\,(\vec{b}\cdot\vec{c})$

$$= \nabla_r(\vec{A}\cdot\vec{r}) - \vec{A}(\nabla_r\cdot\vec{r})$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \Big[\underbrace{\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k}_{\text{对重复下标求和}}\Big]\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$

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利用 $(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{a}\,(\vec{b}\cdot\vec{c})$

$$= \nabla_r(\vec{A}\cdot\vec{r}) - \vec{A}(\nabla_r\cdot\vec{r})$$

$\nabla_r(\vec{A}\cdot\vec{r})$ 中 $\vec{A}$ 视为常矢量

利用 $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ 和 $\nabla\cdot\vec{r} = 3$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 } k \text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$

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利用 $(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{a}\,(\vec{b}\cdot\vec{c})$

$$= \nabla_r(\vec{A}\cdot\vec{r}) - \vec{A}(\nabla_r\cdot\vec{r})$$

$\nabla_r(\vec{A}\cdot\vec{r})$ 中 $\vec{A}$ 视为常矢量

利用 $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ 和 $\nabla\cdot\vec{r} = 3$

$$= -2\vec{A}$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = \Big[\underbrace{\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k}_{\text{对重复下标求和}}\Big]\cdot(x_l\,\hat{e}_l)$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l)$$

利用 $\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

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