高维微分学——隐映照定理的应用
（曲线与曲面的隐式表示）

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1　知识要素

1.1　隐映照定理

定理 1.1 (隐映照定理). 设有映照 $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}):\mathbb{R}^m\times\mathbb{R}^n\supset D_x\times D_y\ni\{\boldsymbol{x},\boldsymbol{y}\}\mapsto\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^n$$

满足:

1. $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})\in\mathscr{C}^1(D_x\times D_y;\mathbb{R}^n)$;
2. $\exists\,(\boldsymbol{x}_0,\boldsymbol{y}_0)\in D_x\times D_y$ 使得 $\begin{cases}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=0\in\mathbb{R}^n,\\ D_y\boldsymbol{f}(\boldsymbol{x}_0,\boldsymbol{y}_0)\in\mathscr{L}(\mathbb{R}^n;\mathbb{R}^n)\text{可逆},\end{cases}$

则有

1. $\exists\,B_\lambda(\boldsymbol{x}_0)\subset D_x, B_\mu(\boldsymbol{y}_0)\subset D_y$, 有 $\forall\,\boldsymbol{x}\in B_\lambda(\boldsymbol{x}_0),\exists!\boldsymbol{y}_x\in B_\mu(\boldsymbol{y}_0)$ 满足 $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}_x)=0\in\mathbb{R}^n$, 由此可作 $\boldsymbol{\xi}(\boldsymbol{x}):B_\lambda(\boldsymbol{x}_0)\ni\boldsymbol{x}\mapsto\boldsymbol{\xi}(\boldsymbol{x})\in\mathbb{R}^n$, 满足 $\begin{cases}\boldsymbol{\xi}(\boldsymbol{x})\in B_\mu(\boldsymbol{y}_0),\\ \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\xi}(\boldsymbol{x}))=0\in\mathbb{R}^n;\end{cases}$
2. $\boldsymbol{\xi}(\boldsymbol{x})\in\mathscr{C}^1(B_\lambda(\boldsymbol{x}_0);\mathbb{R}^n)$.

1.2　曲线的隐式表示

现考虑 $\mathbb{R}^m$ 中的约束

$$\varSigma=\left\{\boldsymbol{x}\in\mathbb{R}^m\;\middle|\;\boldsymbol{f}(\boldsymbol{x})=\begin{pmatrix}f^1\\ \vdots\\ f^{m-1}\end{pmatrix}(\boldsymbol{x})=\boldsymbol{0}\in\mathbb{R}^{m-1}\right\},$$

基于隐映照定理, 针对上述约束, 有以下结论:

如有 $\boldsymbol{x}_0 = \begin{pmatrix} \tilde{x}_0 \\ \hat{\boldsymbol{x}}_0 \end{pmatrix} \in \Sigma$, 其中 $\tilde{x}_0 \in \mathbb{R}, \hat{\boldsymbol{x}}_0 \in \mathbb{R}^{m-1}$, 满足

$$D_{\hat{\boldsymbol{x}}}\boldsymbol{f}(\tilde{x}_0, \hat{\boldsymbol{x}}_0) = \frac{D(f^1, \cdots, f^{m-1})}{D(\hat{x}^1, \cdots, \hat{x}^{m-1})}(\tilde{x}_0, \hat{\boldsymbol{x}}_0) = \frac{D(f^1, \cdots, f^{m-1})}{D(x^2, \cdots, x^m)}(\tilde{x}_0, \hat{\boldsymbol{x}}_0) \in \mathbb{R}^{(m-1)\times(m-1)}$$

非奇异, 则 $\exists B_\lambda(\tilde{x}_0) \subset \mathbb{R}, B_\mu(\hat{\boldsymbol{x}}_0) \subset \mathbb{R}^{m-1}$ 满足

$$\forall\, \tilde{x} \in B_\lambda(\tilde{x}_0), \quad \exists!\, \hat{\boldsymbol{x}} = \boldsymbol{\phi}(\tilde{x}) \in B_\mu(\hat{\boldsymbol{x}}_0), \quad \text{满足约束 } \boldsymbol{f}(\tilde{x}, \boldsymbol{\phi}(\tilde{x})) = \boldsymbol{0} \in \mathbb{R}^{m-1}.$$

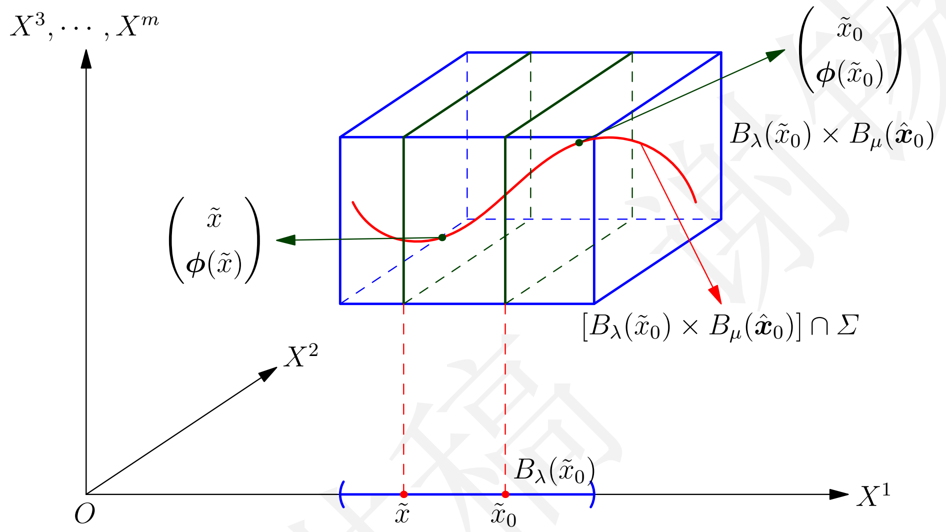

图 1: 曲线的隐映照刻画

隐映照定理结论的几何刻画, 如图1所示, 图中 $B_\mu(\hat{x}_0)$ 示意性地画为矩形. 局部柱体 $B_\lambda(\tilde{x}_0) \times B_\mu(\hat{\boldsymbol{x}}_0) \subset \mathbb{R}^m$ 中, Σ 为隐映照的图像 $\left\{ \begin{pmatrix} \tilde{x} \\ \boldsymbol{\phi}(\tilde{x}) \end{pmatrix} \middle| \forall\, \tilde{x} \in B_\lambda(\tilde{x}_0) \right\} \subset \mathbb{R}^m$. 现为 $\mathbb{R}^m$ 中的曲线

$$\boldsymbol{\Gamma}(\tilde{x}) : \mathbb{R} \supset B_\lambda(\tilde{x}_0) \ni \tilde{x} \mapsto \boldsymbol{\Gamma}(\tilde{x}) = \begin{pmatrix} \tilde{x} \\ \boldsymbol{\phi}(\tilde{x}) \end{pmatrix} \subset \mathbb{R}^m.$$

可确定曲线 $\boldsymbol{\Gamma}(\tilde{x})$ 的切向量为

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\Gamma}(\tilde{x}) &\triangleq \lim_{\Delta\tilde{x}\to 0} \frac{\boldsymbol{\Gamma}(\tilde{x}+\Delta\tilde{x}) - \boldsymbol{\Gamma}(\tilde{x})}{\Delta\tilde{x}} = D\boldsymbol{\Gamma}(\tilde{x}) = \begin{pmatrix} 1 \\ D\boldsymbol{\phi}(\tilde{x}) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -\left(D_{\hat{\boldsymbol{x}}}\boldsymbol{f}\right)^{-1} D_{\tilde{x}}\boldsymbol{f} \end{pmatrix} (\tilde{x}, \boldsymbol{\phi}(\tilde{x})) \in \mathbb{R}^m. \end{aligned}$$

1.3　曲面的隐式表示

现考虑 $\mathbb{R}^m$ 中的约束

$$\Sigma = \{\boldsymbol{x} \in \mathbb{R}^m | f(\boldsymbol{x}) = 0 \in \mathbb{R}\},$$

基于隐映照定理, 针对上述约束, 有以下结论:

如有 $\boldsymbol{x}_0 = \begin{pmatrix} \tilde{\boldsymbol{x}}_0 \\ \hat{x}_0 \end{pmatrix} \in \Sigma$, 其中 $\tilde{\boldsymbol{x}}_0 \in \mathbb{R}^{m-1}, \hat{x}_0 \in \mathbb{R}$, 满足

$$D_{\hat{x}} f(\tilde{\boldsymbol{x}}_0, \hat{x}_0) \triangleq \frac{\partial f}{\partial \hat{x}}(\tilde{\boldsymbol{x}}_0, \hat{x}_0) = \frac{\partial f}{\partial x^m}(\tilde{\boldsymbol{x}}_0, \hat{x}_0) \neq 0 \in \mathbb{R},$$

则 $\exists B_\lambda(\tilde{\boldsymbol{x}}_0) \subset \mathbb{R}^{m-1}, B_\mu(\hat{x}_0) \subset \mathbb{R}$, 满足

$$\forall\, \tilde{\boldsymbol{x}} \in B_\lambda(\tilde{\boldsymbol{x}}_0), \quad \exists!\, \hat{x} = \phi(\tilde{\boldsymbol{x}}) B_\mu(\hat{x}_0), \quad \text{满足约束} f(\tilde{\boldsymbol{x}}, \phi(\tilde{\boldsymbol{x}})) = 0 \in \mathbb{R}.$$

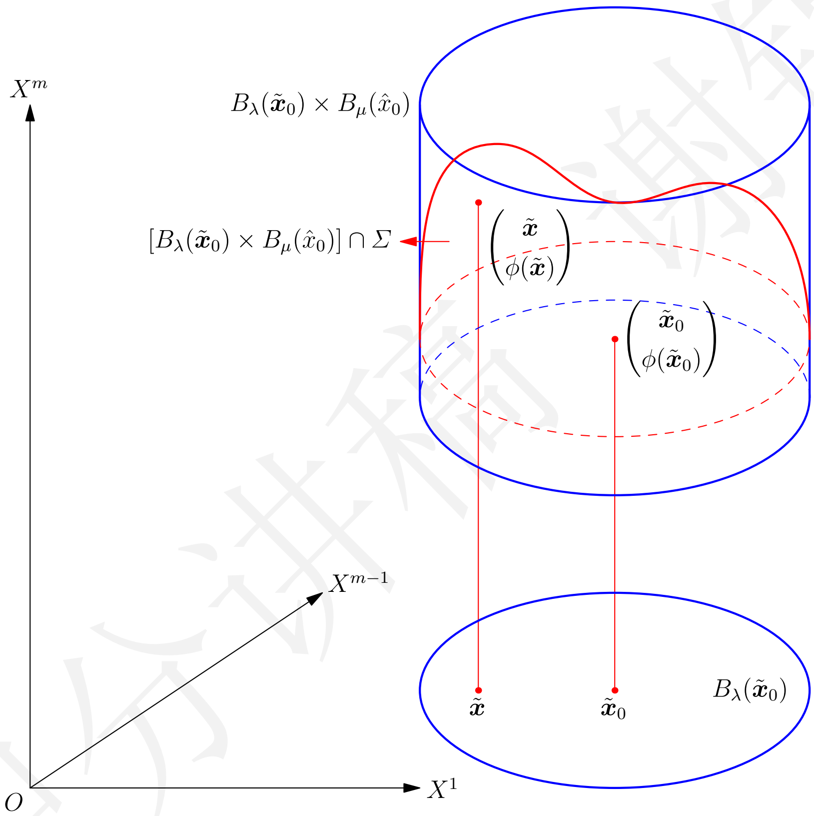

图 2: 曲面的隐映照刻画

隐映照定理结论的几何刻画, 如图2所示. 局部柱体 $B_\lambda(\tilde{\boldsymbol{x}}_0) \times B_\mu(\hat{x}_0) \subset \mathbb{R}^m$ 中, Σ 为隐映照的图像 $\left\{ \begin{pmatrix} \tilde{\boldsymbol{x}} \\ \phi(\tilde{\boldsymbol{x}}) \end{pmatrix} \middle| \forall\, \tilde{\boldsymbol{x}} \in B_\lambda(\tilde{\boldsymbol{x}}_0) \right\} \subset \mathbb{R}^m$. 现为 $\mathbb{R}^m$ 中的曲面

$$\boldsymbol{\Sigma}(\tilde{\boldsymbol{x}}) : \mathbb{R}^{m-1} \supset B_\lambda(\tilde{\boldsymbol{x}}_0) \ni \tilde{\boldsymbol{x}} \mapsto \boldsymbol{\Sigma}(\tilde{\boldsymbol{x}}) = \begin{pmatrix} \tilde{\boldsymbol{x}} \\ \phi(\tilde{\boldsymbol{x}}) \end{pmatrix} \subset \mathbb{R}^m.$$

可确定

$$D\boldsymbol{\Sigma}(\tilde{\boldsymbol{x}}) = \begin{pmatrix} \boldsymbol{I}_{m-1} \\ \boldsymbol{D}\phi(\tilde{\boldsymbol{x}}) \end{pmatrix} := \begin{pmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_{m-1} \end{pmatrix} (\tilde{\boldsymbol{x}}) \in \mathbb{R}^{m \times (m-1)},$$

藉此就确定了切空间

$$T_{\boldsymbol{x}} \Sigma \triangleq \operatorname{span}\left\{ \boldsymbol{g}_1, \cdots, \boldsymbol{g}_{m-1} \right\} (\tilde{\boldsymbol{x}}) \subset \mathbb{R}^m,$$

相应地, 也可以确定法向量 $\boldsymbol{n}(\boldsymbol{x}) \in \mathbb{R}^m$, 满足

$$\begin{cases} (D\boldsymbol{\Sigma})^{\mathrm{T}}(\tilde{\boldsymbol{x}})\boldsymbol{n}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{I}_{m-1} & (D\boldsymbol{\phi})^{\mathrm{T}}(\tilde{\boldsymbol{x}}) \end{pmatrix} \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{0} \in \mathbb{R}^{m-1}, \\ |\boldsymbol{n}(\boldsymbol{x})|_{\mathbb{R}^m} = 1. \end{cases}$$

2 应用事例

2.1 曲线的隐式表示

事例 1 ($\mathbb{R}^3$ 中曲线（隐式表示形式）). $\mathbb{R}^3$ 中的曲线可以表示为

$$\Gamma = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{cases} f(x,y,z) = 0 \in \mathbb{R} \\ g(x,y,z) = 0 \in \mathbb{R} \end{cases} \right\}$$

设有 $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \in \Gamma$，亦即 $\begin{cases} f(x_0,y_0,z_0) = 0 \\ g(x_0,y_0,z_0) = 0 \end{cases}$，且有 $\dfrac{\partial(f,g)}{\partial(x,z)}(x_0,y_0,z_0) \neq 0$，则可构造

$$\boldsymbol{F}\left(y, \begin{bmatrix} x \\ z \end{bmatrix}\right) = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \end{bmatrix} \in \mathbb{R}^2$$

满足

$$\begin{cases} \boldsymbol{F}\left(y_0, \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right) = \begin{bmatrix} f(x_0,y_0,z_0) \\ g(x_0,y_0,z_0) \end{bmatrix} = \boldsymbol{0} \in \mathbb{R}^2 \\ D_{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]}\boldsymbol{F}\left(y_0, \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial z} \end{bmatrix}(x_0,y_0,z_0) \in \mathbb{R}^{2\times 2} \text{ 非奇异} \end{cases}$$

则有 $\exists B_\lambda(y_0) \subset \mathbb{R}, \exists B_\mu\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right) \subset \mathbb{R}^2$，对 $\forall y \in B_\lambda(y_0), \exists \boldsymbol{\xi}(y) = \begin{bmatrix} x(y) \\ z(y) \end{bmatrix} \in B_\mu\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right)$ 满足

$$\boldsymbol{F}\left(y, \begin{bmatrix} x(y) \\ z(y) \end{bmatrix}\right) = \begin{bmatrix} f(x(y),y,z(y)) \\ g(x(y),y,z(y)) \end{bmatrix} = \boldsymbol{0} \in \mathbb{R}^2$$

亦即有 $\begin{bmatrix} x(y) \\ y \\ z(y) \end{bmatrix} \in \Gamma$，所以曲线可以用向量值映照表示为

$$\boldsymbol{\Gamma}(y) : B_\lambda(y_0) \ni y \mapsto \boldsymbol{\Gamma}(y) = \begin{bmatrix} x(y) \\ y \\ z(y) \end{bmatrix} \in \mathbb{R}^3$$

上述分析的几何化可以表示为

曲线 Γ 的切向量为

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}y}(y) = D\boldsymbol{\Gamma}(y) = \begin{bmatrix} x'(y) \\ 1 \\ z'(y) \end{bmatrix} \in \mathbb{R}^3$$

另有

$$\boldsymbol{F}\left(y, \begin{bmatrix} x(y) \\ z(y) \end{bmatrix}\right) = \mathbf{0} \in \mathbb{R}^2, \quad \forall y \in B_\lambda(y_0)$$

则有

$$D_y\boldsymbol{F}\left(y, \begin{bmatrix} x(y) \\ z(y) \end{bmatrix}\right) + D_{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]}\boldsymbol{F}\left(y, \begin{bmatrix} x(y) \\ z(y) \end{bmatrix}\right)\begin{bmatrix} x'(y) \\ z'(y) \end{bmatrix} = \mathbf{0} \in \mathbb{R}^2$$

即有

$$\begin{bmatrix} \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial y} \end{bmatrix}(x(y), y, z(y)) + \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial z} \end{bmatrix}(x(y), y, z(y))\begin{bmatrix} x'(y) \\ z'(y) \end{bmatrix} = \mathbf{0}$$

所以

$$\begin{aligned}\begin{bmatrix} x'(y) \\ z'(y) \end{bmatrix} &= -\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial z} \end{bmatrix}^{-1}(x(y), y, z(y))\begin{bmatrix} \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial y} \end{bmatrix}(x(y), y, z(y)) \\ &= -\frac{1}{\dfrac{\partial(f,g)}{\partial(x,z)}(x(y), y, z(y))}\begin{bmatrix} \dfrac{\partial(f,g)}{\partial(y,z)} \\ \dfrac{\partial(f,g)}{\partial(x,y)} \end{bmatrix}(x(y), y, z(y))\end{aligned}$$

由此可得切向量的表示

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}y}(y) = \begin{bmatrix} -\dfrac{\dfrac{\partial(f,g)}{\partial(y,z)}}{\dfrac{\partial(f,g)}{\partial(x,z)}}(x(y), y, z(y)) \\ 1 \\ -\dfrac{\dfrac{\partial(f,g)}{\partial(x,y)}}{\dfrac{\partial(f,g)}{\partial(x,z)}}(x(y), y, z(y)) \end{bmatrix} \parallel \begin{bmatrix} \dfrac{\partial(f,g)}{\partial(y,z)} \\ \dfrac{\partial(f,g)}{\partial(z,x)} \\ \dfrac{\partial(f,g)}{\partial(x,y)} \end{bmatrix}(x(y), y, z(y)), \quad \forall y \in B_\lambda(y_0)$$

据此，可确定过点 $\begin{bmatrix} x(y_0) \\ y_0 \\ z(y_0) \end{bmatrix} \in \Gamma$ 点的切线可以表示为

$$\boldsymbol{L}(\lambda) : \mathbb{R} \ni \lambda \mapsto \begin{bmatrix} x(y) \\ y \\ z(y) \end{bmatrix} + \lambda \begin{bmatrix} \dfrac{\partial(f,g)}{\partial(y,z)} \\ \dfrac{\partial(f,g)}{\partial(z,x)} \\ \dfrac{\partial(f,g)}{\partial(x,y)} \end{bmatrix}(x(y), y, z(y)) \in \mathbb{R}^3$$

或者表示为

$$\frac{x-x(y_0)}{\dfrac{\partial(f,g)}{\partial(y,z)}(x(y_0),y_0,z(y_0))}=\frac{y-y_0}{\dfrac{\partial(f,g)}{\partial(z,x)}(x(y_0),y_0,z(y_0))}=\frac{z-z(y_0)}{\dfrac{\partial(f,g)}{\partial(x,y)}(x(y_0),y_0,z(y_0))}$$

事例 2 ($\mathbb{R}^4$ 中曲线（隐式表示形式）)**.** $\mathbb{R}^4$ 中的曲线可以表示为

$$\Gamma=\left\{\begin{bmatrix}x\\y\\z\\\theta\end{bmatrix}\in\mathbb{R}^4 \;\middle|\; \begin{cases}f(x,y,z,\theta)=0\in\mathbb{R}\\g(x,y,z,\theta)=0\in\mathbb{R}\\h(x,y,z,\theta)=0\in\mathbb{R}\end{cases}\right\}$$

设有 $\begin{bmatrix}x_0\\y_0\\z_0\\\theta_0\end{bmatrix}\in\Gamma$，亦即 $\begin{cases}f(x_0,y_0,z_0,\theta_0)=0\\g(x_0,y_0,z_0,\theta_0)=0\\h(x_0,y_0,z_0,\theta_0)=0\end{cases}$，且有 $\dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}(x_0,y_0,z_0,\theta_0)\neq 0$，则可构造

$$\boldsymbol{F}\left(z,\begin{bmatrix}x\\y\\\theta\end{bmatrix}\right)=\begin{bmatrix}f(x,y,z,\theta)\\g(x,y,z,\theta)\\h(x,y,z,\theta)\end{bmatrix}\in\mathbb{R}^3$$

满足

$$\begin{cases}\boldsymbol{F}\left(z_0,\begin{bmatrix}x_0\\y_0\\\theta_0\end{bmatrix}\right)=\begin{bmatrix}f(x_0,y_0,z_0,\theta_0)\\g(x_0,y_0,z_0,\theta_0)\\h(x_0,y_0,z_0,\theta_0)\end{bmatrix}=\boldsymbol{0}\in\mathbb{R}^3\\[2ex] D_{\left[\begin{smallmatrix}x\\y\\\theta\end{smallmatrix}\right]}\boldsymbol{F}\left(z_0,\begin{bmatrix}x_0\\y_0\\\theta_0\end{bmatrix}\right)=\begin{bmatrix}\dfrac{\partial f}{\partial x}&\dfrac{\partial f}{\partial y}&\dfrac{\partial f}{\partial\theta}\\\dfrac{\partial g}{\partial x}&\dfrac{\partial g}{\partial y}&\dfrac{\partial g}{\partial\theta}\\\dfrac{\partial h}{\partial x}&\dfrac{\partial h}{\partial y}&\dfrac{\partial h}{\partial\theta}\end{bmatrix}(x_0,y_0,z_0,\theta_0)\in\mathbb{R}^{3\times3}\text{ 非奇异}\end{cases}$$

则有 $\exists B_\lambda(z_0)\subset\mathbb{R}$, $\exists B_\mu\left(\begin{bmatrix}x_0\\y_0\\\theta_0\end{bmatrix}\right)\subset\mathbb{R}^3$, 对 $\forall z\in B_\lambda(z_0)$, $\exists\boldsymbol{\xi}(z)=\begin{bmatrix}x(z)\\y(z)\\\theta(z)\end{bmatrix}\in B_\mu\left(\begin{bmatrix}x_0\\y_0\\\theta_0\end{bmatrix}\right)$
满足

$$\boldsymbol{F}\left(z,\begin{bmatrix}x(z)\\y(z)\\\theta(z)\end{bmatrix}\right)=\begin{bmatrix}f(x(z),y(z),z,\theta(y))\\g(x(z),y(z),z,\theta(y))\\h(x(z),y(z),z,\theta(y))\end{bmatrix}=\boldsymbol{0}\in\mathbb{R}^3$$

亦即有 $\begin{bmatrix}x(z)\\y(z)\\z\\\theta(z)\end{bmatrix}\in\Gamma$，上述分析的几何化可以表示为

所以曲线可以用向量值映照表示为

$$\boldsymbol{\Gamma}(z) : B_\lambda(z_0) \ni z \mapsto \boldsymbol{\Gamma}(z) = \begin{bmatrix} x(z) \\ y(z) \\ z \\ \theta(z) \end{bmatrix} \in \mathbb{R}^4$$

曲线 Γ 的切向量为

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}z}(z) = D\boldsymbol{\Gamma}(z) = \begin{bmatrix} x'(z) \\ y'(z) \\ 1 \\ \theta'(z) \end{bmatrix} \in \mathbb{R}^4$$

另有

$$\boldsymbol{F}\left(z, \begin{bmatrix} x(z) \\ y(z) \\ \theta(z) \end{bmatrix}\right) = \mathbf{0} \in \mathbb{R}^3, \quad \forall z \in B_\lambda(z_0)$$

则有

$$D_z\boldsymbol{F}\left(z, \begin{bmatrix} x(z) \\ y(z) \\ \theta(z) \end{bmatrix}\right) + D_{\left[\begin{smallmatrix} x \\ y \\ \theta \end{smallmatrix}\right]}\boldsymbol{F}\left(z, \begin{bmatrix} x(z) \\ y(z) \\ \theta(z) \end{bmatrix}\right)\begin{bmatrix} x'(z) \\ y'(z) \\ \theta'(z) \end{bmatrix} = \mathbf{0} \in \mathbb{R}^3$$

即有

$$\begin{bmatrix} \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial z} \end{bmatrix}(x(z), y(z), z, \theta(z)) + \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial \theta} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial \theta} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial \theta} \end{bmatrix}(x(z), y(z), z, \theta(z))\begin{bmatrix} x'(z) \\ y'(z) \\ \theta'(z) \end{bmatrix} = \mathbf{0}$$

所以

$$\begin{aligned} \begin{bmatrix} x'(z) \\ y'(z) \\ \theta'(z) \end{bmatrix} &= -\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial \theta} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial \theta} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial \theta} \end{bmatrix}^{-1}(x(z), y(z), z, \theta(z))\begin{bmatrix} \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial z} \end{bmatrix}(x(z), y(z), z, \theta(z)) \\ &= -\frac{1}{\dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}(x(z), y(z), z, \theta(z))}\begin{bmatrix} \dfrac{\partial(f,g,h)}{\partial(z,y,\theta)} \\ \dfrac{\partial(f,g,h)}{\partial(x,z,\theta)} \\ \dfrac{\partial(f,g,h)}{\partial(x,y,z)} \end{bmatrix}(x(z), y(z), z, \theta(z)) \end{aligned}$$

由此可得切向量的表示

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}z}(z)=\begin{bmatrix}-\dfrac{\dfrac{\partial(f,g,h)}{\partial(z,y,\theta)}}{\dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}}(x(z),y(z),z,\theta(z))\\ -\dfrac{\dfrac{\partial(f,g,h)}{\partial(x,z,\theta)}}{\dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}}(x(z),y(z),z,\theta(z))\\ 1\\ -\dfrac{\dfrac{\partial(f,g,h)}{\partial(x,y,z)}}{\dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}}(x(z),y(z),z,\theta(z))\end{bmatrix}=\begin{bmatrix}-\dfrac{\partial(f,g,h)}{\partial(z,y,\theta)}\\ -\dfrac{\partial(f,g,h)}{\partial(x,z,\theta)}\\ \dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}\\ -\dfrac{\partial(f,g,h)}{\partial(x,y,z)}\end{bmatrix}(x(z),y(z),z,\theta(z))$$

$$=\begin{bmatrix}\dfrac{\partial(f,g,h)}{\partial(y,z,\theta)}\\ -\dfrac{\partial(f,g,h)}{\partial(x,z,\theta)}\\ \dfrac{\partial(f,g,h)}{\partial(x,y,\theta)}\\ -\dfrac{\partial(f,g,h)}{\partial(x,y,z)}\end{bmatrix}(x(z),y(z),z,\theta(z)),\quad \forall z\in B_\lambda(z_0)$$

事例 3 ($\mathbb{R}^p$ 中曲线（隐式表示形式）). $\mathbb{R}^p$ 中的曲线可以表示为

$$\Gamma=\left\{\boldsymbol{X}\in\mathbb{R}^p\,|\,\boldsymbol{f}(\boldsymbol{X})=\boldsymbol{0}\in\mathbb{R}^{p-1}\right\}$$

设有 $\boldsymbol{X}_0\in\Gamma$，亦即 $\boldsymbol{f}(\boldsymbol{X}_0)=\begin{bmatrix}f^1\\ \vdots\\ f^{p-1}\end{bmatrix}(\boldsymbol{X}_0)=\boldsymbol{0}\in\mathbb{R}^{p-1}$，且有 $\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^p)}(\boldsymbol{X}_0)\neq 0$，则可构造

$$\boldsymbol{F}\left(X^\alpha,\begin{bmatrix}X^1\\ \vdots\\ \overset{\circ}{X^\alpha}\\ \vdots\\ X^p\end{bmatrix}\right)=\boldsymbol{f}(\boldsymbol{X})\in\mathbb{R}^{p-1}$$

满足

$$\begin{cases} \boldsymbol{F}\left(X_0^\alpha, \begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^p \end{bmatrix}\right) = \boldsymbol{f}(\boldsymbol{X}_0) = \boldsymbol{0} \in \mathbb{R}^{p-1} \\ D_{\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^p \end{bmatrix}} \boldsymbol{F}\left(X_0^\alpha, \begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^p \end{bmatrix}\right) = \dfrac{D(f^1, \cdots, f^{p-1})}{D(X^1, \cdots, \overset{\circ}{X^\alpha}, \cdots, X^p)}(\boldsymbol{X}_0) \in \mathbb{R}^{(p-1)\times(p-1)} \text{ 非奇异} \end{cases}$$

则有 $\exists B_\lambda(X_0^\alpha) \subset \mathbb{R}, \exists B_\mu\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^p \end{bmatrix}\right) \subset \mathbb{R}^{p-1}$，对 $\forall X_0^\alpha \in B_\lambda(X_0^\alpha), \exists \boldsymbol{\xi}(X^\alpha) = \begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^p \end{bmatrix}(X^\alpha) \in$ $B_\mu\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^p \end{bmatrix}\right)$ 满足

$$\boldsymbol{F}\left(X^\alpha, \boldsymbol{\xi}(X^\alpha)\right) = \boldsymbol{0} \in \mathbb{R}^{p-1}$$

亦即有 $\begin{bmatrix} X^1(X^\alpha) \\ \vdots \\ X^\alpha \\ \vdots \\ X^p(X^\alpha) \end{bmatrix} \in \varGamma$，上述分析的几何化可以表示为

所以曲线可以用向量值映照表示为

$$\boldsymbol{\varGamma}(X^\alpha) : B_\lambda(X_0^\alpha) \ni X^\alpha \mapsto \boldsymbol{\varGamma}(X^\alpha) = \begin{bmatrix} X^1(X^\alpha) \\ \vdots \\ X^\alpha \\ \vdots \\ X^p(X^\alpha) \end{bmatrix} \in \mathbb{R}^p$$

曲线 Γ 的切向量为

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}X^\alpha}(X^\alpha) = D\boldsymbol{\Gamma}(X^\alpha) = \begin{bmatrix} \dfrac{\mathrm{d}X^1}{\mathrm{d}X^\alpha}(X^\alpha) \\ \vdots \\ 1 \\ \vdots \\ \dfrac{\mathrm{d}X^p}{\mathrm{d}X^\alpha}(X^\alpha) \end{bmatrix} \in \mathbb{R}^4$$

另有

$$\boldsymbol{F}\left(X^\alpha, \boldsymbol{\xi}(X^\alpha)\right) = \boldsymbol{F}\left(X^\alpha, \begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^p \end{bmatrix}(X^\alpha)\right) = \mathbf{0} \in \mathbb{R}^{p-1}, \quad \forall X^\alpha \in B_\lambda(X_0^\alpha)$$

则有

$$D_{X^\alpha}\boldsymbol{F}\left(X^\alpha, \begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^p \end{bmatrix}(X^\alpha)\right) + D_{\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^p \end{bmatrix}}\boldsymbol{F}\left(X^\alpha, \begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^p \end{bmatrix}(X^\alpha)\right)\begin{bmatrix} \dfrac{\mathrm{d}X^1}{\mathrm{d}X^\alpha} \\ \vdots \\ \dfrac{\mathrm{d}X^{\alpha-1}}{\mathrm{d}X^\alpha} \\ \dfrac{\mathrm{d}X^{\alpha+1}}{\mathrm{d}X^\alpha} \\ \vdots \\ \dfrac{\mathrm{d}X^p}{\mathrm{d}X^\alpha} \end{bmatrix}(X^\alpha) = \mathbf{0} \in \mathbb{R}^{p-1}$$

所以

$$\begin{bmatrix} \dfrac{\mathrm{d}X^1}{\mathrm{d}X^\alpha} \\ \vdots \\ \dfrac{\mathrm{d}X^{\alpha-1}}{\mathrm{d}X^\alpha} \\ \dfrac{\mathrm{d}X^{\alpha+1}}{\mathrm{d}X^\alpha} \\ \vdots \\ \dfrac{\mathrm{d}X^p}{\mathrm{d}X^\alpha} \end{bmatrix}(X^\alpha) = -\begin{bmatrix} \dfrac{\partial f^1}{\partial X^1} & \cdots & \dfrac{\partial f^1}{\partial X^{\alpha-1}} & \dfrac{\partial f^1}{\partial X^{\alpha+1}} & \cdots & \dfrac{\partial f^1}{\partial X^p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f^{p-1}}{\partial X^1} & \cdots & \dfrac{\partial f^{p-1}}{\partial X^{\alpha-1}} & \dfrac{\partial f^{p-1}}{\partial X^{\alpha+1}} & \cdots & \dfrac{\partial f^{p-1}}{\partial X^p} \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{\partial f^1}{\partial X^\alpha} \\ \vdots \\ \dfrac{\partial f^{p-1}}{\partial X^\alpha} \end{bmatrix}$$

$$
= -\frac{1}{\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^p)}}
\begin{bmatrix}
(-1)^{\alpha-2}\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(\overset{\circ}{X^1},\cdots,X^p)} \\
\vdots \\
(-1)^0\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^{\alpha-1}},\cdots,X^p)} \\
(-1)^0\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^{\alpha+1}},\cdots,X^p)} \\
\vdots \\
(-1)^{p-\alpha-1}\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^p})}
\end{bmatrix}
$$

由此可得切向量的表示

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}X^\alpha}(X^\alpha) \parallel
&\begin{bmatrix}
(-1)^{\alpha-2}\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(\overset{\circ}{X^1},\cdots,X^p)} \\
\vdots \\
(-1)^0\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^{\alpha-1}},\cdots,X^p)} \\
(-1)^1\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^{\alpha}},\cdots,X^p)} \\
(-1)^0\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^{\alpha+1}},\cdots,X^p)} \\
\vdots \\
(-1)^{p-\alpha+1}\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^p})}
\end{bmatrix}
\left(\begin{bmatrix} X^1(X^\alpha) \\ \vdots \\ X^\alpha \\ \vdots \\ X^p(X^\alpha) \end{bmatrix}\right) \\
\parallel
&\begin{bmatrix}
(-1)^0\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(\overset{\circ}{X^1},\cdots,X^p)} \\
(-1)^1\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\overset{\circ}{X^2},\cdots,X^p)} \\
\vdots \\
(-1)^{p-1}\dfrac{\partial(f^1,\cdots,f^{p-1})}{\partial(X^1,\cdots,\overset{\circ}{X^p})}
\end{bmatrix}
\left(\begin{bmatrix} X^1(X^\alpha) \\ \vdots \\ X^\alpha \\ \vdots \\ X^p(X^\alpha) \end{bmatrix}\right) \in \mathbb{R}^p
\end{aligned}
$$

2.2 曲面的隐式表示

事例 4 ($\mathbb{R}^3$ 中的 2 维曲面)**.** $\mathbb{R}^3$ 中的 2 维曲面可以表示为

$$
\Sigma = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, f(x,y,z) = 0 \in \mathbb{R} \right\}
$$

设有 $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \in \Sigma$，亦即有 $f(x_0,y_0,z_0)=0$，且有 $\dfrac{\partial f}{\partial y}(x_0,y_0,z_0)\neq 0$，则构造

$$F\left(\begin{bmatrix} x \\ z \end{bmatrix}, y\right) = f(x,y,z) \in \mathbb{R}$$

满足

$$\begin{cases} F\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, y_0\right) = f(x_0,y_0,z_0) = 0 \\ D_y F\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, y_0\right) = \dfrac{\partial f}{\partial y}(x_0,y_0,z_0) \neq 0 \end{cases}$$

则按隐映照定理，$\exists B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right) \subset \mathbb{R}^2$, $\exists B_\mu(y_0) \subset \mathbb{R}$，对 $\forall \begin{bmatrix} x \\ z \end{bmatrix} \in B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right)$, $\exists!\, \xi(x,z) \in B_\mu(y_0) \subset \mathbb{R}$，满足

$$F\left(\begin{bmatrix} x \\ z \end{bmatrix}, \xi(x,z)\right) = f\left(x, \xi(x,z), z\right) = 0$$

此结论几何化，有

故有

$$\boldsymbol{\Sigma}\left(\begin{bmatrix} x \\ z \end{bmatrix}\right) : B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right) \ni \begin{bmatrix} x \\ z \end{bmatrix} \mapsto \boldsymbol{\Sigma}\left(\begin{bmatrix} x \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ \xi(x,z) \\ z \end{bmatrix} \in \mathbb{R}^3$$

进一步可计算

$$D\boldsymbol{\Sigma}\left(\begin{bmatrix} x \\ z \end{bmatrix}\right) = \left[\frac{\partial \boldsymbol{\Sigma}}{\partial x}, \frac{\partial \boldsymbol{\Sigma}}{\partial z}\right](x,z) = \begin{bmatrix} 1 & 0 \\ \dfrac{\partial \xi}{\partial x}(x,z) & \dfrac{\partial \xi}{\partial z}(x,z) \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

由此可确定法向量的方向为

$$\left(\frac{\partial \boldsymbol{\Sigma}}{\partial x} \times \frac{\partial \boldsymbol{\Sigma}}{\partial z}\right)(x,z) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & \dfrac{\partial \xi}{\partial x}(x,z) & 0 \\ 0 & \dfrac{\partial \xi}{\partial z}(x,z) & 1 \end{vmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x}(x,z) \\ -1 \\ \dfrac{\partial \xi}{\partial z}(x,z) \end{bmatrix}, \quad \forall \begin{bmatrix} x \\ z \end{bmatrix} \in B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right)$$

再确定隐函数的偏导数 $\dfrac{\partial \xi}{\partial x}(x,z)$ 和 $\dfrac{\partial \xi}{\partial z}(x,z)$。由

$$F\left(\begin{bmatrix} x \\ z \end{bmatrix}, \xi(x,z)\right) = 0$$

可得

$$D_{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]} F\left(\begin{bmatrix} x \\ z \end{bmatrix}, \xi(x,z)\right) + D_y F\left(\begin{bmatrix} x \\ z \end{bmatrix}, \xi(x,z)\right) D_{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]} \xi(x,z) = 0 \in \mathbb{R}^{1\times 2}$$

即有

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}\right](x, \xi(x,z), z) + \frac{\partial f}{\partial y}(x, \xi(x,z), z)\left[\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial z}\right](x,z) = 0 \in \mathbb{R}^{1\times 2}$$

所以

$$\begin{cases} \dfrac{\partial \xi}{\partial x}(x,z) = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z) \\ \dfrac{\partial \xi}{\partial z}(x,z) = -\dfrac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z) \end{cases}$$

故有

$$\boldsymbol{n} \parallel \left(\frac{\partial \boldsymbol{\Sigma}}{\partial x} \times \frac{\partial \boldsymbol{\Sigma}}{\partial z}\right)(x,z) = (-1)\begin{bmatrix} \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \\ 1 \\ \frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} \end{bmatrix}(x,\xi(x,z),z) \parallel \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix}(x,\xi(x,z),z)$$

现可得 $\begin{bmatrix} x_0 \\ \xi(x_0,z_0) \\ z_0 \end{bmatrix} \in \Sigma$ 点的切平面方程为

$$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ \xi(x_0,z_0) \\ z_0 \end{bmatrix}, \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix}(x,\xi(x,z),z)\right)_{\mathbb{R}^3} = 0$$

亦即对 $\forall \begin{bmatrix} x \\ z \end{bmatrix} \in B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right)$ 有

$$(x-x_0)\frac{\partial f}{\partial x}(x_0,\xi(x_0,z_0),z_0)+(y-\xi(x_0,z_0))\frac{\partial f}{\partial y}(x_0,\xi(x_0,z_0),z_0)+(z-z_0)\frac{\partial f}{\partial z}(x_0,\xi(x_0,z_0),z_0) = 0$$

即点 $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \in \Sigma$ 的切平面方程为

$$(x-x_0)\frac{\partial f}{\partial x}(x_0,y_0,z_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0,z_0) + (z-z_0)\frac{\partial f}{\partial z}(x_0,y_0,z_0) = 0$$

按上述分析，集合 $\Sigma \subset \mathbb{R}^3$ 在局部对应为 $\mathbb{R}^2$ 至 $\mathbb{R}^3$ 的映照，亦即 Σ 在局部为曲面，且可获得其 Monge 型表示形式。进一步可以获得相应各点的切平面方程。

注：关于法向量的获得，也可基于

$$(D\boldsymbol{\Sigma})^T(x,z)\,\boldsymbol{n} = \boldsymbol{0} \in \mathbb{R}^2$$

即为

$$\begin{bmatrix} 1 & -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z) & 0 \\ 0 & -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z) & 1 \end{bmatrix}\begin{bmatrix} n^1 \\ n^2 \\ n^3 \end{bmatrix} = \boldsymbol{0} \in \mathbb{R}^2$$

所以有

$$\begin{bmatrix} 1 & 0 & -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z) \\ 0 & 1 & -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z) \end{bmatrix} \begin{bmatrix} n^1 \\ n^3 \\ n^2 \end{bmatrix} = \mathbf{0} \in \mathbb{R}^2$$

因此

$$\begin{cases} n^1 = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z)n^2 \\ n^3 = \frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z),z)n^2 \end{cases}$$

故有

$$\boldsymbol{n} = \begin{bmatrix} n^1 \\ n^2 \\ n^3 \end{bmatrix} \parallel n^2 \begin{bmatrix} \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \\ 1 \\ \frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} \end{bmatrix} (x,\xi(x,z),z) \parallel \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} (x,\xi(x,z),z)$$

事例 5 ($\mathbb{R}^4$ 中的 3 维曲面). $\mathbb{R}^4$ 中的 3 维曲面可以表示为

$$\varSigma = \left\{ \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix} \in \mathbb{R}^4 \middle| f(x,y,z,\theta) = 0 \in \mathbb{R} \right\}$$

设有 $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \theta_0 \end{bmatrix} \in \varSigma$，亦即有 $f(x_0,y_0,z_0,\theta_0)=0$，且有 $\dfrac{\partial f}{\partial y}(x_0,y_0,z_0,\theta_0) \neq 0$，则构造

$$F\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}, y \right) = f(x,y,z,\theta) \in \mathbb{R}$$

满足

$$\begin{cases} F\left(\begin{bmatrix} x_0 \\ z_0 \\ \theta_0 \end{bmatrix}, y_0 \right) = f(x_0,y_0,z_0,\theta_0) = 0 \\ D_yF\left(\begin{bmatrix} x_0 \\ z_0 \\ \theta_0 \end{bmatrix}, y_0 \right) = \dfrac{\partial f}{\partial y}(x_0,y_0,z_0,\theta_0) \neq 0 \end{cases}$$

则按隐映照定理，$\exists B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \\ \theta_0 \end{bmatrix}\right) \subset \mathbb{R}^3$, $\exists B_\mu(y_0) \subset \mathbb{R}$，对 $\forall \begin{bmatrix} x \\ z \\ \theta \end{bmatrix} \in B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \\ \theta_0 \end{bmatrix}\right)$, $\exists!\, \xi(x,z,\theta) \in B_\mu(y_0) \subset \mathbb{R}$，满足

$$F\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}, \xi(x,z,\theta)\right) = f\left(x, \xi(x,z,\theta), z, \theta\right) = 0$$

此结论几何化，有

故有

$$\boldsymbol{\Sigma}\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}\right) : B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \\ \theta_0 \end{bmatrix}\right) \ni \begin{bmatrix} x \\ z \\ \theta \end{bmatrix} \mapsto \boldsymbol{\Sigma}\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}\right) = \begin{bmatrix} x \\ \xi(x,z,\theta) \\ z \\ \theta \end{bmatrix} \in \mathbb{R}^4$$

进一步可计算

$$D\boldsymbol{\Sigma}\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}\right) = \left[\frac{\partial \boldsymbol{\Sigma}}{\partial x}, \frac{\partial \boldsymbol{\Sigma}}{\partial z}, \frac{\partial \boldsymbol{\Sigma}}{\partial \theta}\right](x,z,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\partial \xi}{\partial x}(x,z,\theta) & \dfrac{\partial \xi}{\partial z}(x,z,\theta) & \dfrac{\partial \xi}{\partial \theta}(x,z,\theta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

上式中涉及隐函数的偏导数，考虑

$$F\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}, \xi(x,z,\theta)\right) = 0$$

可得

$$D_{\left[\begin{smallmatrix} x \\ z \\ \theta \end{smallmatrix}\right]} F\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}, \xi(x,z,\theta)\right) + D_y F\left(\begin{bmatrix} x \\ z \\ \theta \end{bmatrix}, \xi(x,z,\theta)\right) D_{\left[\begin{smallmatrix} x \\ z \\ \theta \end{smallmatrix}\right]} \xi(x,z,\theta) = 0 \in \mathbb{R}^{1\times 3}$$

即有

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \theta}\right]\left(x, \xi(x,z,\theta), z, \theta\right) + \frac{\partial f}{\partial y}\left(x, \xi(x,z,\theta), z, \theta\right)\left[\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial z}, \frac{\partial \xi}{\partial \theta}\right](x,z,\theta) = 0 \in \mathbb{R}^{1\times 3}$$

所以

$$\begin{cases} \dfrac{\partial \xi}{\partial x}(x,z,\theta) = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\left(x, \xi(x,z,\theta), z, \theta\right) \\ \dfrac{\partial \xi}{\partial z}(x,z,\theta) = -\dfrac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}\left(x, \xi(x,z,\theta), z, \theta\right) \\ \dfrac{\partial \xi}{\partial \theta}(x,z,\theta) = -\dfrac{\frac{\partial f}{\partial \theta}}{\frac{\partial f}{\partial y}}\left(x, \xi(x,z,\theta), z, \theta\right) \end{cases}$$

为确定法向量，有

$$(D\boldsymbol{\Sigma})^T(x,z,\theta)\,\boldsymbol{n}=\mathbf{0}\in\mathbb{R}^3$$

即为

$$\begin{bmatrix} 1 & -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta) & 0 & 0 \\ 0 & -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta) & 1 & 0 \\ 0 & -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial \theta}}(x,\xi(x,z,\theta),z,\theta) & 0 & 1 \end{bmatrix}\begin{bmatrix} n^1 \\ n^2 \\ n^3 \\ n^4 \end{bmatrix}=\mathbf{0}\in\mathbb{R}^3$$

所以有

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta) \\ 0 & 1 & 0 & -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta) \\ 0 & 0 & 1 & -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial \theta}}(x,\xi(x,z,\theta),z,\theta) \end{bmatrix}\begin{bmatrix} n^1 \\ n^3 \\ n^4 \\ n^2 \end{bmatrix}=\mathbf{0}\in\mathbb{R}^3$$

因此

$$\begin{cases} n^1=\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta)n^2 \\ n^3=\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta)n^2 \\ n^4=\frac{\frac{\partial f}{\partial \theta}}{\frac{\partial f}{\partial y}}(x,\xi(x,z,\theta),z,\theta)n^2 \end{cases}$$

故有

$$\boldsymbol{n}=\begin{bmatrix} n^1 \\ n^2 \\ n^3 \\ n^4 \end{bmatrix}\parallel\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial \theta} \end{bmatrix}(x,\xi(x,z,\theta),z,\theta)$$

现可得 $\begin{bmatrix} x_0 \\ \xi(x_0,z_0,\theta_0) \\ z_0 \\ \theta_0 \end{bmatrix}\in\Sigma$ 点的切平面方程为

$$\left(\begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix}-\begin{bmatrix} x_0 \\ \xi(x_0,z_0,\theta_0) \\ z_0 \\ \theta_0 \end{bmatrix},\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial \theta} \end{bmatrix}(x,\xi(x,z,\theta),z,\theta)\right)_{\mathbb{R}^4}=0$$

亦即对 $\forall \begin{bmatrix} x \\ z \\ \theta \end{bmatrix} \in B_\lambda\left(\begin{bmatrix} x_0 \\ z_0 \\ \theta_0 \end{bmatrix}\right)$ 有

$$\begin{aligned}&(x-x_0)\frac{\partial f}{\partial x}(x_0,\xi(x_0,z_0,\theta_0),z_0,\theta_0)+(y-\xi(x_0,z_0))\frac{\partial f}{\partial y}(x_0,\xi(x_0,z_0,\theta_0),z_0,\theta_0)\\&\quad+(z-z_0)\frac{\partial f}{\partial z}(x_0,\xi(x_0,z_0,\theta_0),z_0,\theta_0)+(\theta-\theta_0)\frac{\partial f}{\partial \theta}(x_0,\xi(x_0,z_0,\theta_0),z_0,\theta_0)=0\end{aligned}$$

即点 $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \theta_0 \end{bmatrix} \in \Sigma$ 的切平面方程为

$$\begin{aligned}&(x-x_0)\frac{\partial f}{\partial x}(x_0,y_0,z_0,\theta_0)+(y-y_0)\frac{\partial f}{\partial y}(x_0,y_0,z_0,\theta_0)+(z-z_0)\frac{\partial f}{\partial z}(x_0,y_0,z_0,\theta_0)\\&\quad+(\theta-\theta_0)\frac{\partial f}{\partial \theta}(x_0,y_0,z_0,\theta_0)=0\end{aligned}$$

事例 6 ($\mathbb{R}^{p+1}$ 中的 p 维曲面（隐式表示形式）). $\mathbb{R}^{p+1}$ 中的 p 维曲面可以表示为

$$\Sigma=\left\{\boldsymbol{X}\in\mathbb{R}^{p+1}\,\middle|\, f(\boldsymbol{X})=0\in\mathbb{R}\right\}$$

设有 $\boldsymbol{X}_0\in\Sigma$，亦即有 $f(\boldsymbol{X}_0)=0$，且有 $\dfrac{\partial f}{\partial X^\alpha}(\boldsymbol{X}_0)\neq 0$，其中 α 为 $1,\cdots,p+1$ 中某值。则可构造

$$F\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix}, X^\alpha\right)=f(X^1,\cdots,X^\alpha,\cdots,X^{p+1})\in\mathbb{R}$$

此处，$\overset{\circ}{X^\alpha}$ 表示在全部分量中去除这个分量，下同。它满足

$$\begin{cases} F\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^{p+1} \end{bmatrix}, X_0^\alpha\right)=f(X_0^1,\cdots,X_0^\alpha,\cdots,X_0^{p+1})=0 \\ D_{X^\alpha}F\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^{p+1} \end{bmatrix}, X_0^\alpha\right)=\dfrac{\partial f}{\partial X^\alpha}(X_0^1,\cdots,X_0^\alpha,\cdots,X_0^{p+1})\neq 0 \end{cases}$$

则按隐映照定理,$\exists B_\lambda\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^{p+1}\end{bmatrix}\right) \subset \mathbb{R}^p$, $\exists B_\mu(X_0^\alpha) \subset \mathbb{R}$,对 $\forall \begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1}\end{bmatrix} \in B_\lambda\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^{p+1}\end{bmatrix}\right)$,

$\exists!\,\xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1}) \in B_\mu(X_0^\alpha) \subset \mathbb{R}$，满足

$$F\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1}\end{bmatrix}, \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1})\right) = 0$$

亦即有

$$\begin{bmatrix} X^1 \\ \vdots \\ \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1}) \\ \vdots \\ X^{p+1}\end{bmatrix} \in \Sigma$$

此结论几何化，有

故有

$$\boldsymbol{\Sigma}\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1}\end{bmatrix}\right) : B_\lambda\left(\begin{bmatrix} X_0^1 \\ \vdots \\ \overset{\circ}{X_0^\alpha} \\ \vdots \\ X_0^{p+1}\end{bmatrix}\right) \ni \begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1}\end{bmatrix}$$

$$\mapsto \boldsymbol{\Sigma}\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1}\end{bmatrix}\right) = \begin{bmatrix} X^1 \\ \vdots \\ \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1}) \\ \vdots \\ X^{p+1}\end{bmatrix} \in \mathbb{R}^{p+1}$$

进一步可计算

$$D\boldsymbol{\Sigma}\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1}\end{bmatrix}\right) = \left[\frac{\partial \boldsymbol{\Sigma}}{\partial X^1},\cdots,\frac{\partial \boldsymbol{\Sigma}}{\partial X^{\alpha-1}},\frac{\partial \xi}{\partial X^{\alpha+1}},\frac{\partial \boldsymbol{\Sigma}}{\partial X^{p+1}}\right](X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1})$$

$$
= \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\dfrac{\partial \xi}{\partial X^1} & \dfrac{\partial \xi}{\partial X^2} & \cdots & \dfrac{\partial \xi}{\partial X^{\alpha-1}} & \dfrac{\partial \xi}{\partial X^{\alpha+1}} & \dfrac{\partial \xi}{\partial X^{\alpha+2}} & \cdots & \dfrac{\partial \xi}{\partial X^{p+1}} \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{bmatrix} \in \mathbb{R}^{(p+1)\times p}
$$

矩阵为 $p+1$ 行 p 列，其中第 α 行所有元素均为偏导数。考虑

$$
F\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix}, \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1})\right) = 0
$$

可得

$$
D_{\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix}} \boldsymbol{F}\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix}, \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1})\right)
$$
$$
+ D_{X^\alpha} F\left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix}, \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1})\right) D\xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1}) = \mathbf{0} \in \mathbb{R}^{1\times p}
$$

即有

$$
\frac{\partial \xi}{\partial X^\alpha}(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1}) = -\frac{\dfrac{\partial f}{\partial X^\beta}}{\dfrac{\partial f}{\partial X^\alpha}}\left(\begin{bmatrix} X^1 \\ \vdots \\ \xi(X^1,\cdots,\overset{\circ}{X^\alpha},\cdots,X^{p+1}) \\ \vdots \\ X^{p+1} \end{bmatrix}\right), \quad \beta \neq \alpha
$$

为确定法向量，有

$$(D\boldsymbol{\Sigma})^T \left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix} \right) \boldsymbol{n} = \mathbf{0} \in \mathbb{R}^p$$

即为

$$\begin{bmatrix} 1 & \cdots & 0 & \dfrac{\partial \xi}{\partial X^1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \dfrac{\partial \xi}{\partial X^{\alpha-1}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{\partial \xi}{\partial X^{\alpha+1}} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \dfrac{\partial \xi}{\partial X^{p+1}} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} n^1 \\ \vdots \\ n^\alpha \\ \vdots \\ n^{p+1} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^p$$

所以有

$$n^\beta = -\frac{\partial \xi}{\partial X^\beta} \left(\begin{bmatrix} X^1 \\ \vdots \\ \overset{\circ}{X^\alpha} \\ \vdots \\ X^{p+1} \end{bmatrix} \right) n^\alpha = \frac{\dfrac{\partial f}{\partial X^\beta}}{\dfrac{\partial f}{\partial X^\alpha}} \left(\begin{bmatrix} X^1 \\ \vdots \\ \xi(X^1, \cdots, \overset{\circ}{X^\alpha}, \cdots, X^{p+1}) \\ \vdots \\ X^{p+1} \end{bmatrix} \right) n^\alpha$$

故有

$$\boldsymbol{n} = \begin{bmatrix} n^1 \\ \vdots \\ n^\alpha \\ \vdots \\ n^{p+1} \end{bmatrix} \parallel \begin{bmatrix} \dfrac{\partial f}{\partial X^1} \\ \vdots \\ \dfrac{\partial f}{\partial X^{p+1}} \end{bmatrix} \left(\begin{bmatrix} X^1 \\ \vdots \\ \xi(X^1, \cdots, \overset{\circ}{X^\alpha}, \cdots, X^{p+1}) \\ \vdots \\ X^{p+1} \end{bmatrix} \right)$$

现可得 $\begin{bmatrix} X_0^1 \\ \vdots \\ \xi(X_0^1, \cdots, \overset{\circ}{X_0^\alpha}, \cdots, X_0^{p+1}) \\ \vdots \\ X_0^{p+1} \end{bmatrix} \in \Sigma$ 点的切平面方程为

$$\sum_{\beta \neq \alpha} \left(X^\beta - X_0^\beta \right) \frac{\partial f}{\partial X^\beta} + \left(X^\alpha - \xi(X_0^1, \cdots, \overset{\circ}{X_0^\alpha}, \cdots, X_0^{p+1}) \right) \frac{\partial f}{\partial X^\alpha} = 0.$$

3 建立路径