IOP Publishing European Journal of Physics
Eur. J. Phys. 38 (2017) 025209 (12pp) https://doi.org/10.1088/1361-6404/aa5960

Understanding the power reflection and transmission coefficients of a plane wave at a planar interface

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Received 2 November 2016, revised 23 December 2016
Accepted for publication 13 January 2017
Published 3 February 2017

Abstract
In most textbooks, after discussing the partial transmission and reflection of a plane wave at a planar interface, the power (energy) reflection and transmission coefficients are introduced by calculating the normal-to-interface components of the Poynting vectors for the incident, reflected and transmitted waves, separately. Ambiguity arises among students since, for the Poynting vector to be interpreted as the energy flux density, on the incident (reflected) side, the electric and magnetic fields involved must be the total fields, namely, the sum of incident and reflected fields, instead of the partial fields which are just the incident (reflected) fields. The interpretation of the cross product of partial fields as energy flux has not been obviously justified in most textbooks. Besides, the plane wave is actually an idealisation that is only ever found in textbooks, then what do the reflection and transmission coefficients evaluated for a plane wave really mean for a real beam of limited extent? To provide a clearer physical picture, we exemplify a light beam of finite transverse extent by a fundamental Gaussian beam and simulate its reflection and transmission at a planar interface. Due to its finite transverse extent, we can then insert the incident fields or reflected fields as total fields into the expression of the Poynting vector to evaluate the energy flux and then power reflection and transmission coefficients. We demonstrate that the power reflection and transmission coefficients of a beam of finite extent turn out to be the weighted sum of the corresponding coefficients for all constituent plane wave components that form the beam. The power reflection and transmission coefficients of a single plane wave serve, in turn, as the asymptotes for the corresponding coefficients of a light beam as its width expands infinitely.

Keywords: power reflection and transmission coefficients, a plane wave, a light beam, weighted sum, asymptote

(Some figures may appear in colour only in the online journal)

1. Introduction

A wave experiences partial transmission and partial reflection when the medium through which it travels suddenly changes. The power reflection coefficient is defined physically as the normal-to-interface component of energy flux of the reflected wave to that of the incident wave, while the power transmission coefficient describes the normal component of energy flux of the transmitted wave to that of the incident wave. In most textbooks [1–11], for simplicity, such concepts are introduced for the case where a plane wave strikes on a planar interface between two media. The energy flux is obtained by computing the normal-to-interface component of the (period-averaged) Poynting vector. To be more specific, the flows of power incident on and reflected from the interface are evaluated by the Poynting vectors $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ of the incident and reflected plane waves, respectively, and the ratio of whose normal components gives the power (energy) reflection coefficient. However, this simplified choice of illustration is somewhat ambiguous among students for the following reasons. Since the plane wave is of infinite spatial extent, on the incident side of the interface, *the total fields*, based on which the Poynting vector should be computed in order to carry the meaning of energy current density, are actually a superposition of the incident and reflected fields, namely, $\mathbf{E} = \mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{r}$ and $\mathbf{H} = \mathbf{H}_\mathrm{i} + \mathbf{H}_\mathrm{r}$. It is indeed not obviously justified to take partial field, either incident electric and magnetic fields $\mathbf{E}_\mathrm{i}$ and $\mathbf{H}_\mathrm{i}$ or reflected fields $\mathbf{E}_\mathrm{r}$ and $\mathbf{H}_\mathrm{r}$, to compute the Poynting vector and assign the implication of energy flux to each individual part, because the Poynting vector has a quadratic form in field quantities [12, 13][3]. In addition, a plane wave is actually an ideal model that does not exist in the real world since it possesses infinite extent and energy. Then what do the reflection and transmission coefficients evaluated for a plane wave imply in real situation where real beams are limited in extent. Although maybe intuitively known, there is never an explicit numerical demonstration to answer this question.

In this paper, we develop a clear physical understanding by studying the reflection and transmission at a planar interface of a more realistic but still easily tractable model, a fundamental two-dimensional (2D) Gaussian beam. The finite extent of the light beam in space enables the spatial separation of the incident and reflected waves in the regime far enough away from the planar interface. One can therefore compute the Poynting vectors in terms of *the total fields*, $\mathbf{E}$ and $\mathbf{H}$, on two sides of the interface normal, which reduce indeed to the fields of the incident and reflected waves, respectively. By doing so, implication of the Poynting vector as energy current density is truly justified. We then integrate the normal components of the Poynting vectors on both sides of the normal to obtain the total energy fluxes transporting towards and reflected from the interface, the ratio of which defines the power reflection coefficient of a light beam. On the transmitted side, since the refracted wave itself represents the total field, the integration of the normal component of the Poynting vector

[3] In [12, 13], it is demonstrated that the normal component of the Poynting vector evaluated based on the total fields is continuous across a planar interface between two isotropic lossless media, which, together with a proof that the nomral component of the mixed term $\mathbf{E}_\mathrm{i} \times \mathbf{H}_\mathrm{r} + \mathbf{E}_\mathrm{r} \times \mathbf{H}_\mathrm{i}$ vanishes, implies that the Poynting vector of the reflected fields from a planar interface can be understood as the energy flux of the reflected wave. We are obliged to one of the anonymous referees for pointing out this point.

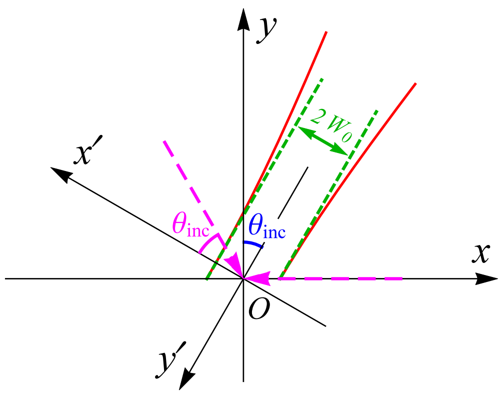

Figure 1. Schematic plot of a two-dimensional Gaussian beam with focal width $2W_0$ incident on a planar interface depicted by $y = 0$ at an incident angle θ_{inc}. The beam is focused on the origin and propagates in direction y'. Two magenta dashed arrows show the range of wave vectors used to describe the beam such that the plane wave components forming the beam with upwards wave vectors are all neglected. See text for more details.

produces naturally energy flux transmitted through the interface, the ratio of which to the total incident energy flux characterises the power transmission coefficient of the beam. We demonstrate that the power reflection and transmission coefficients of a beam of finite transverse spatial extent turn out to be the weighted sum of the corresponding coefficients for the each constituent plane wave component that makes up the beam. On the other hand, by increasing the waist width of the beam, it is found that the beam power reflection and transmission coefficients approach asymptotically to 'the power reflection and transmission coefficients' of a single plane wave that are evaluated based on the procedure in the standard textbooks [1–11], implying that the latter describes actually the energy reflection and transport ratios of a light beam in the limit of infinite beam width. As the beam widths W_0 in usual experiments are typically greater than dozens of microns, while the operating wavelength is only of order of $W_0/100$, the reflection and transmission coefficients of a single plane wave serve as a good approximation to those for the real light beam.

2. The power reflection and transmission coefficients of a light beam

For greatest simplicity, let us consider a 2D transverse electric Gaussian beam with its electric field $\mathbf{E}$ normal to the plane of incidence. Generalisation to three dimensions as well as to the case with $\mathbf{E}$ parallel to the plane of incidence is straightforward. Let the beam of waist width $2W_0$ be focused at the origin of the coordinate system and propagate in direction y', as schematically shown in figure 1. The planar interface is located at $y = 0$ and the incident angle θ_{inc} depicts the angle between the beam propagation direction (y'-axis) and the interface normal (y-axis). The E field polarised along z of such a 2D beam reads [14]

$$E_{\text{inc}}(x', y') = \frac{k_0 W_0}{2\sqrt{\pi}} \int_{-1}^{+1} \exp\left(-\frac{k_0^2 W_0^2 \alpha'^2}{4}\right) \exp[\mathrm{i}k_0(\alpha' x' + \beta' y')]\,\mathrm{d}\alpha', \qquad (1)$$

where the wave number $k_0 = 2\pi/\lambda$ with λ being the wavelength in free space, $2W_0$ is the waist width, and $\beta' = \sqrt{1 - \alpha'^2}$. Here we have excluded the evanescent wave components with $|\alpha'| > 1$, which is well justified for loosely focused beam with $W_0 \geqslant 2\lambda$. Equation (1) actually expresses a beam as a superposition of a series of homogeneous plane waves, which is known as the angular spectrum representation of optical field in optics [15, 16], except that we have excluded the evanescent wave components for simplicity. Based on the transformation between two coordinates (x', y') and (x, y),

$$\begin{cases} x' = y \sin\theta_{\text{inc}} - x \cos\theta_{\text{inc}}, \\ y' = -x \sin\theta_{\text{inc}} - y \cos\theta_{\text{inc}}, \end{cases} \tag{2}$$

the incident E field (1) can be written as

$$E^{\text{in}}(x, y) = \frac{k_0 W_0}{2\sqrt{\pi}} \int_{\alpha_{\min}}^{\alpha_{\max}} \exp\left(-\frac{k_0^2 W_0^2 \alpha'^2}{4}\right) \times \exp[-\text{i}k_0(\alpha x + \beta y)] \frac{\cos\theta'}{\cos(\theta' + \theta_{\text{inc}})} \, \text{d}\alpha, \tag{3}$$

where

$$\begin{aligned} \alpha &= \alpha' \cos\theta_{\text{inc}} + \beta' \sin\theta_{\text{inc}} = \sin(\theta' + \theta_{\text{inc}}), \\ \beta &= \beta' \cos\theta_{\text{inc}} - \alpha' \sin\theta_{\text{inc}} = \cos(\theta' + \theta_{\text{inc}}), \end{aligned} \tag{4}$$

with

$$\begin{cases} \alpha' = \sin\theta', \\ \beta' = \cos\theta', \end{cases} \quad \text{and} \quad \begin{cases} \alpha = \sin\theta, \\ \beta = \cos\theta. \end{cases} \tag{5}$$

The upper and lower bounds of integration are re-set to $\alpha_{\max} = 1$ and $\alpha_{\min} = -\cos(2\theta_{\text{inc}})$ to guarantee that all the plane wave components constituting the incident beam propagate downward and distribute symmetrically with respect to the y'-axis, namely, with the directions of their wave vectors lying between the regime bounded by the two magenta dashed arrows shown in figure 1.

Next we further approximate the integral (3) by a summation over discrete wave vectors. This is done by simply casting the integral in equation (3) into summation

$$E^{\text{in}}(x, y) = \frac{k_0 W_0}{2\sqrt{\pi}} \sum_{j=1}^{M} \exp\left(-\frac{k_0^2 W_0^2 \alpha_j'^2}{4}\right) \exp[-\text{i}k_0(\alpha_j x + \beta_j y)] \times \frac{\cos\theta_j'}{\cos(\theta_j' + \theta_{\text{inc}})} \delta, \tag{6}$$

where $\delta = \frac{(\alpha_{\max} - \alpha_{\min})}{M}$, $\alpha_j = \alpha_{\min} + \left(j - \frac{1}{2}\right)\delta = \sin\theta_j$, $\beta_j = \sqrt{1 - \alpha_j^2}$, $\alpha_j' = \sin\theta_j'$, and $\theta_j' = \theta_j - \theta_{\text{inc}}$. So we have written a beam as a superposition of a finite number M of plane waves

$$E^{\text{in}}(x, y) = \sum_{j=1}^{M} E_j^{\text{in}}, \qquad E_j^{\text{in}} = E_{0j}\, \text{e}^{-\text{i}k_0(\alpha_j x + \beta_j y)}, \tag{7a}$$

each of which has the amplitude E_{0j} given by

$$E_{0j} = \frac{k_0 W_0}{2\sqrt{\pi}} \exp\left(-\frac{k_0^2 W_0^2 \alpha_j'^2}{4}\right) \frac{\cos\theta_j'}{\cos(\theta_j' + \theta_{\text{inc}})} \delta. \tag{7b}$$

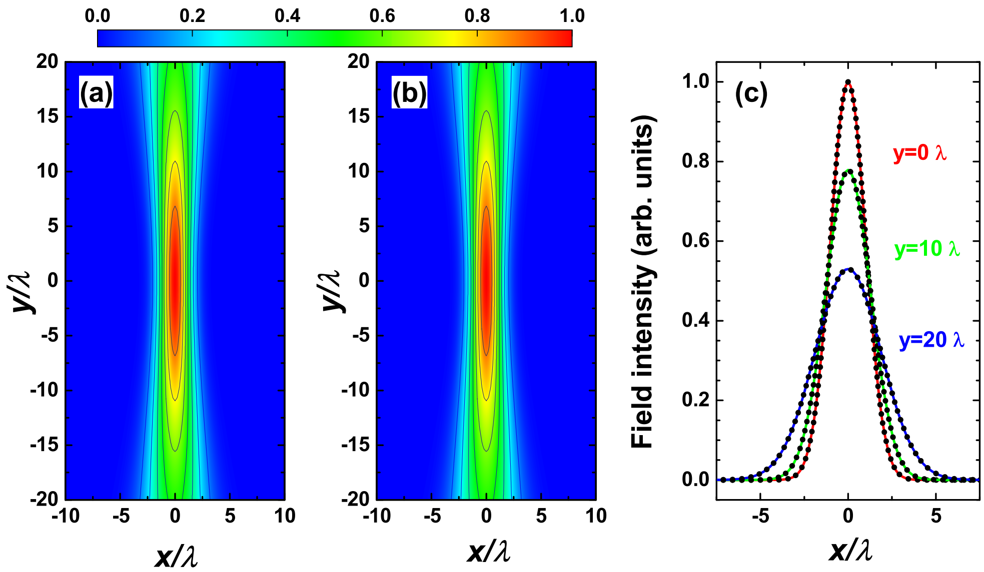

Figure 2. (a) The contour plot of the electric field intensity of the standard Gaussian beam depicted by equation (1). (b) The field intensity profile of a superposition of 100 plane waves used to mimic the beam. (c) The electric field intensity as a function of transverse coordinates x at different longitudinal position y. The solid and dotted lines denote, respectively, the field intensity of the standard Gaussian beam given by equation (1) and that formed by a superposition of 100 plane waves given by (7*a*) and (7*b*).

Figure 2 shows the results for mimicking a 2D Gaussian beam of waist width $2W_0 = 4\lambda$ propagating in free-space by $M = 100$ plane waves given by equations (7*a*) and (7*b*), confirming a quite satisfactory agreement, and, in particular, showing a field pattern confined within a finite transverse extent for our purpose of studying reflection and refraction at an interface.

Next we assume that the 2D Gaussian beam propagates along the direction at angle θ_{inc} with respect to the y axis within the x–y plane, as shown in figure 1. It experiences partial reflection and transmission when it is incident on a planar interface at $y = 0$ between, for simplicity, a free space with both relative permittivity ε_1 and permeability μ_1 equal to 1, and a dielectric with $\varepsilon_2 = 4$ and $\mu_2 = 1$, so that the refractive index $n = \sqrt{\varepsilon_2}$. With the help of the Fresnel coefficients [1–11],

$$r_j = \frac{\sin\theta_j^{\text{t}}\cos\theta_j - \cos\theta_j^{\text{t}}\sin\theta_j}{\sin\theta_j^{\text{t}}\cos\theta_j + \cos\theta_j^{\text{t}}\sin\theta_j} = \frac{\sin(\theta_j^{\text{t}} - \theta_j)}{\sin(\theta_j^{\text{t}} + \theta_j)},$$
$$t_j = \frac{2\sin\theta_j^{\text{t}}\cos\theta_j}{\sin\theta_j^{\text{t}}\cos\theta_j + \cos\theta_j^{\text{t}}\sin\theta_j} = \frac{2\sin\theta_j^{\text{t}}\cos\theta_j}{\sin(\theta_j^{\text{t}} + \theta_j)}, \tag{8}$$

for each single plane wave depicted by the electric field given in (7*a*) and (7*b*), where the incident angle $\theta_j = \sin^{-1}\alpha_j$ and the refracted angle $\theta_j^{\text{t}} = \sin^{-1}(\alpha_j/n)$, the reflected and transmitted waves can be easily worked out to yield

$$E^{\mathrm{re}} = \sum_{j=1}^{M} E_j^{\mathrm{re}}, \qquad E_j^{\mathrm{re}} = r_j E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x - \beta_j y)},$$
$$E^{\mathrm{tr}} = \sum_{j=1}^{M} E_j^{\mathrm{tr}}, \qquad E_j^{\mathrm{tr}} = t_j E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x + \beta_j^{\mathrm{t}} y)}, \tag{9}$$

where $\beta_j^{\mathrm{t}} = \sqrt{n^2 - \alpha_j^2}$. The x and y components of the corresponding incident, reflected and transmitted magnetic fields follow the Maxwell equation

$$\mathbf{H} = \frac{\nabla \times \mathbf{E}}{\mathrm{i}\omega\mu}, \tag{10}$$

reading

$$H_x^{\mathrm{in}} = \sum_{j=1}^{M} H_{jx}^{\mathrm{in}}, \qquad H_{jx}^{\mathrm{in}} = -\frac{\beta_j}{Z_1} E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x + \beta_j y)},$$
$$H_x^{\mathrm{re}} = \sum_{j=1}^{M} H_{jx}^{\mathrm{re}}, \qquad H_{jx}^{\mathrm{re}} = \frac{\beta_j}{Z_1} r_j E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x - \beta_j y)},$$
$$H_x^{\mathrm{tr}} = \sum_{j=1}^{M} H_{jx}^{\mathrm{tr}}, \qquad H_{jx}^{\mathrm{tr}} = -\frac{\beta_j^{\mathrm{t}}}{Z_2} t_j E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x + \beta_j^{\mathrm{t}} y)}, \tag{11a}$$

and

$$H_y^{\mathrm{in}} = \sum_{j=1}^{M} H_{jy}^{\mathrm{in}}, \qquad H_{jy}^{\mathrm{in}} = -\frac{\alpha_j}{Z_1} E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x + \beta_j y)},$$
$$H_y^{\mathrm{re}} = \sum_{j=1}^{M} H_{jy}^{\mathrm{re}}, \qquad H_{jy}^{\mathrm{re}} = -\frac{\alpha_j}{Z_1} r_j E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x - \beta_j y)},$$
$$H_y^{\mathrm{tr}} = \sum_{j=1}^{M} H_{jy}^{\mathrm{tr}}, \qquad H_{jy}^{\mathrm{tr}} = -\frac{\alpha_j}{Z_2} t_j E_{0j}\, \mathrm{e}^{-\mathrm{i}k_0(\alpha_j x + \beta_j^{\mathrm{t}} y)}, \tag{11b}$$

where $Z_1 = \sqrt{\frac{\mu_1\mu_0}{\varepsilon_1\varepsilon_0}}$ and $Z_2 = \sqrt{\frac{\mu_2\mu_0}{\varepsilon_2\varepsilon_0}}$ are, respectively, the wave impedances in media 1 and 2, while ε_0 and μ_0 are the permittivity and permeability in free-space.

The energy current densities in direction normal to the interface are then determined by normal components of the Poynting vectors computed from *the total fields*

$$S_1 = \frac{1}{2}\,\mathrm{Re}[(E^{\mathrm{in}} + E^{\mathrm{re}})(H_x^{\mathrm{in}\,*} + H_x^{\mathrm{re}\,*})],$$
$$S_2 = \frac{1}{2}\,\mathrm{Re}(E^{\mathrm{tr}} H_x^{\mathrm{tr}\,*}), \tag{12a}$$

while the components parallel to the interface (x components) are

$$S_1' = -\frac{1}{2}\,\mathrm{Re}[(E^{\mathrm{in}} + E^{\mathrm{re}})(H_y^{\mathrm{in}\,*} + H_y^{\mathrm{re}\,*})],$$
$$S_2' = -\frac{1}{2}\,\mathrm{Re}(E^{\mathrm{tr}} H_y^{\mathrm{tr}\,*}), \tag{12b}$$

where the superscript $*$ represents the complex conjugate and the subscripts 1 and 2 denote, respectively, the incident and transmitted sides of the interface. For comparison and following the standard textbooks [1–10], we define two quantities evaluated in terms of partial fields by

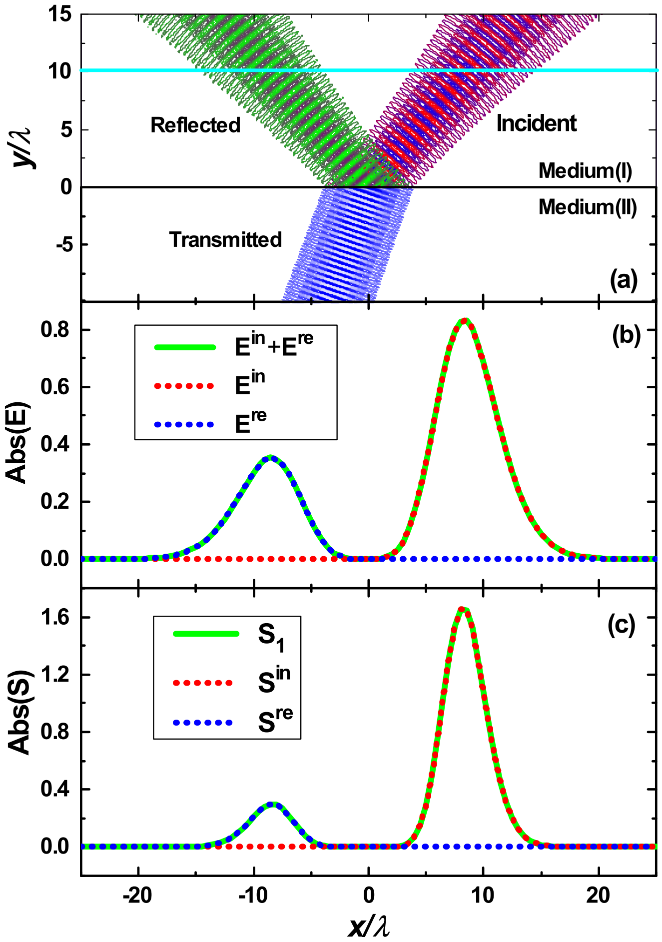

Figure 3. (a) The contour plot of a simulated Gaussian beam incident on plane interface at incident angle $\theta_{\mathrm{inc}} = 40°$. The planar interface locates at $y = 0$. The cyan line at $y = 10\ \lambda$ helps to display spatial separation of incident and reflected waves. (b) and (c) The green line, red line, and blue line denote, respectively, the absolute values of the amplitude of E field (b) and the y component of Poynting vector (c) of the total, incident and reflected waves at $y = 10\ \lambda$ denoted by the horizontal cyan line in panel (a). The complete separation in space of incident and reflected waves shows $S_{\mathrm{I}} = S^{\mathrm{in}}$ ($S_{\mathrm{I}} = S^{\mathrm{re}}$) for $x > 0$ ($x < 0$), and confirms the validity to calculate the incident and reflected power in terms of the total fields.

$$S^{\mathrm{in}} = \frac{1}{2}\ \mathrm{Re}(E^{\mathrm{in}} H_x^{\mathrm{in}\ *}),$$
$$S^{\mathrm{re}} = \frac{1}{2}\ \mathrm{Re}(E^{\mathrm{re}} H_x^{\mathrm{re}\ *}). \tag{13}$$

It is remarked that the implication of S^{in} and S^{re} as energy flux density is not justified by its definition.

Figure 3 shows typically the results for a beam of waist width $2W_0 = 4\lambda$ incident on the interface $y = 0$ at angle $\theta_{\mathrm{inc}} = 40°$. The incident field E^{in}, reflected field E^{re} and transmitted field E^{tr} are exhibited in figure 3(a). Different from a plane wave incidence, the reflected beam and the incident Gaussian beam, although both appear in the incident side of the interface, are spatially well separated from each other at a distance away from the interface, as denoted by the horizontal cyan line. Figure 3(b) displays the absolute values of the electric field amplitude along x parallel to the interface at $y = 10\lambda$, marked by the cyan line in figure 3(a). The green, red, and blue curves denote the total, incident and reflected fields, respectively. The spatial separation of the incident and reflected fields is manifest. On one side ($x > 0$) of the interface normal, the total field reduces completely to incident field, while on the other side of the normal ($x < 0$), the total field is solely determined by the reflected field. In figure 3(c), S_1, S^{in} and S^{re}, defined in (12*a*) and (13), are presented as a function of x at $y = 10\lambda$. It is observed that $S_1 = S^{\text{in}}$ and $S_1 = S^{\text{re}}$ for $x > 0$ and $x < 0$, respectively, justifying the physical implication of S^{in} and S^{re} as energy flux density if they are evaluated at a distance far enough away from the interface for an incidence of light beam with finite transverse spatial extent.

Now we turn to the power reflection and transmission coefficients of a light beam, which are determined in our computation as follows. Define

$$P^{\text{in}} = -\int_0^{+\infty} S_1 \mathrm{d}x, \tag{14a}$$

$$P^{\text{re}} = \int_{-\infty}^{0} S_1 \mathrm{d}x, \tag{14b}$$

$$P^{\text{tr}} = -\int_{-\infty}^{+\infty} S_2 \mathrm{d}x, \tag{14c}$$

where S_1 and S_2 are given by (12*a*), and the integration is evaluated at a fixed value of y far enough away from the interface so that $S_1 = S^{\text{in}}$ for $x > 0$ while $S_1 = S^{\text{re}}$ for $x < 0$. It can be easily understood that P^{in} characterises the energy flux incident perpendicularly downward to the interface, since its integrand S_1 is evaluated in terms of the total fields and carries the implication of energy current density, and, in addition, as the total fields turn out to be identical to the incident field, the energy flux comes solely from the incident wave. Similarly, P^{re} describes the energy flux reflected upward away from the interface, and P^{tr} is the energy flux transmitted to the second medium. All the integrands in equations (14*a*)–(14*c*) are evaluated in terms of local *total fields*, so it is fully justified that so-evaluated y components of the Poynting vectors determine the energy current densities and, consequently, their integrations yield the powers being transported. It follows naturally that the power reflection and transmission coefficients for a light beam, denoted by R_{b} and T_{b}, respectively, can be defined as

$$R_{\text{b}} = \frac{P^{\text{re}}}{P^{\text{in}}}, \qquad T_{\text{b}} = \frac{P^{\text{tr}}}{P^{\text{in}}}. \tag{15}$$

On the other hand, a light beam can be decomposed into a set of plane waves, as we have done for a fundamental Gaussian beam in equations (7*a*) and (7*b*). As each plane wave is of spatially infinite extent, in this case, one chooses to focus on the powers incident, reflected, and transmitted normally on or through the unit length at the interface, which can be evaluated at y far away from the interface by

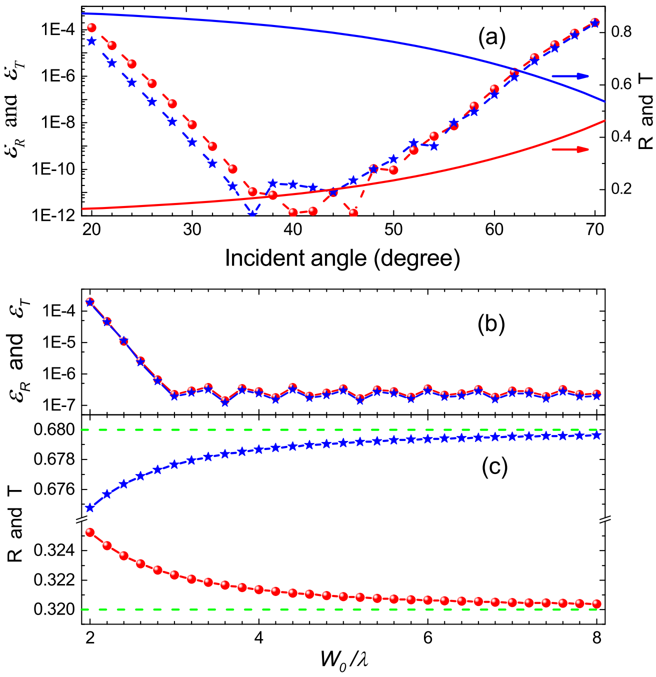

Figure 4. (a) The power reflection coefficient R (solid red line) and transmission coefficient T (solid blue line) of a 2D Gaussian beam with $W_0 = 2\lambda$. Also shown as dashed lines are the corresponding relative discrepancy between R_b and R (red) and T_b and T (blue) defined by equation (21). (b) The relative discrepancy ε_R and ε_T for incident angle $\theta_{inc} = 70°$ as a function of half waist width W_0 of 2D Gaussian beam, showing that the discrepancy decays with beam width until a plateau appears due to the residual numerical errors. (c) The power reflection coefficient R (red) and transmission coefficient T (blue) of a 2D Gaussian beam versus the beam width at $\theta_{inc} = 60°$, suggesting that R and T of a beam approach asymptotically to those of a single plane wave as the beam width tends to infinity. The horizontal dashed green lines display R and T of a single plane wave at the same incident angle θ_{inc} evaluated based on the standard procedure in most textbooks, see, e.g., [1–5].

$$
\begin{aligned}
p^{\mathrm{in}} &= -\lim_{x_m\to\infty}\frac{1}{x_m}\int_0^{x_m} S_1\,\mathrm{d}x = -\lim_{x_m\to\infty}\frac{1}{x_m}\int_0^{x_m} S^{\mathrm{in}}\mathrm{d}x \\
&= -\lim_{x_m\to\infty}\frac{1}{x_m}\int_0^{x_m}\frac{1}{2}\,\mathrm{Re}(E^{\mathrm{in}}H_x^{\mathrm{in}\,*})\mathrm{d}x \\
&= \lim_{x_m\to\infty}\frac{1}{x_m}\int_0^{x_m}\frac{1}{2Z_1}\,\mathrm{Re}\sum_{j,j'}[\beta_{j'}E_{0j}E_{0j'}^{*}\,\mathrm{e}^{-\mathrm{i}k_0(\alpha_j x+\beta_j y-\alpha_{j'}x-\beta_{j'}y)}]\,\mathrm{d}x \\
&= \frac{1}{2Z_1}\sum_j |E_{0j}|^2\beta_j = \frac{1}{2Z_1}\sum_j |E_{0j}|^2\cos\theta_j,
\end{aligned}
\tag{16}
$$

where use has been made of the fact that the mixed terms with $j \neq j'$ vanish in the limit of $x_m \to \infty$, since the integrals for such terms are finite while the denominator x_m tends to infinity. In a similarly way, one derives

$$p^{\mathrm{re}} = \lim_{x_{\mathrm{m}}\to\infty} \frac{1}{x_{\mathrm{m}}} \int_{-x_{\mathrm{m}}}^{0} S_1 \mathrm{d}x = \frac{1}{2Z_1} \sum_j |E_{0j}|^2 r_j^2 \cos\theta_j,$$
$$p^{\mathrm{tr}} = -\lim_{x_{\mathrm{m}}\to\infty} \frac{1}{2x_{\mathrm{m}}} \int_{-x_{\mathrm{m}}}^{x_{\mathrm{m}}} S_2 \mathrm{d}x = \frac{1}{2Z_2} \sum_j |E_{0j}|^2 t_j^2 \cos\theta_j^{\mathrm{t}}. \tag{17}$$

The power reflection and transmission coefficients for a beam are therefore governed also by

$$R = \frac{p^{\mathrm{re}}}{p^{\mathrm{in}}} = \sum_j w_j\, R_j,$$
$$T = \frac{p^{\mathrm{tr}}}{p^{\mathrm{in}}} = \sum_j w_j\, T_j, \tag{18}$$

where

$$R_j = r_j^2 \quad \text{and} \quad T_j = \frac{Z_1 \cos\theta_j^{\mathrm{t}}}{Z_2 \cos\theta_j}\, t_j^2 \tag{19}$$

are, respectively, the power reflection and transmission coefficients for a single plane wave introduced in standard textbooks [1–11], and r_j, t_j are Fresnel coefficients given in equation (8), the weight w_j is given by

$$w_j = \frac{|E_{0j}|^2 \cos\theta_j}{\sum_j |E_{0j}|^2 \cos\theta_j}. \tag{20}$$

It is therefore concluded that the power reflection and transmission coefficients of a beam with finite transverse extent are the weighted sum of the corresponding coefficients of the constituent plane waves that make up the beam.

We have calculated R_{b} and T_{b} given by (15) as well as R and T given by (18) for 2D Gaussian beams with different beam widths. Typical results are shown in figure 4(a) for the case with $W_0 = 2\lambda$. The results for R_{b} and R (also for T_{b} and T) as a function of incident angle θ_{inc} are graphically indiscernible, as can be seen from the relative discrepancy ε_{R} and ε_{T} defined by

$$\varepsilon_{\mathrm{R}} = \frac{|R_{\mathrm{b}} - R|}{R}, \qquad \varepsilon_{\mathrm{T}} = \frac{|T_{\mathrm{b}} - T|}{T}. \tag{21}$$

The discrepancy between R_{b} and R (and also between T_{b} and T) originates from the numerical integration as well as the approximation of the Gaussian beam by a discrete set of plane waves, in which the residual fields away from the beam axis remain oscillating around zero rather than decay exponentially as a standard Gaussian beam should do. The greater discrepancy at smaller incident angle θ_{inc} arises from the larger residual fields near $x = 0$ since the incident and reflected waves come closer in this case, resulting in a somewhat discernible overlap effect. The more serious inconsistency at larger incident angle θ_{inc} comes similarly from the overlap effect between the incident and reflected waves due to larger oblique angle. Besides, it stems also from the fact that, in our calculation, we have removed some plane wave components that propagate upward from the plane wave spectrum of the incident beam. For a beam with greater beam width, the discrepancy can be decreased, since a wider beam has all its constituent plane waves more bent to the direction of beam propagation and thus our simulation by omitting upward wave vectors generates more accurate results. Figure 4(b) evidences this tendency, where the discrepancy decreases with the beam width is exhibited. And finally, in figure 4(c) we show a typical example for the power reflection and transmission coefficients of the 2D Gaussian beam as a function of beam width. It is manifest that as the beam width increases, the power reflection and transmission coefficients approach to those computed using method introduced in standard textbooks [1–10] for a single plane wave at the same incident angle θ_{inc}. As a result, the power reflection and transmission coefficients of a single plane wave serve actually as asymptotes for the corresponding coefficients of a light beam as the beam width is expanded infinitely.

3. Summary

To summarise, the Poynting vector is, in general, interpreted as energy current density only when it is evaluated in terms of *the total fields*. This causes some confusion among students when discussing the power reflection and transmission coefficients of a single plane wave at a planar interface. Because the Poynting vectors therein are evaluated in terms of partial fields, namely, incident and reflected fields, respectively, instead of *the total fields*, they may not be fully justified as energy current densities and the incident and reflected powers based on so obtained Poynting vectors appear confusing. Besides, as a plane wave is actually an ideal model that does not exist in the real world. A question follows naturally about the physical implication of the reflection and transmission coefficients of a plane wave. We propose to resolve this ambiguity and provide a clear physical meaning by computing the power reflection and transmission coefficients of a light beam with limited extent, taking a 2D Gaussian beam as an example. Due to the finite transverse spatial extent of a light beam, on both sides of the interface normal, the total fields reduce, respectively, to those of the incident and reflected waves, and the physical implication of the Poynting vector calculated based on these two individual parts is thus truly justified. Consequently, the energy fluxes for the incident, reflected, and transmitted waves can be unambiguously defined. It is demonstrated that the power reflection and transmission coefficients of a beam turn out to be the weighted sum of the corresponding coefficients of all the constituent plane waves making up the beam. The power reflection and transmission coefficients of a single plane wave can be in turn understood physically as the asymptotes for the corresponding coefficients of a light beam when its width is expanded infinitely. They serve practically as an excellent approximation to the power reflection and transmission coefficients for a real light beam, since in practice the beam widths W_0 are typically several hundreds times of operating wavelength.

Acknowledgments

This work is supported by FDUROP, National University Student Innovation Program, and CURE grant No. 15046. HL was partly supported by Shanghai Adolescents Science and Technology Innovation Project. We thank one of the anonymous referees for pointing out an inaccurate statement in the earlier version of this paper.

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