Electromagnetic Momentum and Radiation Pressure derived from the Fresnel Relations

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Abstract: Using the Fresnel relations as axioms, we derive a generalized electromagnetic momentum for a piecewise homogeneous medium and a different generalized momentum for a medium with a spatially varying refractive index in the Wentzel–Kramers–Brillouin (WKB) limit. Both generalized momenta depend linearly on the field, but the refractive index appears to different powers due to the difference in translational symmetry. For the case of the slowly varying index, it is demonstrated that there is negligible transfer of momentum from the electromagnetic field to the material. Such a transfer occurs at the interface between the vacuum and a homogeneous material allowing us to derive the radiation pressure from the Fresnel reflection formula. The Lorentz volume force is shown to be nil.

OCIS codes: (260.2110) Electromagnetic theory; (260.2160) Energy transfer

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1. Introduction

Since the early years of relativity theory, there has been a controversy regarding the correct relativistic form of the energy–momentum tensor for an electromagnetic field in a linear medium. The energy–momentum tensor proposed by Minkowski [1] was faulted for a lack of symmetry leading Abraham [2] to suggest a symmetric form. Einstein and Laub [3], Peierls [4], Kranys [5], Livens [6], and others [7, 8] have proposed variants of the Abraham and Minkowski tensors while still other workers [9] have endorsed one or the other of the principal results. The element of the energy–momentum tensor related to the electromagnetic momentum density is the main point of contention in the Abraham–Minkowski controversy. While the issue of whether the momentum flux of an electromagnetic field is increased or decreased by the presence of a refractive medium appears to be uncomplicated, experimental measurements [10, 11, 12, 13] have been unable to conclusively identify the electromagnetic momentum density with either the Abraham or the Minkowski formula, or with any of the variant formulas [14, 15]. An inconsistency of this magnitude and persistence in the theoretical and experimental treatment of a simple physical system suggests problems of a fundamental nature.

Noether's theorem connects conservation laws to symmetries [16]. Conservation of energy is associated with invariance with respect to time translation and conservation of linear momentum requires invariance with respect to spatial translation. Because the posited formulas for the momentum density in a dispersionless medium are quadratic in the field, these quantities are, as a matter of linear algebra, either inconsistent or redundant with the electromagnetic energy [17]. In particular, the degeneracy of the energy and momentum of the electromagnetic field in the vacuum is implicated as the crux of the Abraham–Minkowski controversy.

Since there is no general spatial invariance property for dielectrics, one should ask under what conditions a quantity that behaves like momentum can be derived. We show that, by using the Fresnel boundary conditions to connect spatially invariant regions of linear media, a generalized electromagnetic momentum can be derived in the limiting cases of *i*) a piecewise homogeneous medium and *ii*) a medium with a slowly varying refractive index in the WKB limit. Both generalized momenta depend linearly on the field but the refractive index appears to different powers due to the difference in the translational symmetry. Momentum conservation is demonstrated numerically and theoretically in both limiting cases. For the case of a material with a slowly varying index, the momentum of the transmitted field is essentially equal to that of the incident field and no momentum is transferred to the material. However, a field entering a homogeneous medium from the vacuum imparts a permanent dynamic momentum to the material that is twice the momentum of the reflected field, if momentum is to be conserved.

Because the change of the momentum of the material is due to reflection, radiation pressure is deemed to be a surface force acting over the illuminated area of the material and we show that the Lorentz volume force is nil.

2. Piecewise Homogeneous Media

A piecewise homogeneous linear medium represents the special case of a material that, neglecting absorption, can be represented as a composite of finite translationally invariant spacetimes. In a previous short communication [17], we derived a conserved electromagnetic momentum for a piecewise homogeneous medium from the macroscopic effective Hamiltonian and used the result to derive the Fresnel relations. However, the derivation of an effective longitudinal momentum by averaging the dynamics of the harmonic oscillators that comprise the model of the electromagnetic field might be viewed as disconcerting. Here, we take the opposite approach and show that the Fresnel relations imply the continuity of two quantities: the electromagnetic energy flux and the flux of something else. We show that the unknown conserved electromagnetic quantity has the characteristics of a linear momentum. Because the electromagnetic momentum is only determined to within a constant of proportionality, this derivation is not as complete as the Hamiltonian-based theory. However, the Fresnel-based derivation complements the prior work [17], in which it is implicit, by providing a simple direct derivation of the electromagnetic momentum in terms of familiar continuum electrodynamic concepts that can serve as a platform for extensions of the theory.

We consider the boundary conditions at the interface of two homogeneous linear media. A plane electromagnetic wave is normally incident from a medium V_1 with refractive index n_1 into a medium V_2 with index $n_2 > n_1$, where n_1 and n_2 are real. The fields are assumed to be monochromatic plane waves polarized in the x-direction and we write

$$\mathbf{E}_i = \mathbf{e}_x E_i e^{-i(\omega t - k_i z)}$$
$$\mathbf{E}_r = \mathbf{e}_x E_r e^{-i(\omega t + k_r z)}$$
$$\mathbf{E}_t = \mathbf{e}_x E_t e^{-i(\omega t - k_t z)}$$

as the respective amplitudes of the incident, reflected, and refracted waves. If we assume the Fresnel relations

$$E_r = \frac{n_2 - n_1}{n_1 + n_2} E_i \tag{1}$$

$$E_t = \frac{2n_1}{n_1 + n_2} E_i \tag{2}$$

then equivalent Fresnel equations

$$n_1 E_i^2 = n_2 E_t^2 + n_1 E_r^2 \tag{3}$$

$$E_i = E_t + E_r \tag{4}$$

can be derived algebraically.

The Fresnel equations (3) and (4) are recognized as continuity equations in which the rate at which some electromagnetic quantity arrives at the boundary is equal to the rate at which that quantity leaves the boundary. Equation (3) expresses continuity of a flux

$$S = \gamma n |\mathbf{E}|^2 \tag{5}$$

with an undetermined constant γ. The Fresnel continuity equation (4) represents continuity of a flux

$$T = \alpha |\mathbf{E}| \tag{6}$$

that again contains an unknown constant. For a generic property, the continuity law has the form of

$$\nabla \cdot \rho \mathbf{v} = -\frac{\partial \rho}{\partial t} \tag{7}$$

where ρ is the property density and $\rho\mathbf{v}$ is the property flux vector. It is then a simple matter to derive the conservation laws that correspond to the continuous fluxes. We define flux vectors

$$\mathbf{S} = \gamma n|\mathbf{E}|^2 \mathbf{e}_z \tag{8}$$

$$\mathbf{T} = \alpha|\mathbf{E}|\mathbf{e}_z \tag{9}$$

such that $S = |\mathbf{S}|$ and $T = |\mathbf{T}|$. Denoting the respective property densities as u and g, we have

$$u = \mathbf{S}/\mathbf{v} = \frac{n}{c}\gamma n|\mathbf{E}|^2 \tag{10}$$

$$g = \mathbf{T}/\mathbf{v} = \frac{n}{c}\alpha|\mathbf{E}|. \tag{11}$$

Integrating the property densities over the appropriate volume, we obtain the conservation laws

$$\int_{V_1} n_1^2 E_i^2 dv = \int_{V_1} n_1^2 E_r^2 dv + \int_{V_2} n_2^2 E_t^2 dv \tag{12}$$

$$\int_{V_1} n_1 E_i dv = \int_{V_1} n_1 E_r dv + \int_{V_2} n_2 E_t dv. \tag{13}$$

We identify Eq. (12) as the conservation law for electromagnetic energy for a monochromatic plane wave. Equation (13) is the conservation law for the property

$$G = \frac{\alpha}{c}\int_V n|\mathbf{E}|. \tag{14}$$

We only need to show that the conserved quantity G, taken as a vector $\mathbf{G} = G\mathbf{e}_z$, has properties of linear momentum. The second Fresnel continuity equation, Eq. (4), is algebraically equivalent to

$$E_i = E_t - E_r + 2E_r. \tag{15}$$

Likewise, the conservation law (13) can be written as

$$\int_{V_1} n_1 E_i dv \mathbf{e}_z = \int_{V_2} n_2 E_t dv \mathbf{e}_z - \int_{V_1} n_1 E_r dv \mathbf{e}_z + 2\int_{V_1} n_1 E_r dv \mathbf{e}_z. \tag{16}$$

Then the conserved quantity has the characteristics of linear momentum in which the momentum of the reflection is in the negative direction and twice the momentum of the reflection is imputed to the material in the forward direction.

The constants of proportionality for the conserved quantities cannot be determined by the current procedure due to the nature of the Fresnel relations as linear boundary conditions. However, we can identify $\gamma = c/(4\pi)$ based on the known form for the electromagnetic energy for a monochromatic plane wave. By comparison with the prior work [17], the value of α is given in terms of a unit mass density ρ_0 as $\alpha = \sqrt{c^2\rho_0/(4\pi)}$. Then the momentum density

$$\mathbf{g} = \sqrt{\frac{\rho_0}{4\pi}} n|\mathbf{E}|\mathbf{e}_z \tag{17}$$

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can be related to the electric field energy density by

$$u_e = \frac{1}{8\pi} n^2 |\mathbf{E}|^2 = \frac{|\mathbf{g}|^2}{2\rho_0}. \tag{18}$$

We may also write the momentum flux vector

$$\mathbf{T} = |\mathbf{g}|\mathbf{v} = \sqrt{\frac{\rho_0}{4\pi}} n|\mathbf{E}| \frac{c}{n} \mathbf{e}_z \tag{19}$$

and the momentum

$$\mathbf{G} = \int_V \mathbf{g} dv = \sqrt{\frac{\rho_0}{4\pi}} \int_V n|\mathbf{E}| dv \mathbf{e}_z \tag{20}$$

in terms of the momentum density (17) with a concrete coefficient.

3. Slowly Varying Refractive Index

A spatially inhomogeneous medium can be thought of as a sequence of spatially homogeneous media of vanishingly small width. Then the Fresnel relations can be employed in a WKB treatment of electromagnetic momentum in an inhomogeneous medium for which the refractive index varies sufficiently slowly that the reflection is negligible. Expanding the Fresnel relation (2) in a power series, the refracted field can be written in terms of the incident field as

$$\sqrt{n_2} E_t = \sqrt{n_1} E_i \tag{21}$$

for the case in which $\Delta n = n_2 - n_1$ is sufficiently small that reflection can be neglected. Equation (21) represents the continuity of the flux

$$T = \alpha \sqrt{n} |\mathbf{E}| \tag{22}$$

at the interface between the two materials. Starting from the vacuum and repeatedly applying the boundary condition Eq. (21), we obtain the WKB results

$$\mathbf{T}(z) = \alpha \sqrt{n(z)} |\mathbf{E}(z)| \mathbf{e}_z \tag{23}$$

$$g(z) = \mathbf{T}(z)/\mathbf{v}(z) = \frac{\alpha}{c} n^{3/2}(z) |\mathbf{E}(z)|. \tag{24}$$

Integrating the momentum density $g\mathbf{e}_z$ over the volume, we obtain the conserved quantity

$$\mathbf{G} = \int_V g dv \mathbf{e}_z = \sqrt{\frac{\rho_0}{4\pi}} \int_V n^{3/2} |\mathbf{E}| dv \mathbf{e}_z \tag{25}$$

as the momentum of the field in an inhomogeneous linear medium in the slowly varying index limit.

Conservation of momentum requires spatial invariance and we should not necessarily expect a momentum formula to apply in all cases. The significance of the variant momentum (25) is that it provides a clear demonstration that momentum conservation depends on the inhomogeneity of the medium and that momentum conservation laws need to be tested for media with different types of inhomogeneity.

4. Momentum Conservation

Maxwellian continuum electrodynamics describes the interaction of electromagnetic fields and matter in terms of generic material parameters that multiply the fields. As the fields and matter are not treated separately, continuum electrodynamics embraces a degree of indeterminacy in the attribution of electromagnetic quantities between field and matter [18]. In this section, numerical experiments are used to explore the relationships between electromagnetic quantities in a linear dielectric under controlled conditions. In particular, we numerically demonstrate conservation of the generalized momentum in the two limiting cases.

We consider the case of a quasimonochromatic electromagnetic field entering a dielectric medium from the vacuum at normal incidence. It is assumed that dispersion and absorption can reasonably be neglected. The wave equation is solved numerically in one-dimension using a finite-difference time-domain method [19]. For the numerical work, the fields are written in terms of an envelope function and a carrier wave. In one dimension, we write the vector potential as $\mathbf{A}(z,t) = \mathscr{A}(z,t)e^{-i(\omega t - kz)}\mathbf{e}_x$, where $\mathscr{A}$ is an envelope function, ω is the carrier frequency, and k is the carrier wavenumber. Envelope functions for the electric field $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$, the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$, the displacement field $\mathbf{D} = n^2\mathbf{E}$, the magnetic field $\mathbf{H} = \mathbf{B}$, and other quantities can be defined analogously, as required. The basic phenomenology of a propagating electromagnetic field is demonstrated using the Maxwellian model of a dielectric with a macroscopic refractive index n and numerically solving the wave equation as

$$-\frac{\partial^2 \mathscr{A}}{\partial z^2} - 2ik\frac{\partial \mathscr{A}}{\partial z} + \frac{n^2}{c^2}\frac{\partial^2 \mathscr{A}}{\partial t^2} - 2i\omega\frac{n^2}{c^2}\frac{\partial \mathscr{A}}{\partial t} = 0, \tag{26}$$

where $k = n\omega/c$. The approximation of a slowly varying envelope is not made.

In the first example calculation, antireflective layers are used on the entry and exit faces of the dielectric in order to minimize reflections and thereby simplify the propagation analysis. Figure 1 shows a typical case in which the electromagnetic field, represented by the envelope of the vector potential $|\mathscr{A}|$, starts in vacuum, travels to the right, and enters a linear homogeneous dielectric through a thin gradient-index antireflection layer. The figure shows that the dielectric medium affects the refracted field in two distinct ways. First, the refracted field is reduced in width by a factor of the refractive index due to the reduced velocity of the field. Second, the refracted field is reduced in amplitude compared to the incident field due to the creation of the reaction (polarization) field. Both of these effects are reversed upon exiting the medium through a gradient-index antireflection layer, Fig. 2.

Momentum is analyzed using the WKB-based formula (25) because the refractive index varies sufficiently slowly that reflections can be neglected. We find that numerical integration of the generalized momentum (25) provides approximate conservation for any chosen time in the propagation. This result is easily confirmed analytically by treating the field as a square pulse, applying the Fresnel boundary condition in the limit of negligible reflection, and scaling the width of the field in the medium. The theoretical conservation law for a square pulse of width w in the vacuum

$$E_i w = n^{3/2}\frac{E_i}{\sqrt{n}}\frac{w}{n} \tag{27}$$

complements the numerical demonstration of momentum conservation in a medium with a slowly varying refractive index. Because the boundary conditions for this exemplar have been devised to minimize reflections, the incident and transmitted fields are essentially identical. Then, the transmitted field accounts for all the momentum of the incident field and we conclude that no permanent momentum is imparted to a material that does not reflect, or absorb. Further, there is no temporary material momentum because the momentum is fully accounted for at any time that the field is in the medium, in whole or in part.

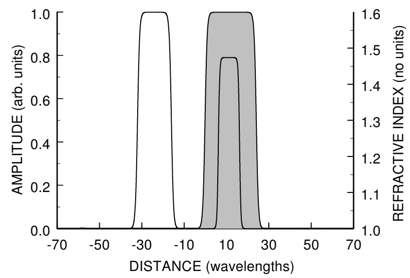

Fig. 1. Propagation of the vector potential from vacuum into a linear medium with a gradient-index antireflection layer on the the entry and exit faces. The shaded region is the profile of the index of refraction. The field travels to the right and the horizontal axis is scaled to the wavelength.

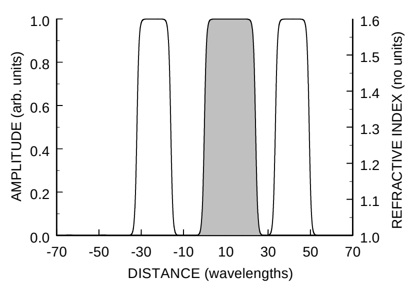

Fig. 2. Same as Fig. 1 after the field has propagated out of the medium through the anti-reflection layer.

We now extend this result to the more usual or realistic case of a step change in the refractive index for a field entering a homogeneous linear medium from the vacuum. The main difference is that a backward traveling wave is generated at the vacuum/dielectric interface and the resulting change in momentum must be incorporated into the conservation law. Using the same incident field as in Fig. 1, Fig. 3 shows the field entering the linear medium. The leading part of the field has entered the medium and a portion of that field has been reflected. The oscillations that are seen in the figure represent the interference of the reflection with the trailing part of the incident field. In Fig. 4, the forward propagating refracted field is entirely within the medium and the reflection has separated from the interface. Figure 5 shows the fields after the refracted field has exited the medium through a gradient-index layer that suppresses the secondary reflection. At the end of the calculation, the momentum of the transmitted field is found to be smaller than that of the incident field by the momentum of the reflection. In order to satisfy conservation of momentum, a permanent forward momentum of twice the momentum of the reflected field must by imputed to the material. In this regard, conservation of electromagnetic momentum is analogous to momentum conservation in the elastic collision of a small object with a wall.

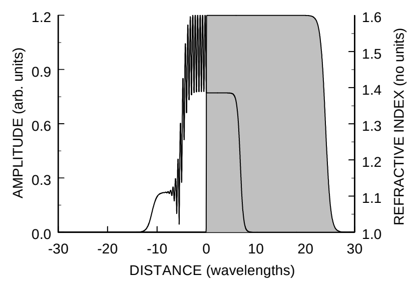

Fig. 3. Propagation of the vector potential from vacuum into a linear medium through a step increase in the refractive index. The shaded region is the profile of the index of refraction. The field travels to the right and the horizontal axis is scaled to the wavelength.

At any point in the calculation, conservation of the momentum (20) can be demonstrated if a forward momentum of twice the momentum of the reflected field is contributed by the material. As soon as the field starts to enter the medium, the momentum of the forward traveling field begins to decrease and the momentum of the backward traveling wave, initially zero, begins to grow as does the momentum of the material. Once the refracted field is entirely within the medium, the momentum of the refracted field remains the same until the field exits the medium through the antireflective layer, and thereafter. Meanwhile, the process of reflection is complete and the reflected field travels through the vacuum and is unchanged. We can conclude that there is no additional transfer of momentum to the material once the field is no longer incident on its surface. Consequently, the permanent transfer of momentum from the field to the material occurs at the point of reflection, the surface of the medium, and only while the field is present at the boundary and undergoing Fresnel reflection. There is no temporary material momentum because the momentum is fully accounted for at any time, particularly as the field exits the medium. For the approximate square pulse of vacuum width w, the momentum conservation

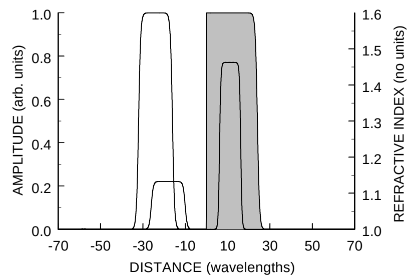

Fig. 4. Propagation of the vector potential from vacuum into a linear medium with a step-index for the entry and a gradient-index antireflection layer on the exit face. The shaded region is the profile of the index of refraction. The field travels to the right and the horizontal axis is scaled to the wavelength.

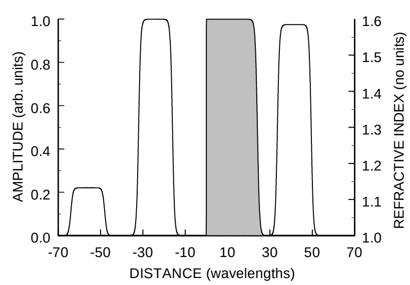

Fig. 5. Same as Fig. 4 after the field has propagated out of the medium through the anti-reflection layer.

law takes the form

$$E_i w = \frac{2n_1 n_2}{n_2+n_1} E_i \frac{w}{n_2} + (2-1)\frac{n_2-n_1}{n_2+n_1} E_i w. \tag{28}$$

Conservation of momentum is guaranteed by matching the boundary condition on the momentum flux.

5. Energy Conservation

It is straightforward to numerically integrate the field at any chosen time in the propagation and show that the electromagnetic energy

$$U = \frac{1}{8\pi}\int_V (n^2\mathbf{E}^2 + \mathbf{B}^2)dv \tag{29}$$

and the Gordon total momentum [22]

$$\mathbf{G}_x = \frac{1}{4\pi c}\int_V n\mathbf{E}\times\mathbf{H}\,dv, \tag{30}$$

are conserved. A more general proof of the conservation of $\mathbf{G}_x$ can be provided by writing a continuity law. Denoting the magnitude of the electromagnetic momentum density

$$\mathbf{g}_x = \frac{1}{4\pi c} n\mathbf{E}\times\mathbf{H} \tag{31}$$

as g_x allows one to write the continuity law (7) as

$$\nabla\cdot g_x\mathbf{v} = -\frac{\partial g_x}{\partial t}. \tag{32}$$

The velocity of the field in the linear medium is c/n in the direction of $\mathbf{E}\times\mathbf{H}$. Substituting the electromagnetic momentum density (31) into Eq. (32) results in a momentum conservation law

$$\nabla\cdot\frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = -c\frac{\partial g_x}{\partial t} \tag{33}$$

that is redundant with Poynting's theorem, where $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{H}$ is the Poynting vector and $u = cg_x$ is the energy density. Therefore, G_x is conserved, but is redundant with the electromagnetic energy from which it was derived [22]. As a matter of linear algebra, the electromagnetic momenta that are quadratic in the field are either redundant or inconsistent with the electromagnetic energy.

6. Radiation Pressure

In a pedagogical example, Stone [23] describes the difference between momentum and pseudomomentum in terms of transverse waves on a string. Momentum is conserved whenever the string, together with any disturbance on it, is translated. When the string is left fixed, but the disturbance is translated, the conserved quantity is pseudomomentum. By analogy, pseudoenergy and pseudomomentum travel with the electromagnetic field as the excitation of spacetime degrees of freedom. Then, because spacetime is not in motion, the electromagnetic momentum and energy that appear in this article in their conventional usage should be reinterpreted as pseudomomentum and pseudoenergy of the electromagnetic field. Pseudoenergy and pseudomomentum can sometimes be converted to real energy and real momentum [23]. In particular, charges, associated with matter, can couple into the internal degrees of freedom and evince real-momentum effects such as radiation pressure.

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The act of changing the momentum of the material requires a force, specifically, a radiation pressure acting over the illuminated area A of the material. In an increment of time, the change in the pseudomomentum of the reflected field is

$$\delta|\mathbf{G}| = \int_A \sqrt{\frac{\rho_0}{4\pi}} \frac{n-1}{n+1} E_i c \delta t da \tag{34}$$

when the field is normally incident on a homogeneous medium from the vacuum. The change in the momentum of the material is twice the pseudomomentum of the reflected field. Then the radiation pressure is

$$P = 2c\sqrt{\frac{\rho_0}{4\pi}} \frac{n-1}{n+1} E_i \tag{35}$$

during the period of illumination. Integrating the radiation pressure (35) over the illuminated surface area yields the surface force

$$\mathbf{F} = \int_A 2c\sqrt{\frac{\rho_0}{2\pi}} \frac{n-1}{n+1} E_i da \mathbf{e}_z. \tag{36}$$

In Sec. 4, we demonstrated that momentum is transferred from the field to the material at the point of reflection. In the absence of absorption or free charges, the force that imparts momentum to the material is a surface force.

The radiation pressure has been attributed previously to a volume Lorentz force [22, 24, 25, 26]. In the continuum limit, the Lorentz force law

$$\mathbf{F} = \sum_i q_i \left(\mathbf{E} + \frac{\mathbf{v}_i}{c} \times \mathbf{B} \right) \tag{37}$$

becomes

$$\frac{d\mathbf{P}_{mech}}{dt} = \int_V \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) dv. \tag{38}$$

Because the volume Lorentz force is nil in the absence of charges and currents, one retains the charge density ρ and the charge current $\mathbf{J}$ in the derivation and, at the end, takes the limit in which these quantities vanish. Then, using the Maxwell equations to eliminate ρ and $\mathbf{J}$, one finds [18]

$$\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{G}_M}{dt} = \frac{1}{4\pi} \int_V \left[\mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{D} \times (\nabla \times \mathbf{E}) - \mathbf{B} \times (\nabla \times \mathbf{B}) \right] dv \tag{39}$$

where

$$\mathbf{G}_M = \int_V \frac{1}{4\pi c} \mathbf{D} \times \mathbf{B} dv \tag{40}$$

is the usual Minkowski momentum. Using the usual constitutive relation $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$, we may write $\mathbf{G}_M$ as the sum of the Abraham momentum

$$\mathbf{G}_A = \int_V \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} dv \tag{41}$$

of the electromagnetic field and a mechanical momentum

$$\mathbf{G}_{mech} = \int_V \frac{1}{c} \mathbf{P} \times \mathbf{B} dv \tag{42}$$

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that is presumed to be associated with bound charges moving within the dielectric material. Then

$$\frac{d\mathbf{P}_{mech}}{dt}+\frac{d\mathbf{G}_{mech}}{dt}+\frac{d\mathbf{G}_A}{dt}=\frac{1}{4\pi}\int_V\left[\mathbf{E}(\nabla\cdot\mathbf{D})-\mathbf{E}\times(\nabla\times\mathbf{E})-4\pi\mathbf{P}\times(\nabla\times\mathbf{E})-\mathbf{B}\times(\nabla\times\mathbf{B})\right]dv \tag{43}$$

can be written as

$$\frac{d\mathbf{P}_{mech}}{dt}+\frac{d\mathbf{G}_{mech}}{dt}=\int_V\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}-\mathbf{P}\times(\nabla\times\mathbf{E})+\frac{d\mathbf{P}}{dt}\times\mathbf{B}\right)dv. \tag{44}$$

In the absence of free currents and charges, we have

$$\frac{d\mathbf{P}_{mech}}{dt}+\frac{d\mathbf{G}_{mech}}{dt}=\frac{1}{c}\int_V\frac{d}{dt}(\mathbf{P}\times\mathbf{B})\,dv \tag{45}$$

or

$$\frac{d\mathbf{P}_{mech}}{dt}=0. \tag{46}$$

The bound-charge volume Lorentz force disappears because it is derived from a tautology. The radiation pressure is therefore a consequence of the surface force (36) in the absence of absorption or free charges.

7. Conclusion

Physics experiments establish the connection between a physical effect or process and its theoretical description. A number of experiments have been performed to distinguish between the Abraham and Minkowski momenta including the Jones–Richards mirror [10], the Ashkin–Dziedzic liquid surface [11], the Gibson photon drag effect [12] and the Lahoz–Graham magnetic cylinder [13]. Because the forces, boundary conditions, and material momenta can be incorporated in different ways, the experiments can be analyzed to support almost any of the conventional momentum formulas [14, 15, 24, 25]. Ultimately, conservation of a momentum of the Abraham–Minkowski form is a matter of a transformation to a quantity that is redundant with the electromagnetic energy.

The results of the numerical experiments of wave propagation that appear in the current work are definitive. Given an initial field in the vacuum and the spatially dependent refractive index, results are generated entirely by the solution of the wave equation. The field is known at every point, both inside and outside of the material, at any time and can be used to calculate electrodynamic quantities. In particular, we demonstrated two limiting cases in which momenta that are not of the Abraham–Minkowski type are conserved. The difference in the conservation properties between the limiting case of a piecewise homogeneous medium and the limiting case of a slowly varying refractive index shows that transfer of momentum to the field the material occurs at the point of reflection. Then, in the absence of charges, radiation pressure is a force acting on the surface of a homogeneous linear material. These results are a simple direct consequence of the Fresnel relations.