Propagation of electromagnetic energy and momentum through an absorbing dielectric

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We calculate the energy and momentum densities and currents associated with electromagnetic wave propagation through an absorbing and dispersive diatomic dielectric, which is modeled by a single-resonance Lorentz oscillator. The relative and center-of-mass coordinates of the dielectric sublattices and the electromagnetic field vectors are treated as dynamical variables, while the dielectric loss is modeled by a phenomenological damping force. The characteristics of the energy propagation agree with previous work, including the form of the energy velocity. The treatment of momentum propagation extends previous work to lossy media, and it is found that the damping plays an important role in the transfer of momentum from the electromagnetic field to the center of mass of the dielectric. We discuss the significances of the momentum, the pseudomomentum, and their sum, the wave momentum. For each of these quantities we derive the density, the current density, and the appropriate conservation or continuity equation. The general expressions are illustrated by applications to a steady-state monochromatic wave and to an excitation in the form of a localized Gaussian pulse. The velocities associated with propagation of the various kinds of momentum are derived and discussed. [S1063-651X(97)04901-5]

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I. INTRODUCTION

The nature of electromagnetic energy and the characteristics of its propagation through dielectric media have been studied since the early years of electromagnetic theory. For propagation through the simplest kind of linear, isotropic, and homogeneous medium, the energy density W and energy current density, or Poynting vector $\mathbf{S}$ are routinely treated in standard texts [1,2]. The forms of these energy densities and their conservation law have also been evaluated for much more general dielectric media [3]. For propagation through absorbing or scattering materials, the classic treatment of electromagnetic wave propagation, and particularly the identification of the several distinct velocities that are associated with an optical pulse, was provided by Sommerfeld and Brillouin [4]. The detailed theory for lossy dielectrics is quite complicated, but the main features of the energy density and current, and of energy propagation, are correctly predicted by a simple calculation [5], based on the standard model of electromagnetic waves in a Lorentzian dielectric with a single resonance. The essential feature of this theory is the inclusion of contributions to the total energy density W and energy current density $\mathbf{S}$ of the optical excitation from both the electromagnetic field and the dielectric medium.

It is interesting to determine whether there is an analogous theory for electromagnetic momentum propagation in a lossy medium, and this is the primary purpose of the present paper. Such an inquiry is particularly topical because recent work [6] on dispersive, but lossless, dielectrics has proposed a resolution of the long-standing Minkowski-Abraham controversy concerning the correct expressions for the densities of electromagnetic momentum and current, denoted here by $\mathbf{G}$ and $\mathbf{T}$, respectively (see [7,8] for reviews). These densities are well understood for electromagnetic fields in free space, where the momentum density is a vector quantity proportional to the energy current, $\mathbf{G}=\mathbf{S}/c^2$, and the momentum current density is a second rank tensor, or 3×3 matrix, related to the Maxwell stress tensor [1,2,9]. For the electromagnetic momentum in material media, it is necessary to take account of contributions from both the electromagnetic field and the dielectric medium. The momentum current in a lossless dielectric was obtained by this approach as a modified form of the Maxwell stress tensor. In addition, the natures of the momentumlike quantities that have been defined for the coupled system of electromagnetic field and dielectric material, including the densities proposed by Abraham and Minkowski, were identified [6].

The controversy has always revolved around a linear light wave for which deformation of the dielectric medium is irrelevant, but a key ingredient of its recent resolution is the inclusion of deformation of the medium. This necessitates the use of both spatial (Eulerian) and material (Lagrangian) coordinates, and it allows the deduction of conservation laws from Noether's theorem. Thus the momentum conservation law follows from invariance to displacements of the spatial coordinates (homogeneity of free space), and the pseudomomentum conservation law follows from invariance to displacements of the material coordinates (homogeneity of the material medium). This approach [6] found the electromagnetic momentum density $\mathbf{G}_m$ to be $\varepsilon_0\mathbf{E}\times\mathbf{B}$, close to, but in general different from, the Abraham form $\varepsilon_0\mu_0\mathbf{E}\times\mathbf{H}$. It also found the pseudomomentum density $\mathbf{G}_{\rm psm}$ to be $\mathbf{P}\times\mathbf{B}$ plus a dispersive term. Thus the sum $\mathbf{G}$ of the momentum and pseudomomentum densities, which we call the *wave momentum*, is the generalization of the Minkowski momentum $\mathbf{D}\times\mathbf{B}$ to include dispersion. However, the Minkowski momentum was proposed as being the ordinary momentum, while this derivation shows instead that it is the sum of ordinary momentum and pseudomomentum. The name ''wave momentum'' was introduced for this reason.

While the inclusion of material deformation has played an essential role in the clarification of what is momentum and what is pseudomomentum, it also acts as a barrier to simple physical understanding. The aim of the present paper is thus to find simplified versions of the conservation laws for momentum and pseudomomentum, even after generalization of previous work to include loss. This is achieved by the simplification of the dielectric to a nonmagnetic diatomic crystal with cubic isotropy, essentially the single-resonance Lorentz model. The ions are assumed to be coupled to the electromagnetic field only by an electric-dipole interaction.

Before proceeding to the main calculations, we present a simplified discussion of energy and momentum propagation in Sec. II. An improved version of previous calculations of the energy continuity equations and the velocity of energy propagation [5] leads to essentially the same results as before, but our method facilitates parallel discussions of the momentum propagation characteristics. It is found, however, that the propagation of momentum involves both the center of mass and relative coordinates of the diatomic dielectric, whose proper treatment requires a Lagrangian formulation. Thus it is shown in Sec. III that the momentum density and current obey a conservation law when the center-of-mass momentum is included, but that the pseudomomentum, and hence the wave momentum, suffer dissipation on account of the dielectric loss. The various electromagnetic densities derived in Secs. II and III are evaluated for a steady-state monochromatic wave in Sec. IV, where the velocities of propagation of energy and wave momentum are derived, and for an optical pulse in Sec. V. The results are discussed in Sec. VI.

II. SIMPLE THEORY OF ELECTROMAGNETIC ENERGY AND MOMENTUM PROPAGATION

The present section is devoted to a derivation of some basic results for electromagnetic fields in a dielectric material treated in the Lorentz model. We present a simple derivation of the equations that describe the propagation of energy, and show that the corresponding description of momentum propagation cannot be obtained by so simple a theory. The detailed derivations of the equations that describe momentum propagation and the identification of the different characters of the momentumlike contributions are given careful consideration in Sec. III.

A. Basic equations

The fundamental energy and momentum properties of electromagnetic fields in matter are governed by Maxwell's equations and by the equations of motion for the matter. We consider a nonmagnetic dielectric material that has no free charges or currents. The Maxwell-Lorentz forms of the equations in conventional notation and Système International (SI) units are then

$$\nabla \cdot E = \rho/\varepsilon_0, \tag{2.1}$$

$$\nabla \cdot B = 0, \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.4}$$

where the fields are functions of position and time, $\mathbf{E} \equiv \mathbf{E}(\mathbf{r},t)$, and so on. The bound charge and current densities, ρ and $\mathbf{j}$, respectively, are also functions of position and time; they can be expressed in terms of the dielectric polarization $\mathbf{P}$ as

$$\rho = -\nabla \cdot P \tag{2.5}$$

and

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t}. \tag{2.6}$$

The electric displacement is defined in the usual way,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{2.7}$$

and the magnetic field is given by

$$\mathbf{H} = \mathbf{B}/\mu_0. \tag{2.8}$$

Note that in the view implicit in these equations, $\mathbf{E}$ and $\mathbf{B}$ are the fundamental electromagnetic fields, while $\mathbf{P}$ describes the response of the matter, and Eqs. (2.7) and (2.8) are constitutive equations for $\mathbf{D}$ and $\mathbf{H}$.

We consider a polar, diatomic, cubic, crystal lattice in which the relative spatial displacement field of the two ions in the unit cell is denoted $\mathbf{s} \equiv \mathbf{s}(\mathbf{r},t)$. The long-wavelength optic modes of vibration have a basic threefold degeneracy which is lifted by the long-range electrical forces to form a twofold-degenerate transverse mode and a nondegenerate longitudinal mode [10]. Then, if the frequency of the transverse mode is denoted ω_T and its damping rate is denoted Γ, the standard form of the Lorentz equation for the ith Cartesian component of the internal coordinate of the ionic motion is

$$m\ddot{s}_i + m\Gamma \dot{s}_i + m\omega_T^2 s_i = \varsigma E_i. \tag{2.9}$$

Here m is the reduced mass density of the two ions, of masses M_1 and M_2, in the primitive unit cell of volume Ω,

$$m = \frac{M_1 M_2}{\Omega(M_1 + M_2)} \tag{2.10}$$

and the charge density ς associated with the internal motion is given by

$$\varsigma = e/\Omega, \tag{2.11}$$

where e and $-e$ are the charges on the two kinds of ion. The polarization is expressed in terms of the internal coordinate by

$$\mathbf{P} = \varsigma \mathbf{s}. \tag{2.12}$$

B. Energy propagation

The flow of electromagnetic energy through the dielectric is determined by the energy current density, or Poynting vector, given by

$$\mathbf{S}_{\mathrm{em}}=\mathbf{E}\times\mathbf{B}/\mu_0 \tag{2.13}$$

for the cubic isotropic material assumed here. It is straightforward to show with the use of Eqs. (2.3) and (2.4) that

$$\frac{\partial}{\partial r_i}(\mathbf{S}_{\mathrm{em}})_i+\frac{\partial}{\partial t}W_{\mathrm{em}}=-\mathbf{E}\cdot\mathbf{j}, \tag{2.14}$$

where the repeated index i is summed over the Cartesian coordinates x, y, and z, and

$$W_{\mathrm{em}}=\tfrac{1}{2}\varepsilon_0E^2+\tfrac{1}{2}\mu_0H^2 \tag{2.15}$$

is the usual electromagnetic energy density. Equation (2.14) expresses the continuity of the electromagnetic energy, and the term on the right represents the rate of loss of energy from the field by transfer to the dielectric.

Multiplication of (2.9) by $\dot{s}_i$ gives

$$m\ddot{s}_i\dot{s}_i+m\Gamma\dot{s}_i^2+m\omega_T^2s_i\dot{s}_i=\varsigma E_i\dot{s}_i=\mathbf{E}\cdot\mathbf{j}, \tag{2.16}$$

where Eqs. (2.6) and (2.12) have been used, similar to a calculation in [4]. The rate of loss on the right of the electromagnetic energy continuity equation (2.14) is thus balanced by the rate of gain of energy represented by the term on the right of Eq. (2.16) for the dielectric lattice mode. The sum of Eqs. (2.14) and (2.16) can be written in the form of an energy continuity equation for the coupled electromagnetic field and dielectric lattice,

$$\frac{\partial S_i}{\partial r_i}+\frac{\partial W}{\partial t}=-m\Gamma\dot{s}^2, \tag{2.17}$$

where the total-energy current density

$$\mathbf{S}=\mathbf{S}_{\mathrm{em}}=\mathbf{E}\times\mathbf{B}/\mu_0 \tag{2.18}$$

is the same as the electromagnetic current density (2.13), but the total energy density is

$$W=\tfrac{1}{2}\{\varepsilon_0E^2+\mu_0H^2+m\dot{s}^2+m\omega_T^2s^2\}. \tag{2.19}$$

The excitation of the dielectric lattice, that is of the Lorentzian oscillator or optic mode, thus makes no explicit contribution to the energy current density. The lattice does, though, have an implicit effect via the scaling of the ratio of the magnetic and electric fields by the complex refractive index of the medium [see Eq. (4.10)]. However, the energy density explicitly contains the kinetic and potential energies of the optic vibrational mode in addition to the electromagnetic energy density (2.15). The term on the right of Eq. (2.17) represents the rate of loss of energy density from the coupled field-lattice system by the optic mode damping.

C. Momentum propagation

The flow of electromagnetic momentum is determined by the momentum current density, whose components are given by [1,2,6,9,10]

$$(\mathbf{T}_{\mathrm{em}})_{ji}=-\varepsilon_0E_jE_i+\tfrac{1}{2}\varepsilon_0E^2\delta_{ji}-\frac{B_jB_i}{\mu_0}+\frac{B^2}{2\mu_0}\delta_{ji}. \tag{2.20}$$

This quantity is usually identified as the negative of the Maxwell stress tensor [1,2], but occasionally as the Maxwell stress tensor [9]. The momentum continuity equation for the electromagnetic field is obtained from Maxwell's equations by forming the vector products of $\varepsilon_0\mathbf{E}$ with Eq. (2.1), $\mathbf{B}/\mu_0$ with Eq. (2.2), $\mathbf{E}$ with Eq. (2.3), and $\mathbf{B}$ with Eq. (2.4), and then adding the four equations. The result after use of standard vector operator identities is

$$\frac{\partial}{\partial r_i}(\mathbf{T}_{\mathrm{em}})_{ji}+\frac{\partial}{\partial t}(\mathbf{G}_{\mathrm{em}})_j=-\rho E_j-(\mathbf{j}\times\mathbf{B})\equiv-F_j, \tag{2.21}$$

where

$$\mathbf{G}_{\mathrm{em}}=\varepsilon_0\mathbf{E}\times\mathbf{B} \tag{2.22}$$

is the electromagnetic momentum density. Equation (2.21) expresses continuity of electromagnetic momentum. The terms on the right represent the rate of loss of momentum from the field by transfer to the dielectric, in the form of minus the usual Lorentz force density, denoted F_j.

The transfer of momentum from the electromagnetic field implies that the dielectric as a whole is set into motion. The internal relative displacement field $\mathbf{s}$ is itself invariant under a uniform displacement of the crystal, and cannot therefore carry momentum. The dielectric momentum is carried by the motion of the spatial displacement field $\mathbf{R}\equiv\mathbf{R}(\mathbf{r},t)$ defined by the position of the center of mass of the two ions in the unit cell. A treatment of the propagation of momentum through the dielectric thus requires a theoretical framework that includes both the relative and center-of-mass coordinates $\mathbf{s}$ and $\mathbf{R}$; this is provided by the Lagrangian formalism presented in Sec. III.

The effect of the dissipation term in the internal equation of motion (2.9) is to remove energy from the optic modes of vibration. The sink for this energy is provided by a reservoir, whose nature is determined by the microscopic mechanism of the dissipation. For example, anharmonic forces in the lattice transfer the optic-mode energy into continuous distributions of other vibrational modes which are not directly coupled to the electromagnetic field. Thus an initial excitation of the coupled electromagnetic field and optic modes decays to a steady state in which all of the energy is transferred to the reservoir. This transfer has implications for both the momentum and the kinetic energy associated with the motion of the dielectric crystal.

Suppose that the initial excitation has N quanta of wave vector k and frequency ω per unit volume. The magnitude of the momentum density acquired by the dielectric crystal as a whole, when all of the energy has been transferred to the reservoir, is

$$M\dot{R}=N\hbar k=N\hbar\omega/c, \tag{2.23}$$

where M is the dielectric mass density,

$$M=(M_1+M_2)/\Omega, \tag{2.24}$$

and the relation between the frequency and wave vector has been taken in its free-space form, for the purpose of an order-of-magnitude estimate. Clearly it is important to include the center-of-mass momentum of the crystal in any theory of momentum propagation through an absorbing dielectric.

The transfer of momentum to the dielectric must be accompanied by a growth in its kinetic energy density, whose value for the momentum density given by Eq. (2.23) is

$$\frac{M\dot{R}^2}{2}=N\hbar\omega\,\frac{N\hbar\omega}{2Mc^2}. \tag{2.25}$$

The rest-mass energy density Mc^2 of the crystal is always very much larger than the initial energy density $N\hbar\omega$. The crystal kinetic energy is thus completely negligible compared to $N\hbar\omega$. This justifies the neglect of center-of-mass motion in the theory of energy propagation given in Sec. II B, despite its importance in the theory of momentum propagation.

III. LAGRANGIAN THEORY OF ELECTROMAGNETIC MOMENTUM PROPAGATION

This section is devoted to a rigorous derivation of the various momentum densities associated with the propagation of electromagnetic waves through absorbing dielectrics. The basic dielectric model is the same as that used in Sec. II, but it is necessary to generalize the model to include center-of-mass motion in order to describe momentum propagation. It is also necessary to distinguish the contributions of momentum and pseudomomentum. The continuum mechanics background to the calculations is described in detail in Ref. [10]. It is assumed throughout that the dielectric material fills all of space; the effects of crystal boundaries are excluded from the calculations.

A. Lagrangian formulation

The system of dielectric material (M) and electromagnetic field (F) coupled by an electric-dipole interaction (I) is described by a Lagrangian density

$$\mathcal{L}=\mathcal{L}_M+\mathcal{L}_I+\mathcal{L}_F, \tag{3.1}$$

where the Lagrangian itself is formed by integration over the Lagrangian density in the usual way.

The Lagrangian density of the electromagnetic field is

$$\mathcal{L}_F=\frac{\varepsilon_0}{2}\,\mathbf{E}^2-\frac{1}{2\mu_0}\,\mathbf{B}^2, \tag{3.2}$$

where the electric and magnetic fields are determined by the scalar potential ϕ and the vector potential $\mathbf{A}$ in the usual way,

$$\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{\partial\mathbf{A}}{\partial t} \tag{3.3}$$

and

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}. \tag{3.4}$$

The interaction Lagrangian density is

$$\mathcal{L}_I=\mathbf{j}\cdot\mathbf{A}-\rho\phi, \tag{3.5}$$

where the charge and current densities are related to the dielectric polarization by Eqs. (2.5) and (2.6). However, when the center-of-mass motion is included, the latter expression should be augmented by inclusion of the Röntgen current [11], to give a total current density

$$\mathbf{j}=\frac{\partial\mathbf{P}}{\partial t}+\boldsymbol{\nabla}\times(\mathbf{P}\times\dot{\mathbf{R}}), \tag{3.6}$$

where $\mathbf{R}$ is again the continuum center-of-mass coordinate. The interaction Lagrangian density (3.5) can be converted with the use of this expression to

$$\mathcal{L}_I=\mathbf{P}\cdot(\mathbf{E}+\dot{\mathbf{R}}\times\mathbf{B}), \tag{3.7}$$

where some perfect space and time derivative terms, which make no contribution to the Lagrange equations of motion, have been discarded [10]. The material Lagrangian density for a rigid body is

$$\mathcal{L}_M=\tfrac{1}{2}M\dot{\mathbf{R}}^2+\tfrac{1}{2}m\dot{\mathbf{s}}^2-\tfrac{1}{2}m\omega_T^2\mathbf{s}^2, \tag{3.8}$$

where the dielectric parameters are as defined in Sec. II. The theory also needs to include a term that allows for damping of the internal motion at a rate proportional to Γ. This is conveniently implemented by a Rayleigh dissipation function of the form

$$\mathcal{R}=\tfrac{1}{2}\,m\Gamma\dot{\mathbf{s}}^2, \tag{3.9}$$

which is incorporated into the Euler-Lagrange equations by an appropriate additional term [12].

The equations of motion for the electromagnetic and material field variables are obtained by the standard Lagrangian procedures. Thus the Maxwell-Lorentz equations (2.1) and (2.4) are rederived straightforwardly, while Eqs. (2.2) and (2.3) are satisfied automatically from the definitions (3.3) and (3.4) of the fields in terms of the potentials. It should however be noted that the Röntgen term in the current density (3.6) causes a generalization of relation (2.8) between magnetic field and magnetic induction to [see, for example, Eq. (76.11) of Ref. [2]]

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{P}\times\dot{\mathbf{R}}. \tag{3.10}$$

The new term is a function of both the internal relative-displacement coordinate and the center-of-mass coordinate of the dielectric material.

For the dielectric spatial displacement variables, the equation of motion for the relative position of the two ions in the unit cell is obtained with the use of Eqs. (3.7)–(3.9) as

$$m\ddot{s}_i+m\Gamma\dot{s}_i+m\omega_T^2 s_i=\varsigma(E_i+(\dot{\mathbf{R}}\times\mathbf{B})_i), \tag{3.11}$$

which is identical to Eq. (2.9) except for the addition of the term proportional to the center-of-mass velocity $\dot{\mathbf{R}}$. The equation of motion for the continuum center-of-mass coordinate is obtained similarly as

$$M\ddot{R}_j - \varsigma s_i \frac{\partial}{\partial r_j}(E_i + (\dot{\mathbf{R}}\times\mathbf{B})_i) - \varsigma \frac{d}{dt}(\mathbf{s}\times\mathbf{B})_j = 0. \tag{3.12}$$

A more convenient form of this equation is found after considerable manipulation [10], using Eqs.(2.2), (2.5) and (3.6), to be

$$M\ddot{R}_j - \frac{\partial}{\partial r_i}\{\varsigma(E_j + (\dot{\mathbf{R}}\times\mathbf{B})_j)s_i\} = \rho E_j + (\mathbf{j}\times\mathbf{B})_j = F_j. \tag{3.13}$$

The significances of the terms on the left are clarified by their contributions to the momentum densities defined in Sec. III B below.

As discussed in connection with Eq. (2.25), the kinetic energy of the dielectric is negligible compared to the energy densities associated with the electromagnetic field and the optic mode vibration of the lattice. The contribution of the center-of-mass motion to the Lorentz force on the right-hand side of (3.11) is also small, as $\dot{R}B$ is of order $\dot{R}E/c$, which is certainly much smaller than E. With terms in $\dot{\mathbf{R}}$ removed, the Lagrangian theory reproduces the results of Sec. II B as Eq. (3.11) reduces to Eq. (2.9), and the Maxwell equations are unchanged. The energy continuity equation (2.17) and the energy densities given in Eqs. (2.18) and (2.19) thus continue to hold. The derivations that follow consider the more subtle problems and distinctions associated with the propagation of momentum and pseudomomentum through the absorbing dielectric.

B. Momentum conservation

The law of momentum conservation is a consequence of the invariance of the laws of physics to arbitrary infinitesimal displacements of the spatial coordinates. The momentum is thus defined with respect to the Maxwell equations (2.1)–(2.4) and the center-of-mass equation (3.13). The continuity equation (2.21) for the electromagnetic momentum, which is based entirely on Maxwell's equations, therefore remains valid. The Lorentz force densities F_j on the right-hand sides of Eqs. (2.21) and (3.13) are equal and opposite, demonstrating the action and reaction of the forces between the electromagnetic and material parts of the coupled system. Addition of these equations, using definition (2.12) of the polarization, gives

$$\frac{\partial}{\partial r_i}(\mathbf{T}_m)_{ji} + \frac{\partial}{\partial t}(\mathbf{G}_m)_j = 0, \tag{3.14}$$

which is the conservation law for the momentum of the field-material system. Here

$$(\mathbf{T}_m)_{ji} = -(E_j + (\dot{\mathbf{R}}\times\mathbf{B})_j)P_i + (\mathbf{T}_{\text{em}})_{ji} \tag{3.15}$$

is the momentum current density, and

$$\mathbf{G}_m = M\dot{\mathbf{R}} + \mathbf{G}_{\text{em}} \tag{3.16}$$

is the momentum density of the coupled field and material. Expressions for the electromagnetic contributions to the momentum current density and the momentum density are given in Eqs. (2.20) and (2.22), respectively. A tensor contribution with the same magnitude as the center-of-mass kinetic energy has been omitted from the momentum current density (3.15) in accordance with the discussion that follows Eq. (2.25).

It should be noted that the equation of motion (3.11) for the internal coordinate plays no role in the above derivation, on account of the inability of the relative coordinate to carry momentum [6]. As the dissipation described by Eq. (3.9) acts only on the internal coordinate, the absence of this coordinate from the derivation of Eqs. (3.14)–(3.16) accounts for the lack of damping terms in these results. The fulfilment of momentum conservation in the presence of energy dissipation, as described by the term on the right-hand side of Eq. (2.17), is of course a common occurrence in mechanics.

The momentum density (3.16) is clearly separated into a contribution $M\dot{\mathbf{R}}$ from the center-of-mass motion and a contribution $\varepsilon_0\mathbf{E}\times\mathbf{B}$ from the electromagnetic field. The latter differs from the Abraham form $\varepsilon_0\mu_0\mathbf{E}\times\mathbf{H}$ on account of the generalized relation (3.10) between the magnetic field and the induction. It is seen that, in contrast to the magnetic induction $\mathbf{B}$, which is purely a property of the electromagnetic field, the magnetic field $\mathbf{H}$ contains a contribution that depends on the material internal coordinate contained in $\mathbf{P}$. The occurrence of $\mathbf{B}$ rather than $\mathbf{H}$ ensures that the electromagnetic momentum density is properly independent of any material variables.

The total momentum is obtained by integration of $\mathbf{G}_m(\mathbf{r},t)$ over all space, and this quantity is conserved for a closed system with no flow of momentum through its boundaries. Thus integration of Eq. (3.14) over the effectively-infinite dielectric material gives

$$\frac{\partial}{\partial t}\int d\mathbf{r}\,\mathbf{G}_m(\mathbf{r},t) = 0, \tag{3.17}$$

provided that $\mathbf{T}_m(\mathbf{r},t)$ vanishes at $\mathbf{r}=\infty$. The total momentum is therefore conserved, and only its division between the electromagnetic field and the dielectric center-of-mass motion changes with time as, for example, in the propagation of a pulse of excitation through the crystal.

Although the material motion makes an important contribution to the momentum density, its contribution to the momentum current density (3.15) is generally less important. Thus $\dot{R}B$ is again of order $\dot{R}E/c$, which is much smaller than E, and, with the corresponding term removed, Eq. (3.15) reduces to

$$(\mathbf{T}_m)_{ji} = -E_jP_i + (\mathbf{T}_{\text{em}})_{ji}. \tag{3.18}$$

C. Pseudomomentum

Pseudomomentum has been a much neglected quantity in continuum mechanics, and a regularly misinterpreted quantity in quantum mechanics. Quantum-mechanical treatments of excitations in solids have often called $\hbar\mathbf{k}$ the pseudomomentum of an excitation quantum. This was shown to be wrong [6] on the basis of an unambiguous definition of the pseudomomentum as the momentumlike quantity that is conserved by virtue of the homogeneity of the material body.

Noether's theorem [10] can be used with a Lagrangian formulation to obtain a rigorous derivation of the pseudomomentum conservation law for a homogeneous body. This is a body of infinite, or at least large, extent compared to the interaction volume considered, so that any boundary effects can be ignored. The invariance used in the application of the theorem to pseudomomentum is with respect to an arbitrary infinitesimal displacement of the material coordinates of a point that moves with the dielectric. This is completely analogous to the arbitrary infinitesimal displacement of the spatial coordinates, namely, the position of a point with respect to the free-space vacuum, which is used with Noether's theorem to obtain the momentum conservation law. Based on these definitions, it has been proved [6] that $\hbar\mathbf{k}$ is a quantum of the wave momentum, that is, of the sum of momentum and pseudomomentum. The pseudomomentum conservation law derived below follows directly from the equations of motion, without the use of Noether's theorem. However, Noether's theorem relates the conservation law to the invariance property that gives rise to it and thus provides a firm identification of the conserved quantity.

The pseudomomentum of the system considered here is not in fact a conserved quantity, because of the loss introduced by the optic mode damping. The conservation equation is thus replaced by a continuity equation. To find it from the equations of motion, we need to combine only the internal coordinate equation (3.11) and the version (3.12) of the center-of-mass continuum equation, that is, the material equations. The electromagnetic field equations, which essentially describe vacuum-based rather than material-based quantities, do not contribute. Thus addition of Eq. (3.11), after multiplication by $\partial s_i/\partial r_j$, to Eq. (3.12) gives a result that can be written in the form of the continuity equation

$$\frac{\partial}{\partial r_i}(\mathbf{T}_{\rm psm})_{ji}+\frac{d}{dt}(\mathbf{G}_{\rm psm})_j=m\Gamma\dot{\mathbf{s}}\cdot\frac{\partial\mathbf{s}}{\partial r_j}, \qquad (3.19)$$

where

$$(\mathbf{T}_{\rm psm})_{ji}=\{\tfrac{1}{2}m(\dot{s}^2-\omega_T^2s^2)+(\mathbf{E}+\dot{\mathbf{R}}\times\mathbf{B})\cdot\mathbf{P}\}\delta_{ji} \qquad (3.20)$$

is the pseudomomentum current density, with a term of the same magnitude as the center-of-mass kinetic energy again neglected. The pseudomomentum density is given by

$$(\mathbf{G}_{\rm psm})_j=-M\dot{R}_j-m\dot{\mathbf{s}}\cdot\frac{\partial\mathbf{s}}{\partial r_j}+(\mathbf{P}\times\mathbf{B})_j. \qquad (3.21)$$

The term on the right-hand side of Eq. (3.19) represents the rate of loss of pseudomomentum caused by damping of the optic mode. The total time derivative occurs in accordance with the role of $\mathbf{G}_{\rm psm}$ as a momentum defined relative to the moving medium.

The pseudomomentum current density simplifies when the term that includes the center-of-mass velocity $\dot{R}$ is neglected, and Eq. (3.20) reduces to

$$(\mathbf{T}_{\rm psm})_{ji}=\{\tfrac{1}{2}m(\dot{s}^2-\omega_T^2s^2)+\mathbf{E}\cdot\mathbf{P}\}\delta_{ji}. \qquad (3.22)$$

However, the center-of-mass term in the pseudomomentum density (3.21) is comparable to the other terms, and it must be retained.

D. Wave momentum

It is difficult, if not impossible, to observe the pseudomomentum alone. Any interaction in a large homogeneous body involves a conserved pseudomomentum but, because the underlying vacuum space is also homogeneous, the interaction involves a conserved momentum as well. As similar quantities with identical dimensions, it is not surprising that the momentum and pseudomomentum should combine together in summation to form the wave momentum. Such a summation is valid when the deformation of the material body is negligibly small. Thus the experiments of Jones and coworkers [13] on liquid dielectrics and of Gibson *et al.* [14] on semiconductors observed momentum transfers proportional to the refractive index, in agreement with the form of the unit of wave momentum $\hbar\mathbf{k}$ in a nonabsorbing medium.

The wave momentum current density and momentum density are denoted $\mathbf{T}$ and $\mathbf{G}$, respectively. Dielectric deformation is negligible in the vicinity of the optic mode frequency, and we may add the momentum conservation equation (3.14) to the pseudomomentum continuity equation (3.19), to form the wave momentum continuity equation

$$\frac{\partial T_{ji}}{\partial r_i}+\frac{\partial G_j}{\partial t}=m\Gamma\dot{\mathbf{s}}\cdot\frac{\partial\mathbf{s}}{\partial r_j}. \qquad (3.23)$$

In this relation, the expression for T_{ji} is obtained by addition of the approximations (3.18) and (3.22) for small center-of-mass velocity ($\dot{R}\ll c$) as

$$T_{ji}=-E_jD_i+\tfrac{1}{2}\mathbf{E}\cdot\mathbf{D}\delta_{ji}-\frac{B_jB_i}{\mu_0}+\frac{B^2}{2\mu_0}\delta_{ji}$$
$$+\tfrac{1}{2}(m\dot{s}^2-m\omega_T^2s^2+\mathbf{E}\cdot\mathbf{P})\delta_{ji}, \qquad (3.24)$$

and the expression for G_j is obtained by addition of the general results (3.16) and (3.21) as

$$G_j=(\mathbf{D}\times\mathbf{B})_j-m\dot{\mathbf{s}}\cdot\frac{\partial\mathbf{s}}{\partial r_j}, \qquad (3.25)$$

where $\mathbf{D}$ is the electric displacement defined in Eq. (2.7). It is seen that the center-of-mass momentum cancels in the formation of the wave momentum. The term on the right of Eq. (3.23) represents the rate of loss of wave momentum from the coupled field-lattice system caused by the optic mode damping.

The first four terms in the current density (3.24) have the form of the negative of the Maxwell stress tensor in a dielectric [1]. The remaining bracketted terms are additional, but their cycle-averaged contributions vanish in the examples of monochromatic and pulsed excitations of the system considered in Secs. IV and V, respectively (see also [6]). The wave momentum density (3.25) is the same as the Minkowski expression plus a dispersive term. However, the Minkowski momentum was proposed as an expression for the momentum, not for the sum of momentum and pseudomomentum as embodied in the wave momentum. It is seen from the above expressions that, unlike the energy densities (2.18) and (2.19), neither the momentum current density (3.24) nor the momentum density (3.25) separates into distinct electromagnetic and material contributions, as the polarization **P**, expressed in the form (2.12), is a material variable.

The final results (3.23)–(3.25) can also be obtained directly from the simple theory of Sec. II. Thus multiplication of Eq. (2.9) by $\partial s_i/\partial r_j$ gives

$$m\ddot{s}_i\frac{\partial s_i}{\partial r_j}+m\Gamma\dot{s}_i\frac{\partial s_i}{\partial r_j}+m\omega_T^2 s_i\frac{\partial s_i}{\partial r_j}=\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial r_j}, \qquad (3.26)$$

where Eq. (2.12) has been used. Subtraction of Eq. (3.26) from Eq. (2.21) gives

$$\frac{\partial}{\partial r_i}\{(\mathbf{T}_{\mathrm{em}})_{ji}+\tfrac{1}{2}\,m(\dot{s}^2-\omega_T^2 s^2)\delta_{ji}\}+\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial r_j}+\frac{\partial}{\partial t}\left\{(\mathbf{G}_{\mathrm{em}})_j-m\dot{\mathbf{s}}\cdot\frac{\partial\mathbf{s}}{\partial r_j}\right\}+\rho E_j+(\mathbf{j}\times\mathbf{B})_j=m\Gamma\dot{\mathbf{s}}\cdot\frac{\partial\mathbf{s}}{\partial r_j}. \qquad (3.27)$$

It is not difficult to show with the use of Eqs. (2.3), (2.5), (2.6) and standard vector operator identities that

$$\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial r_j}+\rho E_j+(\mathbf{j}\times\mathbf{B})_j=\frac{\partial}{\partial t}(\mathbf{P}\times\mathbf{B})_j+\frac{\partial}{\partial r_j}(\mathbf{E}\cdot\mathbf{P})-\frac{\partial}{\partial r_i}(E_jP_i). \qquad (3.28)$$

Thus Eq. (3.27) can be written in the form of the continuity equation (3.23) with the same definitions (3.24) and (3.25) of the wave momentum densities. However, in contrast to this direct derivation, the Lagrangian formulation establishes the nature of the wave momentum. Its two distinct contributions, arising from the conserved momentum and the dissipating pseudomomentum, are unambiguously identified. Their separate conservation and continuity properties are expressed by Eqs. (3.14) and (3.19) respectively.

IV. MONOCHROMATIC WAVE

No assumptions have so far been made about the time dependences of the fields. We now evaluate the various densities that have been derived in the previous two sections for the simple example of a monochromatic plane wave. The cycle averages of the various energy and momentum densities associated with the electromagnetic fields and the internal motion of the crystal lattice are all independent of the time in this example, and they have exponentially decaying spatial dependences. The electromagnetic and internal parts of the system are thus subjected to a steady-state excitation, and the monochromatic case usefully displays the full frequency dependences of the energy and momentum densities. However, the constant supply of electromagnetic energy, which maintains the steady state, results in a center-of-mass momentum density that grows linearly with the time, provided that the velocity remains nonrelativistic. The results derived in the present section are valid for arbitrarily strong damping.

A. Dielectric function

Consider a plane wave of frequency ω and wave vector k that is propagated parallel to the z axis with its electric and magnetic vectors oriented in the directions of the x and y axes, respectively. The real electric field is written conventionally as a sum of positive- and negative-frequency contributions,

$$E(z,t)=E^+(z,t)+E^-(z,t)=E^+(\omega)\exp(-i\omega t+ikz)+E^-(\omega)\exp(i\omega t-ikz). \qquad (4.1)$$

Here $E^+(\omega)$ is the complex amplitude at $z=0$, $t=0$,

$$E^-(\omega)=[E^+(\omega)]^*, \qquad (4.2)$$

and a similar notation is used for the other fields. The amplitude of the field at $z=0$ is assumed to be time independent, and the model therefore provides for a constant supply of energy at the coordinate origin. It is emphasized that the dielectric material is still assumed to fill all of space, and the boundary condition at $z=0$ does not imply the existence of any real boundary.

It follows from Eq. (2.9) that

$$s_i^+(\omega)=\frac{\varsigma E_i^+(\omega)/m}{\omega_T^2-\omega^2-i\omega\Gamma}. \qquad (4.3)$$

The electric displacement $D(\omega)$ and the dielectric function $\varepsilon(\omega)$ are defined by

$$D_i^+(\omega)=\varepsilon_0\varepsilon(\omega)E_i^+(\omega)=\varepsilon_0E_i^+(\omega)+P_i^+(\omega), \qquad (4.4)$$

and use of Eqs. (2.12) and (4.3) leads to the explicit expression

$$\varepsilon(\omega)=1+\frac{\varsigma^2}{\varepsilon_0 m}\frac{1}{\omega_T^2-\omega^2-i\omega\Gamma}. \qquad (4.5)$$

The refractive index $\eta(\omega)$ and extinction coefficient $\kappa(\omega)$ are defined in the usual way by

$$\varepsilon(\omega)=[\eta(\omega)+i\kappa(\omega)]^2, \qquad (4.6)$$

and it follows from Eq. (4.5) that

$$\eta(\omega)^2-\kappa(\omega)^2=1+\frac{\varsigma^2}{\varepsilon_0 m}\frac{\omega_T^2-\omega^2}{(\omega_T^2-\omega^2)^2+\omega^2\Gamma^2} \qquad (4.7)$$

and

$$2\eta(\omega)\kappa(\omega)=\frac{\varsigma^2}{\varepsilon_0 m}\frac{\omega\Gamma}{(\omega_T^2-\omega^2)^2+\omega^2\Gamma^2}. \qquad (4.8)$$

The wave vector is given by the usual expression,

$$k=[\eta(\omega)+i\kappa(\omega)]\omega/c, \qquad (4.9)$$

and the complex magnetic and electric field amplitudes are related by

$$B^+(\omega)=[\eta(\omega)+i\kappa(\omega)]E^+(\omega)/c. \qquad (4.10)$$

It is convenient to simplify the expressions that occur in the remainder of the section by removal of explicit ω dependence from the notation for the dielectric properties and field amplitudes.

B. Energy propagation

The total-energy current density (2.18) has only a nonzero z component for the geometry assumed here, and its cycle average is

$$\langle S_z \rangle = 2\varepsilon_0 c\,\eta |E^+|^2 e^{-2\omega\kappa z/c} = 2\varepsilon_0 c\,\eta |E^+|^2 e^{-z/L}, \tag{4.11}$$

where

$$L = c/2\omega\kappa \tag{4.12}$$

is the attenuation length, the distance after which the intensity of an electromagnetic wave in the dielectric decays to $1/e$ of its initial value. The total-energy density (2.19) has a cycle average

$$\langle W \rangle = 2\varepsilon_0 \left(\eta^2 + \frac{2\omega\eta\kappa}{\Gamma} \right) |E^+|^2 e^{-z/L}, \tag{4.13}$$

and we note that it is not possible to express this quantity entirely in terms of macroscopic electromagnetic functions, independent of the parameters of the optic mode. These expressions for the total energy densities agree with a previous derivation [5]. The cycle average of the energy dissipation rate on the right of Eq. (2.17) is

$$-\langle m\Gamma \dot{s}^2 \rangle = -4\varepsilon_0 \omega\,\eta\kappa |E^+|^2 e^{-z/L} = -\frac{2\varepsilon_0 c\,\eta}{L} |E^+|^2 e^{-z/L}. \tag{4.14}$$

The cycle average of the energy continuity equation (2.17) takes the form

$$\frac{\partial \langle S_z \rangle}{\partial z} = -\langle m\Gamma \dot{s}^2 \rangle, \tag{4.15}$$

and it is readily verified from Eqs. (4.11), (4.12) and (4.14) that this relation is indeed satisfied. The energy that is constantly supplied at $z=0$ in the example considered here steadily drains into the reservoir associated with the dissipation, until none is left for propagation distances $z \gg L$. The kinetic energy delivered to the dielectric material also grows steadily in this example, but the material velocity is assumed to be always sufficiently small that the accumulated kinetic energy is negligible.

Just as the ratio of the values of $\langle S_z \rangle$ and $\langle W \rangle$ in a lossless dielectric gives the ray or energy velocity [10], so the ratio of the energy densities is taken to define the velocity v_e of energy transport through the absorbing dielectric as

$$v_e = \frac{\langle S_z \rangle}{\langle W \rangle} = \frac{c}{\eta + (2\omega\kappa/\Gamma)}. \tag{4.16}$$

This can be rearranged in the form

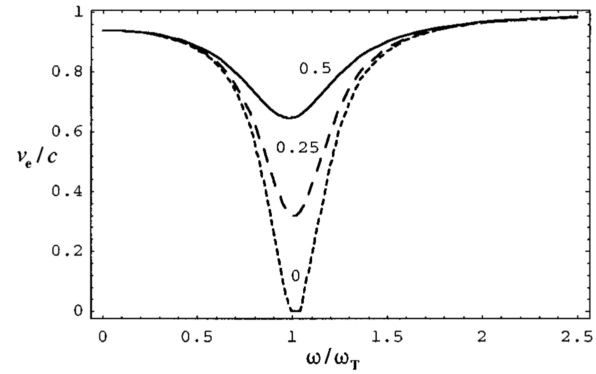

FIG. 1. Frequency dependence of the energy velocity (4.16) for the dielectric function (4.5) with $\varsigma^2/\varepsilon_0 m \omega_T^2 = 0.136$; this gives the ratio of longitudinal to transverse optic mode frequencies $\omega_L/\omega_T = 1.066$ found in GaAs. The curves are labeled with the appropriate values of Γ/ω_T.

$$\frac{1}{v_e} = \frac{1}{v_p} + \frac{1}{L\Gamma}, \tag{4.17}$$

where

$$v_p = c/\eta, \tag{4.18}$$

is the phase velocity. Figure 1 shows the frequency dependence of the energy velocity in the vicinity of the transverse resonance for several values of the damping.

Although derived for a specific model, relation (4.17) between energy and phase velocities, decay length and damping rate is found to apply to a wide range of systems, including the propagation of pulsed optical signals through dielectrics in regions of resonant absorption [15,16] and in regions of resonant amplification [17], self-induced transparency in two-level atoms [18], and energy transport in media containing randomly distributed scatterers [19]. A similar relation is also valid for the propagation of ultrasonic signals [20]. Each of these systems has a detailed theoretical treatment for the relevant attenuation or amplification process, but the derived and measured propagation velocities generally agree with the common form of energy velocity given by Eq. (4.17).

C. Momentum propagation

Momentum. The momentum current density given by Eqs. (3.18) and (2.20) has three nonzero components for the geometry assumed here. The two transverse components have cycle-averaged values

$$\langle (T_m)_{xx} \rangle = \varepsilon_0 (1 - \eta^2 + 3\kappa^2) |E^+|^2 e^{-z/L} \tag{4.19}$$

and

$$\langle (T_m)_{yy} \rangle = \varepsilon_0 (1 - \eta^2 - \kappa^2) |E^+|^2 e^{-z/L}. \tag{4.20}$$

The cycle-averaged longitudinal component is

$$\langle (T_m)_{zz} \rangle = \langle (T_{\rm em})_{zz} \rangle = \varepsilon_0 (1 + \eta^2 + \kappa^2) |E^+|^2 e^{-z/L}. \tag{4.21}$$

The momentum density is given by Eqs. (3.16) and (2.22), and only the z component is nonzero, with the cycle-averaged value

$$\langle (G_m)_z\rangle = \langle M\dot{R}_z\rangle + \langle (G_{\rm em})_z\rangle = \langle M\dot{R}_z\rangle + \frac{2\varepsilon_0\eta}{c}|E^+|^2 e^{-z/L}. \tag{4.22}$$

The cycle averages of the continuity equation (2.21) for the electromagnetic momentum and of the conservation equation (3.14) for the momentum lead to the equalities

$$\langle F_z\rangle = -\frac{\partial}{\partial z}\langle (T_{\rm em})_{zz}\rangle = -\frac{\partial}{\partial z}\langle (T_m)_{zz}\rangle = \frac{\partial}{\partial t}\langle M\dot{R}_z\rangle \tag{4.23}$$

for the monochromatic wave excitation considered here. It is readily verified that the explicit expression for the cycle-averaged Lorentz force density, obtained from Eq. (2.21) as $\langle jB\rangle$, agrees with that obtained from the momentum current density component given in Eq. (4.21). The time dependence of the center-of-mass momentum density is thus obtained by integration of Eq. (4.23) as

$$\langle M\dot{R}_z(z,t)\rangle = (\varepsilon_0 t/L)(1+\eta^2+\kappa^2)|E^+|^2 e^{-z/L}, \tag{4.24}$$

and this quantity vanishes in the limit of a lossless dielectric as $L\rightarrow\infty$. The dielectric material is here assumed to be a rigid body, and the total momentum transferred to the unit cross-sectional area at time t is obtained by integration of Eq. (4.24) as

$$\int_0^\infty dz\langle M\dot{R}_z(z,t)\rangle = \varepsilon_0 t(1+\eta^2+\kappa^2)|E^+|^2. \tag{4.25}$$

The material center-of-mass momentum thus grows linearly with the time, as momentum is steadily transferred from field to dielectric. The total momentum transfer vanishes in the limit of a lossless dielectric, and the apparent nonzero result obtained from Eq. (4.25) for $\kappa\rightarrow 0$ is an artifact of the prior integration over an infinite extent of the medium. The conservation law for the momentum density given in Eq. (3.17) does not hold for the open system considered here, where there is a steady input of electromagnetic energy and momentum.

Pseudomomentum. The pseudomomentum current density given by Eq. (3.22) is the same for all three diagonal components, and its cycle average is

$$\langle (T_{\rm psm})_{ii}\rangle = \varepsilon_0(-1+\eta^2-\kappa^2)|E^+|^2 e^{-z/L}, \quad i=x,y,z. \tag{4.26}$$

The cycle average of the pseudomomentum density given by Eq. (3.21) has only the z component

$$\langle (G_{\rm psm})_z\rangle = -\langle M\dot{R}_z\rangle + \frac{2\varepsilon_0\eta}{c}\left(-1+\eta^2+\kappa^2+\frac{2\omega\eta\kappa}{\Gamma}\right) \times |E^+|^2 e^{-z/L}. \tag{4.27}$$

The cycle average of the pseudomomentum dissipation rate that appears on the right-hand side of Eq. (3.19) also has only the z component

$$\left\langle m\Gamma\dot{s}\frac{\partial s}{\partial z}\right\rangle = -\frac{4\varepsilon_0\omega\eta^2\kappa}{c}|E^+|^2 e^{-z/L} = -\frac{2\varepsilon_0\eta^2}{L}|E^+|^2 e^{-z/L}. \tag{4.28}$$

The cycle average of the pseudomomentum continuity equation Eq. (3.19) thus takes the form

$$\frac{\partial}{\partial z}\langle (T_{\rm psm})_{zz}\rangle - \frac{\partial}{\partial t}\langle M\dot{R}_z\rangle = \left\langle m\Gamma\dot{s}\frac{\partial s}{\partial z}\right\rangle, \tag{4.29}$$

and this is seen to agree with Eqs. (4.23) and (4.24) when the cycle averages (4.26) and (4.28) are substituted.

Wave momentum. The cycle averages of the various wave momentum densities are now obtained by summation of the momentum and pseudomomentum contributions, and the results are

$$\langle T_{xx}\rangle = -\langle T_{yy}\rangle = 2\varepsilon_0\kappa^2|E^+|^2 e^{-z/L} \tag{4.30}$$

and

$$\langle T_{zz}\rangle = 2\varepsilon_0\eta^2|E^+|^2 e^{-z/L} \tag{4.31}$$

for the wave momentum current density and

$$\langle G_z\rangle = \frac{2\varepsilon_0\eta}{c}\left\{\eta^2+\kappa^2+\frac{2\omega\eta\kappa}{\Gamma}\right\}E^+|^2 e^{-z/L} \tag{4.32}$$

for the wave momentum density. The cycle average of the wave momentum continuity equation (3.23) takes the form

$$\frac{\partial\langle T_{zz}\rangle}{\partial z} = \left\langle m\Gamma\dot{s}\frac{\partial s}{\partial z}\right\rangle, \tag{4.33}$$

and it is readily verified from Eqs. (4.12), (4.28), and (4.31) that this relation is indeed satisfied.

As with the energy velocity (4.16), it is possible to define a velocity $v_{\rm wm}$ of wave momentum transport through the absorbing dielectric in the direction of the z axis as

$$v_{\rm wm} = \frac{\langle T_{zz}\rangle}{\langle G_z\rangle} = \frac{c\eta}{\eta^2+\kappa^2+(2\omega\eta\kappa/\Gamma)}. \tag{4.34}$$

This can be rearranged in the form

$$\frac{1}{v_{\rm wm}} = \frac{1}{v_p} + \frac{1}{L\Gamma} + \frac{\kappa^2}{c\eta} = \frac{1}{v_e} + \frac{\kappa^2}{c\eta}, \tag{4.35}$$

with a term additional to expression (4.17) for the energy velocity. The wave momentum velocity is thus in general smaller than the energy velocity, and Fig. 2 shows the frequency dependence of the difference between the two. The differences are small for the chosen parameters, but they could be significant for larger damping.

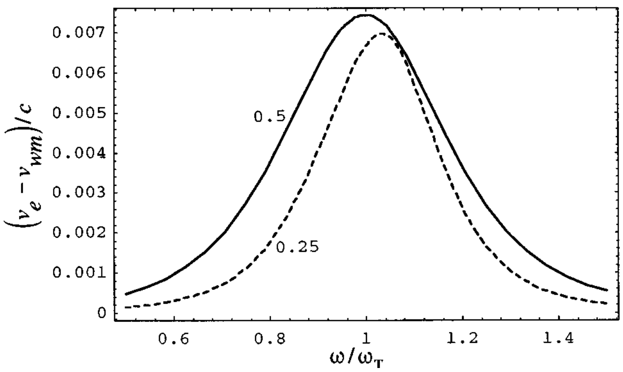

FIG. 2. Frequency dependence of the difference between wave momentum and energy velocities for the same parameters as Fig. 1.

D. Limit of zero damping

It is instructive to consider the forms of the various densities defined above in the limit of zero damping $[\Gamma \to 0, \kappa(\omega) \to 0]$, when the refractive index is obtained from Eq. (4.7) as

$$\eta^2 = 1 + \frac{s^2}{\varepsilon_0 m} \frac{1}{\omega_T^2 - \omega^2}. \tag{4.36}$$

Thus, for a lossless dielectric, the cycle-average energy current density (4.11) can be simply expressed in terms of the phase velocity (4.18) as

$$\langle S_z \rangle = \frac{2\varepsilon_0 c^2}{v_p} |E^+|^2. \tag{4.37}$$

The group velocity is defined by

$$\frac{c}{v_g} = \frac{\partial}{\partial \omega}(\omega \eta) = \frac{c}{v_p} + \frac{v_p}{c} \frac{s^2}{\varepsilon_0 m} \frac{\omega^2}{(\omega_T^2 - \omega^2)^2}, \tag{4.38}$$

and it is easily verified with the use of Eqs. (4.7) and (4.8) that

$$\underset{\Gamma \to 0}{Lt} \left(\eta^2 + \frac{2\omega \eta \kappa}{\Gamma} \right) = \eta \frac{\partial}{\partial \omega}(\omega \eta) = \frac{c^2}{v_p v_g}. \tag{4.39}$$

The lossless limit of the cycle-average energy density (4.13) is thus

$$\langle W \rangle = \frac{2\varepsilon_0 c^2}{v_p v_g} |E^+|^2. \tag{4.40}$$

The cycle-averaged dissipation rate (4.14) of course vanishes, and the energy velocity (4.16) becomes the same as the group velocity in the limit of zero damping. This is shown as the $\Gamma = 0$ curve in Fig. 1.

In the absence of any material boundaries, and hence of any reflection of the incident electromagnetic wave, no momentum is transferred from the electromagnetic field to the center-of-mass motion of a lossless dielectric. Thus $\dot{R}$ can everywhere be set equal to zero. The cycle-average momentum densities (4.21) and (4.22) become

$$\langle (T_m)_{zz} \rangle = \varepsilon_0 \left(1 + \frac{c^2}{v_p^2} \right) |E^+|^2 \tag{4.41}$$

and

$$\langle (G_m)_z \rangle = \frac{2\varepsilon_0}{v_p} |E^+|^2. \tag{4.42}$$

With no motion of the center of mass, it is possible to define a momentum velocity as

$$v_m = \frac{\langle (T_m)_{zz} \rangle}{\langle (G_m)_z \rangle} = \left(1 + \frac{c^2}{v_p^2} \right) \frac{v_p}{2}. \tag{4.43}$$

The momentum densities both take their usual free-space values when the phase velocity is set equal to c, and the momentum velocity also becomes equal to c.

The pseudomomentum densities (4.26) and (4.27) become

$$\langle (T_{\mathrm{psm}})_{zz} \rangle = \varepsilon_0 \left(-1 + \frac{c^2}{v_p^2} \right) |E^+|^2 \tag{4.44}$$

and

$$\langle (G_{\mathrm{psm}})_z \rangle = \frac{2\varepsilon_0}{v_p} \left(-1 + \frac{c^2}{v_p v_g} \right) |E^+|^2, \tag{4.45}$$

where the limit given in Eq. (4.39) is used in the latter. A pseudomomentum velocity can thus be defined by

$$v_{\mathrm{psm}} = \frac{\langle (T_{\mathrm{psm}})_{zz} \rangle}{\langle (G_{\mathrm{psm}})_z \rangle} = \frac{c^2 - v_p^2}{c^2 - v_g v_p} \frac{v_g}{2}. \tag{4.46}$$

The pseudomomentum densities both vanish in free space.

The cycle-averaged wave momentum densities (4.31) and (4.32) become

$$\langle T_{zz} \rangle = \frac{2\varepsilon_0 c^2}{v_p^2} |E^+|^2 = \frac{\langle S_z \rangle}{v_p} \tag{4.47}$$

and

$$\langle G_z \rangle = \frac{2\varepsilon_0 c^2}{v_p^2 v_g} |E^+|^2 = \frac{\langle W \rangle}{v_p}. \tag{4.48}$$

The cycle-averaged dissipation rate (4.28) vanishes in the limit of zero damping, and the wave momentum velocity (4.34) reduces, like the energy velocity, to the group velocity

$$v_{\mathrm{wm}} = v_e = v_g. \tag{4.49}$$

This velocity can be expressed in the form

$$v_{\mathrm{wm}} = \frac{\langle (G_m)_z \rangle v_m + \langle (G_{\mathrm{psm}})_z \rangle v_{\mathrm{psm}}}{\langle (G_m)_z \rangle + \langle (G_{\mathrm{psm}})_z \rangle} \tag{4.50}$$

of a sum of the momentum and pseudomomentum velocities weighted by their respective densities. However, although the wave momentum velocity has the well-behaved form shown by the $\Gamma = 0$ curve in Fig. 1, the momentum velocity (4.41) diverges at both the transverse and longitudinal frequencies while the pseudomomentum velocity (4.46) diverges at the longitudinal frequency and is negative there and at all higher frequencies. The unphysical aspects of these velocities are discussed in Sec. VI.

V. OPTICAL PULSE

The example of a constantly sustained monochromatic wave treated in Sec. IV is somewhat untypical; the energy and momentum densities are independent of the time, except for the center-of-mass momentum which increases linearly. We now consider the more realistic example of a finite optical pulse which is initiated at the origin of coordinates, and left to decay as the dissipation takes effect. It is difficult to treat the propagation of a general optical pulse. We here choose a Gaussian envelope whose parameters have relative magnitudes that are convenient both for evaluation of the various integrals that occur in the theory and for illustration of the effects of dissipation. Specifically, the frequency spread of the pulse is assumed to be much smaller than its central frequency, and its spatial length is assumed to be much smaller than the optical attenuation length of the medium. The effects of loss are included only in the decay of the optical pulse and in the eventual transfer of its initial momentum to the medium. The results derived in the present section are thus valid only for weak damping.

A. Gaussian pulse

The coordinate axes are as defined in Sec. IV A. The real electric field associated with an optical pulse continues to have the form in the first line of Eq. (4.1), but its positive-frequency part is generalized to

$$E^{+}(z,t)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}d\omega\; E^{+}(z,\omega)\exp(-i\omega t+ikz), \tag{5.1}$$

and the negative-frequency part is given by the complex conjugate expression. For a Gaussian pulse, we choose

$$E^{+}(z,\omega)=\frac{lE^{+}}{\sqrt{2}c}\exp\left(-\frac{l^{2}(\omega-\omega_{0})^{2}}{4c^{2}}\right), \tag{5.2}$$

where l is the spatial length of the pulse. Its central frequency is assumed to be very much larger than the frequency width,

$$\omega_{0}\gg c/l, \tag{5.3}$$

and possible numerical values of these and other parameters are considered in the Appendix. It is also assumed that the refractive index and extinction coefficient satisfy the inequality

$$\eta(\omega)\gg\kappa(\omega). \tag{5.4}$$

The refractive index is assumed to vary slowly over the frequency spread c/l, so that it can be expanded around ω_0 as

$$\omega\eta(\omega)\approx\omega_{0}\eta(\omega_{0})+(\omega-\omega_{0})\left.\frac{\partial(\omega\eta(\omega))}{\partial\omega}\right|_{\omega_{0}}$$

$$=\omega_{0}\eta_{0}+(\omega-\omega_{0})\frac{c}{v_{g}}, \tag{5.5}$$

where $\eta_0\equiv\eta(\omega_0)$ and the group velocity v_g is defined in Eq. (4.38). The extinction coefficient is assumed to have negligible dispersion around its value κ_0 at frequency ω_0.

With these assumptions and approximations, and with k given by Eq. (4.9), the positive-frequency field (5.1) can be put in the form

$$E^{+}(z,t)=\frac{lE^{+}}{2\sqrt{\pi}c}\exp\left\{-i\omega_{0}\left(t-\frac{z}{v_{p}}\right)-\frac{\omega_{0}\kappa_{0}z}{c}+\frac{\kappa_{0}^{2}z^{2}}{l^{2}}+\frac{2ic\kappa_{0}z}{l^{2}}\left(t-\frac{z}{v_{g}}\right)\right\}\times\int_{-\Omega_{0}}^{\infty}d\Omega\,\exp\left\{-i\Omega\left(t-\frac{z}{v_{g}}\right)-\frac{l^{2}\Omega^{2}}{4c^{2}}\right\}, \tag{5.6}$$

where $v_p=c/\eta_0$ is the phase velocity, and $\Omega=\omega-\Omega_0$ with

$$\Omega_{0}=\omega_{0}-\frac{2c\kappa_{0}z}{l^{2}}. \tag{5.7}$$

As is discussed in the Appendix, we may assume that the second term on the right of Eq. (5.7) is much smaller than the first, and with the inequality (5.3), the lower limit on the integral in Eq. (5.6) is effectively $-\infty$. The third and fourth terms in the first exponent of Eq. (5.6) are correspondingly negligible compared to the second term. The positive-frequency field thus takes the form

$$E^{+}(z,t)=E^{+}\exp\left\{-i\omega_{0}\left(t-\frac{z}{v_{p}}\right)-\frac{z}{2L_{0}}-\frac{c^{2}}{l^{2}}\left(t-\frac{z}{v_{g}}\right)^{2}\right\} \tag{5.8}$$

to a very good approximation. It shows the familiar properties of a phase that propagates with the phase velocity, a magnitude that diminishes with the characteristic attenuation length $2L_0=c/\omega_0\kappa_0$ and a peak that propagates with the group velocity. The corresponding magnetic field is obtained from the Maxwell equation (2.3) and, with the various inequalities assumed above, it takes the approximate form

$$B^{+}(z,t)=E^{+}(z,t)/v_{p}. \tag{5.9}$$

A good approximation to the relative spatial displacement field of the two ions in the primitive cell is obtained from Eq. (2.9) as

$$s^{+}(z,t)=\frac{\mathrm{s}}{m(\omega_{T}^{2}-\omega_{0}^{2})}E^{+}(z,t). \tag{5.10}$$

B. Energy propagation

The cycle-average value of the total-energy current density obtained from Eq. (2.18) with use of the fields (5.8) and (5.9) is

$$\langle S_z\rangle=\frac{2\varepsilon_0 c^2}{v_p}|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.11}$$

The cycle-average energy density is obtained from Eq. (2.19) with the use of the fields (5.8)–(5.10), the low-damping form (4.36) of the refractive index, and the group velocity (4.38) as

$$\langle W\rangle=\frac{2\varepsilon_0 c^2}{v_p v_g}|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.12}$$

It is seen that the energy current density and the energy density are given by the zero-damping expressions for a monochromatic wave, Eqs. (4.37) and (4.40) respectively, but with additional exponential factors that describe the effects of the damping to lowest order and the Gaussian shape of the pulse envelope. The energy velocity defined as in Eq. (4.16) is the same as the group velocity in the low-damping limit considered here.

The energy continuity equation (2.17) is easily verified for the Gaussian pulse. Thus the left-hand side of the continuity equation obtained with the use of Eqs. (5.11) and (5.12) is

$$\frac{\partial\langle S_z\rangle}{\partial z}+\frac{\partial\langle W\rangle}{\partial t}=-4\varepsilon_0\omega_0\eta_0\kappa_0|E^+|^2 \times\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}, \tag{5.13}$$

and the same expression is obtained for the right-hand side of Eq. (2.17) with the use of Eqs. (4.8) and (5.10) in the low-damping limit.

C. Momentum propagation

Momentum. We consider only the zz component of the momentum current density, whose cycle average obtained from Eqs. (3.18) and (2.20) with the use of the fields (5.8) and (5.9) is

$$\langle (T_m)_{zz}\rangle=\langle (T_{\rm em})_{zz}\rangle =\varepsilon_0\left(1+\frac{c^2}{v_p^2}\right)|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.14}$$

The momentum density obtained from Eqs. (3.16) and (2.22) is

$$\langle (G_m)_z\rangle=\langle M\dot{R}_z\rangle+\langle (G_{\rm em})_z\rangle =\langle M\dot{R}_z\rangle+\frac{2\varepsilon_0}{v_p}|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.15}$$

The total field momentum at time t is therefore

$$\int_{-\infty}^{\infty}dz\langle (G_{\rm em})_z\rangle=\frac{\sqrt{2\pi}\varepsilon_0 l v_g}{c v_p}|E^+|^2\exp\left\{-\frac{v_g t}{L_0}+\frac{l^2 v_g^2}{8c^2L_0^2}\right\}. \tag{5.16}$$

The second term in the exponent can be neglected, as it is of order 10^{-3} for the parameter values given in the Appendix. The total field momentum is therefore approximately

$$\int_{-\infty}^{\infty}dz\langle (G_{\rm em})_z\rangle=\frac{\sqrt{2\pi}\varepsilon_0 l v_g}{c\nu_p}|E^+|^2e^{-v_g t/L_0}, \tag{5.17}$$

and the transfer of momentum from the electromagnetic field to the center-of-mass motion is characterized by the time scale L_0/v_g.

The relation (4.23) between the cycle-averaged densities of the Lorentz force and the various momenta and momentum currents is generalized to

$$\langle F_z\rangle=-\frac{\partial}{\partial z}\langle (T_{\rm em})_{zz}\rangle-\frac{\partial}{\partial t}\langle (G_{\rm em})_z\rangle=\frac{\partial}{\partial t}\langle M\dot{R}_z\rangle \tag{5.18}$$

for the pulse excitation. This equation can in principle be integrated to determine the time dependence of the center-of-mass momentum, analogous to Eq. (4.24) for the monochromatic wave, but the integral cannot be performed analytically for the Gaussian pulse. However, with the dielectric material assumed to be in the form of a rigid body, its total momentum at time t is again the main quantity of interest and this can be calculated by integration of Eq. (5.18). More straightforwardly, the pulse excitation satisfies the conditions for validity of Eq. (3.17) and, if the center of mass is stationary at time $t=0$, the total center-of-mass momentum at time t is simply obtained as

$$\int_{-\infty}^{\infty}dz\langle M\dot{R}_z\rangle=\int_{-\infty}^{\infty}dz\langle (G_{\rm em}(t))_z-(G_{\bf em}(0))_z\rangle =\frac{\sqrt{2\pi}\varepsilon_0 l v_g}{c v_p}|E^+|^2(1-e^{-v_g t/L_0}). \tag{5.19}$$

Thus, after a time much longer than L_0/ν_g, the center of mass acquires all of the momentum initially carried by the electromagnetic field.

Pseudomomentum. The cycle average of the zz component of the pseudomomentum current density is obtained from Eq. (3.22) with the use of Eqs. (2.12) and (5.10) as

$$\langle (T_{\rm psm})_{zz}\rangle=\varepsilon_0\left(-1+\frac{c^2}{v_p^2}\right)|E^+|^2 \times\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.20}$$

The cycle average of the z component of the pseudomomentum density is similarly obtained from Eq. (3.21), where only the dominant terms are retained in the derivatives of the relative spatial displacement field given by Eq. (5.10). Then with the use of the explicit expression for the group velocity given in Eq. (4.38),

$$\langle (G_{\rm psm})_z\rangle = -\langle M\dot{R}_z\rangle + \frac{2\varepsilon_0}{v_p}\left(\frac{c^2}{v_p v_g}-1\right)|E^+|^2 \times \exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.21}$$

The cycle average of the pseudomomentum dissipation rate on the right-hand side of Eq. (3.19) is

$$\left\langle m\Gamma\dot{s}\,\frac{\partial s}{\partial z}\right\rangle = -\frac{2\varepsilon_0 c^2}{v_p^2 L_0}\,|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.22}$$

Insertion of these expressions into the pseudomomentum continuity equation (3.19) reproduces essentially the same equation of motion for the center-of-mass momentum density as is found from Eq. (5.18).

Wave momentum. The cycle averages of the wave momentum densities are again obtained by addition of the momentum and pseudomomentum contributions, and the results are

$$\langle T_{zz}\rangle = \frac{2\varepsilon_0 c^2}{v_p^2}\,|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\} \tag{5.23}$$

and

$$\langle G_z\rangle = \frac{2\varepsilon_0 c^2}{v_p^2 v_g}\,|E^+|^2\exp\left\{-\frac{z}{L_0}-\frac{2c^2}{l^2}\left(t-\frac{z}{v_g}\right)^2\right\}. \tag{5.24}$$

The cycle-averaged wave momentum dissipation rate is given by Eq. (5.22), and it is easily verified that these expressions satisfy the wave momentum continuity equation (3.23). The wave momentum velocity defined in Eq. (4.34) is the same as the group velocity for the pulse excitation in the low-damping limit considered here.

VI. DISCUSSION

A recent study based on a Lagrangian formulation and use of Noether's theorem has elucidated the meanings and relationships of momentum, pseudomomentum, and wave momentum [6], but it has led to new questions that we have attempted to answer in the present work. If optical loss is added, how are the conservation laws of the various momentumlike quantities affected? Can the earlier general results be clarified by restriction to a linear light wave? Can velocities of the various momentumlike quantities be defined, interpreted, and related to energy, group, and phase velocities? Some time ago, similar questions posed about energy propagation [5] were addressed, and these are reconsidered in Sec. II. It is sensible to relate the queries concerning momentum, and their solutions, to those for the energy.

The general conclusions of the recent study [6] can be briefly summarized. Momentum conservation results, by Noether's theorem, from an invariance of the laws of physics to an arbitrary infinitesimal displacement of the spatial coordinates, which expresses the homogeneity of space. Similarly, pseudomomentum conservation results from an invariance to an arbitrary infinitesimal displacement of the material coordinates that label a mass point, which expresses the homogeneity of the material body. As there are only two frames of reference, there can be only two such conservation laws, and they are naturally expressed in different coordinate systems. However, they can be added after each has been specialized to the interaction under study. It is believed that pseudomomentum cannot be observed alone because underlying every homogeneous body lies the homogeneous space, and momentum and pseudomomentum should be observed in summation inside a homogeneous body. The sum of the two for a light wave in a homogeneous body was shown to have quanta of $\hbar\mathbf{k}$ per photon, and for this reason the sum is named wave momentum. This property demonstrates the primary importance of wave momentum, because $\hbar\mathbf{k}$ often plays a crucial role in determining phase-matching conditions in wave and quantum interactions.

The energy, momentum, and pseudomomentum conservation laws are derived in Secs. II and III of the present paper for a linear light wave in a homogeneous and isotropic dielectric medium with a single group of three dipole-active optic modes. Optical loss enters through the equation of motion for the optic mode. The wave momentum conservation law is found by addition of those for the momentum and pseudomomentum. It is found that the energy loss does not affect the momentum conservation, a situation frequently encountered in mechanics. It does however affect the energy, pseudomomentum, and wave momentum conservation laws by adding a loss term and converting the conservation laws to continuity relations.

The momentum conservation law and the three continuity relations are examined in Sec. IV for a plane light wave using cycle averaging. This permits analysis of the density, the flow, and, where appropriate, the dissipation of these quantities. A meaningful velocity for a wave is often found from the ratio of the cycle-averaged flow to the cycle-averaged density of a quantity. This is true for the energy, where it is called the energy or ray velocity. An interesting generalization of the energy velocity for the lossy medium is reproduced in Sec. IV from previous work [5]. The energy velocity can never exceed the velocity of light in vacuum because of relativity. In the lossless limit this velocity becomes the group velocity. It is worth remarking that the energy and group velocities are distinct concepts, and are equal only for systems that conserve energy, momentum, and pseudomomentum. For nonlinear systems, this is only so if the cycle-averaged Lagrangian divided by the frequency is held constant during the differentiation in the calculation of the group velocity [10].

The definition of the group velocity on the left of Eq. (4.38) remains valid for a lossy medium, with $\eta(\omega)$ taken to be the real refractive index as in Eq. (4.6). The group velocity is not restricted to be smaller than the velocity of light, and its most basic interpretation is the velocity of the peak of a pulse. This peak velocity has been observed in some remarkable experiments [21] to exceed the velocity of light, become infinite, and then negative, which corresponds to the emergence of the peak of the pulse from a slab before its entry into it. It has been shown [22] that none of those occurrences violates Maxwell's equations, causality, or relativity. Similar observations of propagation at the group velocity have been made for single-photon pulses [23].

We find that the presence of loss causes a transfer of momentum and pseudomomentum from the electromagnetic field to the center-of-mass motion of the body where, for convenience, a rigid body is assumed. As this motion is quite distinct from, and slower than, that of the electromagnetic field, the transferred contributions to the momentum and pseudomomentum densities prevent meaningful definitions of velocities for either of these momenta in the presence of loss. However, the center-of-mass contribution cancels from the wave momentum density. Consequently a meaningful definition of a velocity of wave momentum in the presence of loss can be formed from the ratio of its flow and density. For a monochromatic wave, this velocity is found to have a very simple relationship, Eq. (4.35), to the energy velocity in the presence of loss.

The center-of-mass motion disappears in the absence of loss, and in this situation it may be expected that meaningful velocities can be defined for all three kinds of momentum. However, we have shown that the momentum velocity (4.43) and the pseudomomentum velocity (4.46) suffer from unphysical infinities, and the latter from negative values, which are difficult to interpret. These behaviors seem to be consequences of the delicate and unphysical nature of the zero damping limit. As we discuss after Eq. (4.25), momentum transfer to the medium remains in the limit of zero loss if the size of the medium is made infinite before the damping is removed. Further, it would be surprising for a velocity to be meaningless in the presence of ever-diminishing loss and then to become suddenly meaningful in the zero-loss limit. More generally, the presence of dispersion without loss violates the Kramers-Kronig relations [2]; the momentum is evidently peculiarly sensitive to such a violation. The wave momentum velocity remains well behaved in the lossless limit when both it and the energy velocity become the group velocity. The existence of the wave momentum velocity in both absorbing and nonabsorbing dielectrics, in the absence of valid momentum and pseudomomentum velocities, is additional evidence for the uniquely important role of wave momentum in interactions within homogeneous bodies.

The monochromatic wave treated in Sec. IV gives rise to steady-state energy and momentum densities, except that the center-of-mass momentum grows linearly with the time as the constant energy supplied to the electromagnetic wave is dissipated in the medium. The Gaussian pulse treated in Sec. V gives an additional perspective to the transfer of momentum. Here the pulse is assumed to be short compared to the characteristic decay length of the medium, and the loss is taken into account only in the decay of the initial pulse and the transfer of its momentum to the medium. In contrast to the monochromatic wave, all of the densities are now time dependent, and all of the initial field momentum is transferred to the center of mass after a sufficiently long time. Once again only the energy and wave momentum velocities can be defined, and, because of the restricted inclusion of loss, both propagate at the group velocity.

Finally we comment on the relation of our calculations to an earlier paper on the conservation of momentum and pseudomomentum, which considers only a nondispersive and lossless medium. The general definition of momentum in Sec. 2 of [24] is similar to ours, but the definition of pseudomomentum involves simultaneous displacements of both spatial and material coordinates to give a quantity more akin to our wave momentum. We believe, however, that the invariances with respect to the two coordinate systems, and the corresponding momenta, are generally distinct and should be treated separately. Their combination to form the wave momentum is valid only in special conditions, as discussed in Sec. IV D. Nevertheless, with the electrostrictive term removed, expression (10.5) in [24] for the momentum associated with a travelling wave agrees with our wave momentum density (4.48), rather than our momentum density (4.42). Also, it appears from (10.4) and the equation following it in [24], that momentum can be transferred from a light wave to a lossless dielectric with no boundaries, contrary to our conclusions. We are thus unable to make an unambiguous comparison of the two formalisms. We believe, however, that the treatment given in the present paper provides a rigorous and detailed account of the propagation of electromagnetic momentum through dispersive and absorbing dielectrics.

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APPENDIX

We consider reasonable values for the parameters of a Gaussian pulse envelope that justify the approximations made in Sec. V. For a central frequency in the visible region of the spectrum, we take

$$\omega_0 \approx 3\times 10^{15}\ \mathrm{s}^{-1} \tag{A1}$$

and a much smaller frequency width is typically

$$c/l \approx 3\times 10^{10}\ \mathrm{s}^{-1}. \tag{A2}$$

The spatial length of the pulse is thus

$$l \approx 10^{-2}\ \mathrm{m}. \tag{A3}$$

The refractive index and extinction coefficient are taken to be

$$\eta_0 \approx 1.5 \quad \text{and} \quad \kappa_0 \approx 5\times 10^{-7}. \tag{A4}$$

The intensity decay length is then

$$L_0 = c/2\omega_0\kappa_0 \approx 10^{-1}\ \mathrm{m}, \tag{A5}$$

which is conveniently much larger than the length of the pulse. The optical pulse has effectively totally dissipated after a propagation distance z of 1 m, and, for this distance,

$$2c\kappa_0 z/l^2 \approx 3\times 10^{6}\ \mathrm{s}^{-1}, \tag{A6}$$

which is negligible compared to ω_0.

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