Physical cause of group velocity in normally dispersive, nondissipative media

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A physical explanation for group velocity is given for the very common case of group velocity less than the phase velocity with negligible absorption. The underlying physics is that a nonpropagating or immobilized energy density must be present in the medium associated with the presence of the wave field. The individual waves carry energy forward in a wave packet and energize the medium in the forward portion of the packet. In the rear portion of the packet, the individual waves grow at the expense of the energy immobilized in the medium, the particle motions in the medium being in phase with the wave field and therefore capable of generating new waves in phase with the group of waves that produced them. The physics is conveniently described in terms of deep water waves, but the principles involved apply equally well to other dispersive, nonabsorbing media, including plasmas, dielectrics, and waveguides. The flux of energy can be expressed as the phase velocity times that part of the energy density that propagates with the individual waves or, averaged over a wave period, as the total energy density (including the energy immobilized in the medium) times the group velocity. This eliminates the confusion commonly present when an attempt is made to interpret the Poynting flux for electromagnetic waves in terms of the product of an energy density and the group velocity. The group velocity is just the weighted average velocity.

Group velocity in wave fields is well described mathematically in terms of interference between frequency components that propagate with slightly different velocities. However, this conceals the basic physical cause of the phenomenon. Considerable insight into the physics of propagation of a wave packet or group can be provided, at least for the case of normal dispersion in nonabsorbing media, by considering energy densities and fluxes. The purpose here is not to set forth a discussion of group velocity applicable to all situations, but rather to describe its physical cause in a relatively simple but frequently encountered situation—that of normal dispersion in a nondissipative medium.

The basic cause of group velocity in the very commonly encountered case of normal dispersion with negligible absorption is that, in order for individual waves to propagate, the medium has to be energized with energy that does not propagate with the individual waves. This nonpropagating energy is proportional to the energy that does propagate, and it appears in different forms in different types of waves (e.g., deep water waves and electromagnetic waves) and in different media (e.g., dielectrics and plasmas). With normal dispersion, the group velocity is less than the phase velocity, and within a group of waves the individual waves move forward in the group, dying out in the forward portion of the group while new waves form and grow in the trailing portion of the group. The waves in the forward portion of the group advance into regions where the medium has not yet been energized sufficiently to correspond to the amplitude of the approaching waves; here, the wave field attenuates by conversion of propagating into nonpropagating energy. As individual waves move forward in the trailing portion of the group, excess organized energy would be left behind in the medium except for the fact that it is converted into new waves. These new waves are in phase with the wave field that preceded them, the phase of the particle motions in the medium having been determined by the wave field that has already passed by. This explains why a wave packet, which is a localized phenomenon, can be described in terms of wave trains of infinite extent.

There is an ambiguity in the term propagation; it may refer to the propagation of individual waves or to the propagation of a group of waves—a wave packet. Here, we use the term to mean wave propagation, which occurs at the phase velocity. The term is also appropriately applied to the movement of a wave packet, which occurs at the group velocity, but here we will avoid such usage. Failure to recognize the ambiguity can lead to confusion.

I. DEEP WATER WAVES

We will describe the phenomenon of group velocity of a wave group first in terms of deep water waves, which are easily observed; the same concepts apply to electromagnetic waves, although the ratios of propagating to immobilized energy, and of phase velocity to group velocity, are different. The particle motions in small-amplitude deep-water waves are circular, and the particle velocities remain constant in magnitude as the individual waves pass by. The velocities decay exponentially with depth, the decay length being equal to the wavelength divided by 2π. Thus the waves in a constant-amplitude portion of a wave packet have the peculiar property of having no variation in kinetic energy W_s in the direction of propagation, and the progress of the individual waves involves no average forward transport of kinetic energy. The energy that is transported by the individual waves is potential energy W_p, which has maxima at the wave crests and troughs. The ratio of phase velocity v_p to group velocity v_g is 2, independent of the wavelength.

The upper portion of Fig. 1 represents a wave group containing five waves. The lower portion shows the instantaneous distribution of potential energy density W_p within the wave group, as well as its value $\langle W_p \rangle$ averaged over a wave period. As this group of waves passes a fixed point, ten individual waves pass that point, not just the five that are visible at any one time. This is the natural consequence of the ratio of phase to group velocity being equal to two. Crawford[1] has suggested a simple experiment that permits observation of this property. If a series of ten waves is launched in a reflecting pool, the result is a wave group containing only five individual waves. But, as the group passes any fixed point, ten individual waves pass that point.

The above presents what at first sight appears to be an anomaly; the energy transported past a point by the individual waves as the group passes by is twice the total amount of energy shown in Fig. 1 and identified as propagating with the waves. The individual waves transferring potential energy in the direction of propagation, by their growth and dissipation, move the kinetic energy forward; the kinetic energy does not propagate along with the individual waves, but it is created in the forward portion of the group and converted back into propagating energy in the rear portion of the group. In that way, the pattern of kinetic energy is transported along with the group. The kinetic energy varies with position within the group and it is equal to the potential energy averaged over a wave period (see Appendix A for this and other mathematical properties of deep water waves); hence the curve in Fig. 1 for average potential energy $\langle W_p \rangle$ also represents the instantaneous value of the kinetic energy W_s. Thus the total energy passing a point as the group passes by is equal to the total energy in the group—the sum of the immobilized energy plus the propagating energy, or the kinetic energy plus the potential. The rate of energy flow averaged over a wave period can be represented equally well as the product of the phase velocity and the average propagating energy density $\langle W_p \rangle$ or as the product of the group velocity and the average total energy density $\langle W_t \rangle = \langle W_p \rangle + W_s$. The instantaneous rate of energy flow cannot be expressed in terms of the group velocity; it is given by the product of the phase velocity and the instantaneous propagating energy density. (There is in addition a to and fro horizontal flow of kinetic energy that averages to zero. See Appendix A.)

The relationship between group and phase velocity can be likened to the forward progress of the links of the track of a crawler-type tractor. The moving links move forward at what corresponds to the phase velocity and are converted into stationary links at the forward end of the track. At the rear end of the track, the stationary links are converted into moving links. The average rate of forward motion of the whole set of links making up the track corresponds to the group velocity. The forward flux of links can be expressed either as the product of the velocity of the tractor and the total density of links (moving and stationary) or as the product of the velocity of the moving links and the density of the moving links.

A beautiful example of group and phase velocity for deep water waves can be observed by looking at the bow wave produced by a moving ship. Crawford[2] has given a very readable and illuminating account of the structure of the bow wave and the overall wave pattern around a moving ship in terms of interference between differing frequency components in the wave group. One can see a succession of new individual waves forming in the trailing portion of the bow wave, growing in amplitude as they move forward through the bow wave until they reach their peak amplitude near the middle of the group of waves making up the bow wave. Beyond that point, the individual waves diminish in amplitude as they move toward the leading edge of the wave group. In Crawford's description, this is due to interference between infinitely long wavetrains that make up the wave group. However, in a physical sense, it is due to conversion between immobilized and propagating forms of energy density.

The concept presented here is complementary to and fully consistent with Crawford's description in terms of interference between frequency components. A wave group of the sort illustrated by the bow wave or a wave group launched by rocking a canoe can be regarded as a self-contained group without the requirement of assuming infinitely long wavetrains for the individual frequency components; the wavetrains appear to be infinitely long because new waves are continually generated in the rear portion of the group in phase with the waves that preceded them, and waves die out in the forward portion of the group by being converted into nonpropagating kinetic energy.

Figure 2 shows a sketch of a bow wave by Froude,[3] and Fig. 3 shows a photograph of a bow wave.[4] Cross sections along lines perpendicular to the individual waves are simi-

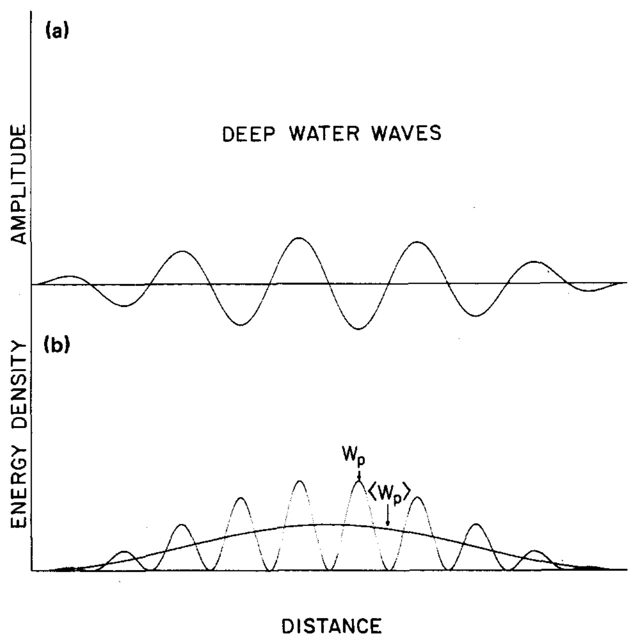

Fig. 1. (a) The contour of a wave group with five wave crests in the group, based on the superposition of two sine waves. (b) The instantaneous distribution of potential energy in the above wave group (W_t) and the potential energy averaged over a wavelength ($\langle W_t \rangle$), for the case of deep water waves.

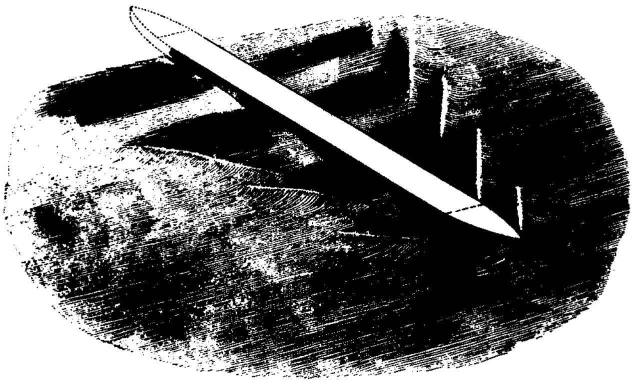

Fig. 2. A sketch by Froude of the individual waves making up the bow wave of a ship under way.

lar to the profile presented in Fig. 1, although both the sketch and the photograph indicate that the number of wave crests in the particular group being portrayed is three, rather than five as indicated in Fig. 1. In the case of the photograph, there is also a clearly discernible stern wave, but attention should be focused on the bow wave. The generation of new waves and the decay of old ones is usually most clearly evident in the group making up the bow wave.

Fig. 3. A photograph of a bow wave. The stern wake is also prominent. (Courtesy of Jack A. C. Kaiser, U. S. Naval Research Laboratory, Washington, D.C.).

II. ELECTROMAGNETIC WAVES IN DIELECTRICS

The rate of energy flow in electromagnetic waves is given by the Poynting vector, but attempts to relate this energy flux to an energy density and group velocity have always proved troublesome, as one can see by seeking explanations in existing textbooks. These difficulties have led in the past to proposals for alternatives to the Poynting vector,[5,6] but these have not won widespread acceptance. The problem was well stated by Hines: "We shall confine our attention for the moment to nonabsorbing media. In these, Poynting's density is a function of the phase, $k(nz - ct)$. This suggests a flow having a speed in the z direction equal to the phase speed c/n, a result that is at variance with the generally accepted group speed, $c/(dkn/dk)$, if the medium is dispersive." Hines went on to point out that the group speed could be obtained, on average, by adding a constant of integration to Poynting's density. Here, we identify that constant of integration as the immobilized energy density that must be present in the medium as the wavetrain passes by.

Consider the case of normal dispersion associated with a single resonant frequency well above the wave frequency, with negligible absorption at the wave frequency. This is characteristic of many transparent optical materials. The passage of an electromagnetic wave energizes the medium (i.e., it causes the electrons to oscillate in response to the wave). The immobilized energy density consists of the sum of the kinetic energy of the oscillating electrons plus an equal amount, on average, of potential energy associated with the restoring forces on the electrons. The remaining part of the potential energy acts as if it were electric field energy and is conventionally taken into account by the use of a relative permittivity ϵ_r in the evaluation of the electric field energy density. The velocities and displacements of the electrons are out of phase with each other by $\pi/2$, and hence the sum of the potential energy (not including the part that is taken into account by ϵ_r) and the kinetic energy is constant, and that sum constitutes the nonpropagating energy density. The energy of the individual electrons alternates between kinetic and potential, and this energy (all of the kinetic energy and part of the potential energy equal on average to the kinetic energy) does not propagate with the individual waves. The nonpropagating energy density is constant in a constant amplitude portion of a wave group and equal to $2\langle W_k \rangle$, where $\langle W_k \rangle$ is the average value of the kinetic energy density. The remaining part of the potential energy propagates with the phase velocity. These and other properties of electromagnetic waves in transparent dielectrics are set forth in detail in Appendix B.

The flow of electromagnetic energy is given by the Poynting vector **S**, and it is equal to the product of the phase velocity and the propagating energy density W_p, which is the sum of the energy densities of the electric and magnetic fields, $W_e + W_m$, evaluated in the usual manner using relative permittivity and permeability (the latter usually being regarded as equaling unity for dielectrics).

Sometimes, the immobilized energy density is included in the definition of the electromagnetic energy density, which then can be related to a velocity of energy flow only in terms of averages over a wave period; the average Poynting flux is then the product of the group velocity and the average electromagnetic energy density. *Thus it is important to be explicit in the definition of electromagnetic energy density if it is to be related to energy flux and a velocity.*

The average value of the Poynting flux is

$$\langle S \rangle = \langle W_e + W_m \rangle v_p = \langle W_t \rangle v_g,$$

where v_p and v_g are the phase and group velocities, and $\langle W_t \rangle$ is the average value of the total energy density, including the immobilized part; i.e.,

$$\langle W_t \rangle = \langle W_e + W_m \rangle + 2 \langle W_k \rangle.$$

Havelock[7] derived an equivalent relationship for $\langle S \rangle$ involving group velocity and total energy in 1914, expressing it in terms of the energy density of the aethereal electromagnetic field and the kinetic and potential energy of the vibrators in the material medium. Brillouin[8] also derived an equivalent equation, expressible in the terminology used here as $v_p \langle W_{\text{em}} \rangle = v_g \langle W_t \rangle$, terming it a curious relation. In the light of the present work, where it is recognized that only an identifiable part, W_{em}, of the total energy, W_t, propagates with the individual waves, the result no longer appears curious; v_g is just the weighted average velocity of the propagating and immobilized energy densities.

The influence of the electron mass on the amplitude of the electron oscillation affects the electrical polarization of the dielectric; this effect is appropriately taken into account through the use of a frequency-dependent relative permittivity ϵ_r in the computation of electric field energy density. For the case that we are considering—wave frequency well below the single resonant frequency—the phase of the velocity oscillation for the electrons lags the phase of the electric field by $\pi/2$, and the phase of the displacement of the electrons lags the field by π. The Maxwell displacement current is

$$J_{\text{disp}} = \frac{\partial P}{\partial t} + \epsilon_0 \frac{\partial E}{\partial t},$$

where P is the polarization of the medium due to the displacement of the electrons. In dielectrics, the polarization current $\partial P / \partial t$ (the current due to the motion of the electrons) is in phase with the $\epsilon_0 \ (\partial E / \partial t)$ term; the two terms augment one another in producing the displacement current. The magnetic field is related to the displacement current, and its value is enhanced by the polarization relative to its value in vacuum for a wave field carrying the same energy flux.

III. ELECTROMAGNETIC WAVES IN PLASMAS

In plasmas, the energy relationships are more complicated than in dielectrics, as the restoring force on the oscillating electrons is due to the macroscopic electric field. The electric field energy density consists of two components—propagating and nonpropagating. The nonpropagating part relates to the forced oscillations of the plasma, and the remaining part propagates with the wave field at the phase velocity. The immobilized energy density consists of the kinetic energy of the electrons and, on average, an equal amount of electric field energy; these two components of the immobilized energy density are out of phase with one another and they have a constant sum—twice the average kinetic energy density. As part of the electric field energy density is associated with the immobilized energy density in the plasma, the average propagating energy density is less than the sum of the average densities of the electric and magnetic fields; it is $\langle W_p \rangle = \langle W_{\text{em}} - W_k \rangle$, where W_{em} is the sum of the energy densities of the electric and magnetic fields and W_k is the kinetic energy density. The total immobilized energy density, which is partly electric field energy density and which does not vary with the phase of the wave, is $W_s = 2 \langle W_k \rangle$. These properties are developed in Appendix C.

For plasmas, the kinetic energy must be evaluated if the Poynting vector is to be expressed in terms of an energy density and a velocity. The average Poynting flux can be expressed equally conveniently in terms of the phase velocity or the group velocity as

$$\langle S \rangle = \langle W_{\text{em}} - W_k \rangle v_p = \langle W_{\text{em}} + W_k \rangle v_g.$$

The group and phase velocities are related by $v_p v_g = c^2$, where c is the velocity of electromagnetic waves in vacuum.

The current due to the motions of the electrons (analogous to the polarization current in dielectrics) opposes the $\epsilon_0 (\partial E / \partial t)$ term of the displacement current and acts to decrease the magnetic induction, making it less than in vacuum for wave fields carrying the same energy flux. The electric field energy density exceeds the magnetic field energy density, but only as much of the electric field energy density as equals the magnetic field energy density propagates with the individual waves. The excess electric field energy density is out of phase with but equal in magnitude to the kinetic energy density; this part of the electric field energy density and the kinetic energy density make up the nonpropagating energy density. (If the plasma were treated as a dielectric, its electrical susceptibility would be negative.) The electron oscillations in the plasma produce no total contribution to the magnetic field, their conduction current being cancelled by their contribution to the displacement current; this is consistent with the statement that the energy density associated with the forced plasma oscillations does not propagate with the individual waves. For the forced plasma oscillations alone, the Poynting vector is zero, there being no magnetic field associated with them. The instantaneous value of the Poynting flux is discussed in Appendix C.

IV. ELECTROMAGNETIC WAVES IN WAVEGUIDES

Propagation of electromagnetic waves in a waveguide provides another interesting example of group velocity. It resembles a plasma in that $v_p v_g = c^2$, but there is no kinetic energy involved in this case, only the electric and magnetic field energies. The immobilized energy density is associated with a standing wave in the transverse direction. The average energy density of the standing wave in the transverse direction is $\langle W_s \rangle = (\epsilon_0 E_0^2/4) \cos^2 \theta$ and the average energy density of the propagating component is $\langle W_p \rangle = (\epsilon_0 E_0^2/4) \sin^2 \theta$, where θ is the angle of incidence of the rays within the waveguide. In this case, the averages designated by $\langle \ \rangle$ are over both space and time. The average total density is $\epsilon_0 E_0^2/4 = \langle W_{\text{em}} \rangle$. The Poynting flux averaged over space and a wave period can be expressed either in terms of the phase or group velocity,

$$\langle S \rangle = \langle W_{\text{em}} \rangle v_g = \langle W_{\text{em}} - W_s \rangle v_p.$$

The properties are developed in Appendix D. If the Poynting flux is to be related to an energy density and a velocity, the use of total average energy density and group velocity is probably more convenient than is the use of propagating energy density and phase velocity. However, if instantaneous values of the energy flux are desired, it is necessary to express the flux in terms of the product of the phase velocity and the instantaneous value of the propagating energy density $(\epsilon_0 E^2/2)\sin^2\theta$.

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APPENDIX A: DEEP WATER WAVES

This treatment follows Sommerfeld.[9] The y axis is vertical, positive downward. Propagation is in the plus x direction, the phase being $\varphi = kx - \omega t$. Unit width in the z direction is used in evaluating the energy content and flow. The velocity potential for small-amplitude waves in deep water is

$$\phi = A\cos\varphi\; e^{-ky}, \tag{A1}$$

where the units are L^2/T. The velocity components of the wave motions are

$$v_x = -\frac{\partial\phi}{\partial x} = Ak\sin\varphi\; e^{-ky}, \tag{A2}$$

and

$$v_y = -\frac{\partial\phi}{\partial y} = Ak\cos\varphi\; e^{-ky}. \tag{A3}$$

The square of the velocity is

$$v^2 = v_x^2 + v_y^2 = A^2k^2e^{-2ky}. \tag{A4}$$

Thus v^2 and the kinetic energy of the wave motion are not functions of time or horizontal position except to the extent that A depends upon the phase of the group envelope, as portrayed in Fig. 1. *As the kinetic energy density is constant, the individual waves do not transport kinetic energy, although they create it from potential energy when they move into an area of undisturbed water, and they consume it in the decaying portion of a wave group where the individual waves are growing in amplitude at the expense of kinetic energy.* The lack of transport of kinetic energy by individual waves is most easily visualized in a portion of the wavetrain where the amplitude is constant.

The particle displacements are

$$x_d = \int v_x\, dt = \frac{Ak}{\omega}\cos\varphi\, e^{-ky}$$

and

$$y_d = \int v_y\, dt = -\frac{Ak}{\omega}\sin\varphi\, e^{-ky} \tag{A5}$$

The phase velocity is $v_p = \omega/k$. Sommerfeld[9] derives the dispersion relation

$$\omega^2 = gk. \tag{A6}$$

The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{g}{2\omega} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_p,$$

and the ratio of group to phase velocity is

$$v_g/v_p = \tfrac{1}{2}. \tag{A7}$$

The y component of the surface undulations for small-amplitude waves is given by

$$\eta \approx -(Ak/\omega)\sin\varphi = -(\omega/g)A\sin\varphi, \tag{A8}$$

where the approximation is that $e^{-k\eta} \approx 1$.

The potential energy per unit surface area is

$$W_p = \int_0^\eta g\rho y\, dy = \frac{g\rho\eta^2}{2} = \frac{g\rho k^2A^2}{2\omega^2}\sin^2\varphi, \tag{A9}$$

where ρ is the density. Averaged over a wavelength (or a wave period),

$$\begin{aligned}\langle W_p\rangle &= \frac{1}{\lambda}\frac{g\rho k^2A^2}{2\omega^2}\int_0^\lambda \sin^2(kx-\omega t)dx\\ &= \frac{g\rho k^2A^2}{4\omega^2} = \frac{\rho kA^2}{4}.\end{aligned} \tag{A10}$$

The depth-integrated kinetic energy in a distance element dx is

$$\begin{aligned}dW_s &= \int_\eta^\infty \frac{\rho}{2}v^2\,dy\,dx = \frac{\rho k^2A^2}{2}\int_\eta^\infty e^{-2ky}\,dy\,dx\\ &\approx \rho kA^2\,dx/4.\end{aligned} \tag{A11}$$

The approximation involved in the last step is that $e^{-2k\eta} \approx 1$ for small-amplitude waves. The kinetic energy per unit surface area is therefore

$$W_s = \rho kA^2/4. \tag{A12}$$

This is just equal to the potential energy density averaged over a wavelength or a wave period, the energy densities being energy per unit surface area.

The pressure distribution can be obtained from the approximate Bernoulli equation for small-amplitude perturbations,

$$p = \rho\frac{\partial\phi}{\partial t} + \rho gy + p_0, \tag{A13}$$

where p_0 is the atmospheric pressure. The energy transport in time dt through a vertical cross section of unit width in the z direction and depth dy at any fixed value of x and y is

$$dW = pv_x\,dy\,dt = -p\frac{\partial\phi}{\partial x}\,dy\,dt. \tag{A14}$$

Thus the depth-integrated energy flux per unit distance in the z direction is

$$S = -\int_\eta^\infty p\frac{\partial\phi}{\partial x}\,dy. \tag{A15}$$

We will evaluate separately the contributions to S by the three terms in Eq. (A13) as S_1, S_2, and S_3. From the first term, we have

$$S_1 = -\int_\eta^\infty \rho \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x}\,dy = \rho A^2 \omega k \sin^2\varphi \int_\eta^\infty e^{-2ky}\,dy$$
$$\approx (\rho\omega/2) A^2 \sin^2 \omega t. \qquad (A16)$$

The last step makes the evaluation at $x = 0$ and involves the approximation that $e^{-2k\eta} \approx 1$. This is just the flux of potential energy.

The second term in Eq. (A13) makes a contribution to S equal to

$$S_2 = -\int_\eta^\infty \rho g y \frac{\partial\phi}{\partial x}\,dy = \rho g k A \sin\varphi \int_\eta^\infty y e^{-ky}\,dy.$$

As

$$\int_\eta^\infty y e^{-ky}\,dy = \left(-\frac{e^{-ky}}{k^2}(1+ky)\right)_\eta^\infty$$
$$= (e^{-k\eta}/k^2)(1+k\eta) \approx 1/k^2,$$

at $x = 0$,

$$S_2 = -\rho g (A/k)\sin\omega t = -\rho(\omega^2/k^2) A \sin\omega t$$
$$= -\rho v_p^2 A \sin\omega t. \qquad (A17)$$

The third term in Eq. (A13) gives

$$S_3 - \int_\eta^\infty p_0 \frac{\partial\phi}{\partial x}\,dy = p_0 A k \sin\varphi \int_\eta^\infty e^{-ky}\,dy$$
$$\approx -p_0 A \sin\omega t \qquad (A18)$$

at $x = 0$.

Thus the depth-integrated energy flux density per unit distance in the z direction is

$$S = (\rho\omega A^2/2)\sin^2\omega t - \rho g (A/k)\sin\omega t - p_0 A \sin\omega t. \qquad (A19)$$

The last two terms on the right average to zero, so the flux density averaged over a wave period is

$$\langle S\rangle = \rho\omega A^2/4 = v_p \rho k A^2/4 = v_p \langle W_p\rangle. \qquad (A20)$$

As $\langle W_p\rangle = W_s$ and $v_p = 2v_g$, this can also be written

$$\langle S\rangle = v_g(\langle W_p\rangle + W_s) = v_g\langle W_t\rangle, \qquad (A21)$$

where $\langle W_t\rangle = \langle W_p\rangle + W_s$ is the depth-integrated total energy density per unit area averaged over a wave period.

We have identified the kinetic energy density as the nonpropagating energy density that must be present in the medium in order for the wave field to exist. The kinetic energy does not propagate except in the sense that it is continually created in the forward portion of a wave group and used up by creating propagating energy in the trailing portion of the wave group, causing the pattern of kinetic energy density to move along with the group. The energy density that is carried forward by the wave field, at the phase velocity, is the potential energy density of the wave field. The group velocity is not only the velocity at which the group moves forward; it is the weighted mean velocity of all the energy associated with the wave field—propagating and nonpropagating. The wave group is a self-contained entity, even though the individual waves continually advance through the group. The various Fourier components making up the group appear to be of infinite extent because new waves are continually generated, in phase with the preceding waves, in the trailing portion of the group.

APPENDIX B: LOSSLESS DIELECTRICS

For an electromagnetic wave propagating through a dielectric with normal dispersion and well removed from absorption bands, the electron displacements are π out of phase with the electric vector of the electromagnetic wave[10] (out of phase because the charge is negative). The nonpropagating energy consists of the kinetic energy of the vibrating electrons plus the associated potential energy due to the restraining forces on the electrons.

For dielectrics in general, the electric field energy density is considered to be larger than in free space (for the same field intensity) by the factor ϵ_r, the relative permittivity. It is worth recalling how this comes about. Linear isotropic dielectrics may be considered as consisting of N electrons per unit volume, each bound to its equilibrium position by a restraining force described by a force constant κ. In the presence of a static electric field of intensity E, the electrons are displaced from their equilibrium positions a distance x such that $\kappa x = -eE$, where $-e$ is the charge on an electron. The total potential energy per unit volume associated with the displacement of electrons is

$$W_d = N\frac{\kappa x^2}{2} = \frac{N}{2}\frac{e^2E^2}{\kappa} = \frac{Ne^2}{\epsilon_0\kappa}\frac{\epsilon_0 E^2}{2}. \qquad (B1)$$

This potential energy density is produced as the result of electrical work done as the electric field intensity rises from zero to E. When the field returns to zero, the potential energy is returned as electrical energy, so the potential energy density can conveniently be considered to be electric field energy density; this must be added to the "true" electrical field energy density $\epsilon_0 E^2/2$ to obtain the electric field energy density as conventionally expressed,

$$W_e = \frac{\epsilon_0 E^2}{2} + \frac{Ne^2}{\epsilon_0\kappa}\frac{\epsilon_0 E^2}{2}$$
$$= \frac{\epsilon_0 E^2}{2} + \chi_e\frac{\epsilon_0 E^2}{2} = \frac{\epsilon_0\epsilon_r E^2}{2}, \qquad (B2)$$

where the electrical susceptibility is

$$\chi_e = Ne^2/\epsilon_0\kappa, \qquad (B3)$$

and the relative permittivity is

$$\epsilon_r = 1 + \chi_e = 1 + Ne^2/\epsilon_0\kappa. \qquad (B4)$$

If the applied electric field intensity is not static (or nearly so), it is necessary to take into account the fact that each electron in the dielectric is an independent oscillator whose natural frequency is $\omega_0 = \sqrt{\kappa/m}$, where m is the mass of an electron. The oscillators are uncoupled in the approximation that we are using. If the exciting function is an electromagnetic wave, each electron is forced to oscillate at the wave frequency, and there are phase relationships among the various oscillators only because each is forced to maintain a fixed phase relationship to the wave field.

Consider the case where the wave frequency is much less than the resonant frequency, i.e., $\omega \ll \omega_0$. (We consider the absorption band at ω_0 to be the only one significantly influencing the propagating waves, and the absorption at the wave frequency to be negligible.) Let the wave field be described by

$$E = E_x = E_0 e^{i\varphi}, \qquad (B5)$$

where $\varphi = \omega t - kz$. The displacements and velocities of the electrons are

$$x = [(-e/m)/(\omega_0^2 - \omega^2)]E \tag{B6}$$

and

$$v = [(-i\omega e/m)/(\omega_0^2 - \omega^2)]E. \tag{B7}$$

It is worthwhile deriving the potential energy in terms of the restoring force constant κ. Designate the potential energy density stored in the force field by W_{pot}; only part of this is regarded as electric field energy density, as will be shown below. The potential energy density stored in the force field is

$$W_{\text{pot}} = N\frac{\kappa x^2}{2} = \left(\frac{N\kappa}{2}\right)\left(\frac{-eE/m}{\omega_0^2 - \omega^2}\right)^2. \tag{B8}$$

As $\omega_0 = \sqrt{\kappa/m}$,

$$\begin{aligned}
W_{\text{pot}} &= N\frac{m\omega_0^2}{2}\frac{e^2}{m^2(\omega_0^2 - \omega^2)^2}E^2 \\
&= \frac{Ne^2}{m\epsilon_0}\frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E^2}{2} = \frac{\omega_0^2\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E^2}{2} \\
&= \frac{\omega^2\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E^2}{2} + \frac{(\omega_0^2 - \omega^2)\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E^2}{2} \\
&= \frac{\omega^2\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E_0^2}{2}\cos^2\varphi \\
&\quad + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\frac{\epsilon_0 E_0^2}{2}\cos^2\varphi,
\end{aligned} \tag{B9}$$

where $\omega_0^2 = Ne^2/\epsilon_0 m$ is the plasma frequency. The second term on the right is conventionally considered to be part of the electric field energy density through the use of a relative permittivity different from unity. The first term on the right is that part of the potential energy that interacts with the kinetic energy in the forced oscillations of the charges. Designate the first term W_{pe}; it pairs with the kinetic energy density,

$$W_k = \frac{Nmv^2}{2} = \frac{\omega^2\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E_0^2}{2}\sin^2\varphi, \tag{B10}$$

with which it is out of phase, to produce the total immobilized energy density in the medium. That is, the immobilized energy density associated with the excitation of the oscillators is

$$W_s = W_{\text{pe}} + W_k = \frac{\omega^2\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E_0^2}{2}, \tag{B11}$$

which is constant so long as E_0 is constant. It is twice the average value of the kinetic energy density, i.e., $W_s = 2\langle W_k\rangle$. The total electric field energy density, including that part of W_{pot} that is considered to be part of the electric field density, is

$$\begin{aligned}
W_e &= \frac{\epsilon_0 E_0^2}{2}\cos^2\varphi + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\frac{\epsilon_0 E_0^2}{2}\cos^2\varphi \\
&= \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)\frac{\epsilon_0 E^2}{2} \\
&= (1 + \chi_e)\frac{\epsilon_0 E^2}{2} = \frac{\epsilon_0\epsilon_r E^2}{2},
\end{aligned} \tag{B12}$$

where

$$\chi_e = \omega_p^2/(\omega_0^2 - \omega^2), \tag{B13}$$

which agrees with the value $Ne^2/\epsilon_0\kappa$ given earlier for χ_e [Eq. (B3)] if ω^2 is negligible by comparison with ω_0^2.

The polarization current J_p is the current due to the motions of the electrons, and

$$\begin{aligned}
J_p &= -Nev = [i\omega Ne^2/\epsilon_0 m(\omega_0^2 - \omega^2)]\epsilon_0 E \\
&= [i\omega\omega_p^2/(\omega_0^2 - \omega^2)]\epsilon_0 E.
\end{aligned} \tag{B14}$$

J_p is just part of the displacement current J_{disp}; let J_d represent the remaining part, so that $J_{\text{disp}} = J_p + J_d$, where

$$J_d = \epsilon_0\frac{\partial E}{\partial t} = i\omega\epsilon_0 E. \tag{B15}$$

Then

$$J_{\text{disp}} = i\omega\epsilon_0 E[1 + \omega_p^2/(\omega_0^2 - \omega^2)]. \tag{B16}$$

Note that J_p and J_d are in phase and that their ratio is

$$J_p/J_d = \omega_p^2/(\omega_0^2 - \omega^2) = \chi_e. \tag{B17}$$

The polarization is

$$\begin{aligned}
P &= -Nex = [(Ne^2/m)/(\omega_0^2 - \omega^2)]E \\
&= [\omega_p^2/(\omega_0^2 - \omega^2)]\epsilon_0 E = \chi_e\epsilon_0 E.
\end{aligned} \tag{B18}$$

The relative permittivity is

$$\epsilon_r = 1 + \chi_e = 1 + \omega_p^2/(\omega_0^2 - \omega^2) = J_{\text{disp}}/J_d. \tag{B19}$$

The contribution of the polarization to the electric field is

$$E_p = \frac{Nex}{\epsilon_0} = -\frac{P}{\epsilon_0} = -\frac{\omega_p^2}{\omega_0^2 - \omega^2}E, \tag{B20}$$

and this is π out of phase with E.

E, x, v, J_p, J_d, and E_p, along with H, are shown as phasors in Fig. 4 for $\chi_e = 0.25$.

The electric field is related to $\partial\mathbf{B}/\partial t$ by $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, or

$$\hat{\mathbf{j}}(-ik)E = -\hat{\mathbf{j}}(i\omega)B = -\hat{\mathbf{j}}i\omega\mu_0 H.$$

Therefore,

$$E = (\omega/k)\mu_0 H = v_p B$$

and

$$E/H = \omega\mu_0/k = \mu_0 v_p, \text{ or } E = v_p B, \tag{B21}$$

where μ_r is taken to be unity, as it usually is for dielectrics. Alternatively, the magnetic field intensity H is related to the current density by

$$\nabla\times\mathbf{H} = \mathbf{J}_p + \mathbf{J}_d = -\hat{\mathbf{i}}(-ik)H.$$

Therefore,

$$ikH = i[\omega\omega_p^2/(\omega_0^2 - \omega^2)]\epsilon_0 E + i\omega\epsilon_0 E,$$

and

$$H = (\omega/k)[1 + \omega_p^2/(\omega_0^2 - \omega^2)]\epsilon_0 E = v_p\epsilon_r\epsilon_0 E.$$

Therefore, as before, $E/H = 1/v_p\epsilon_r\epsilon_0 = \mu_0 v_p$.

The dispersion relationship is

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_r\epsilon_0}} = \frac{c}{\sqrt{1 + \omega_p^2/(\omega_0^2 - \omega^2)}}, \tag{B22}$$

or

$$k = (\omega/c)\sqrt{1 + \omega_p^2/(\omega_0^2 - \omega^2)}. \tag{B23}$$

Thus the index of refraction is

$$n = c/v_p = \sqrt{1 + \omega_p^2/(\omega_0^2 - \omega^2)} = \sqrt{\epsilon_r}. \tag{B24}$$

The group velocity is given by $\partial\omega/\partial k$; therefore,

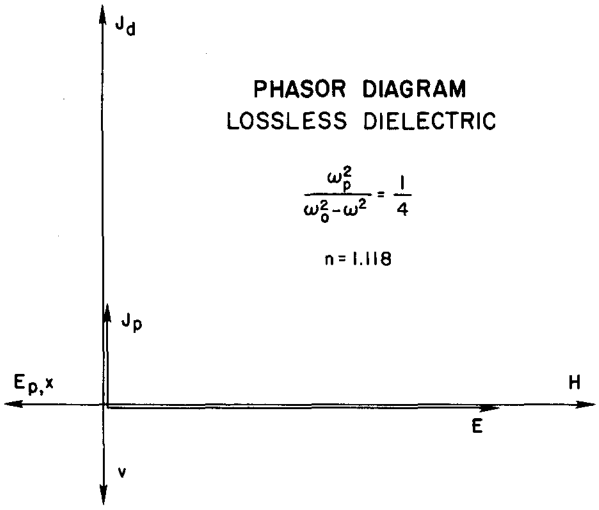

Fig. 4. A phasor diagram illustrating the phase relationships for various quantities involved in electromagnetic wave propagation in a dielectric.

$$v_g = \frac{\partial \omega}{\partial k} = c\{n(\omega_0^2 - \omega^2)^2/[(\omega_0^2 - \omega^2)^2 + \omega_0^2\omega_p^2]\} \tag{B25a}$$

$$= v_p\{n^2(\omega_0^2 - \omega^2)^2/[(\omega_0^2 - \omega^2)^2 + \omega_0^2\omega_p^2]\} \tag{B25b}$$

$$= v_p\{(\omega_0^2 - \omega^2 + \omega_p^2)(\omega_0^2 - \omega^2)/ [(\omega_0^2 - \omega^2)^2 + \omega_0^2\omega_p^2]\}. \tag{B25c}$$

To force the oscillators to oscillate at frequency ω when their natural frequency is ω_0 requires a cyclic interchange of energy between the oscillators and the forcing field, and it is worthwhile looking at this explicitly to see where it appears in the equations. After the electrons pass through their equilibrium positions with maximum velocity, they are slowed down by the mechanical restoring force. Their velocities would be reduced to zero in a quarter of the natural period except for the electrical force upon them, which causes them to slow down more slowly and extends their excursion. An expenditure of electrical energy is required to accomplish this. After reaching their maximum excursions, the mechanical force would return them to their equilibrium position too quickly, and an electrical force is required to prevent this; in this case electrical energy is produced at the expense of mechanical energy and the energy extracted from the potential energy is just equal to that put in earlier during the half-cycle.

The rate at which electrical energy is expended per unit volume in doing mechanical work on the oscillators is

$$-NevE = -Ne[(\omega e/m)/(\omega_0^2 - \omega^2)] \times E_0 \sin\varphi E_0 \cos\varphi = [-\omega\omega_p^2/(\omega_0^2 - \omega^2)](\epsilon_0 E_0^2/2)\sin 2\varphi. \tag{B26}$$

The amount of electrical energy that has been converted to mechanical energy per unit volume at any time is obtained by integration, and it is

$$W_{\text{chi}} = [\omega_p^2/(\omega_0^2 - \omega^2)](\epsilon_0 E_0^2/2)(\cos 2\varphi + 1) = [\omega_p^2/(\omega_0^2 - \omega^2)]\epsilon_0 E_0^2 \cos^2\varphi = \chi_e \epsilon_0 E^2, \tag{B27}$$

where the constant of integration has been selected on the basis that no mechanical energy due to this interchange is present when x (and E) $= 0$. (There is, of course, mechanical energy present in the oscillators at this time in the form of kinetic energy, which interchanges cyclically with that part of the potential energy not considered to be electric field energy.) Thus the electrical energy expended and recovered each half-cycle by doing mechanical work in forcing the oscillation at frequency ω is just the mechanical potential energy that is conventionally considered to be electric field energy and taken into account by the use of an electrical susceptibility different from zero or a relative permittivity different from unity.

The electric field energy density is

$$W_e = \frac{1}{2}\epsilon_0\epsilon_r E^2 = \frac{\omega_0^2 - \omega^2 + \omega_p^2}{\omega_0^2 - \omega^2}\frac{\epsilon_0 E_0^2}{2}\cos^2\varphi, \tag{B28}$$

and the magnetic field energy density is

$$W_m = \tfrac{1}{2}\mu_0\mu_r H^2 = \tfrac{1}{2}\epsilon_0\epsilon_r E^2 = W_e. \tag{B29}$$

Let W_{em} be the sum of electric and magnetic field energy densities. Its average value over a wave period is

$$\langle W_{\text{em}}\rangle = 2\langle W_e\rangle = \epsilon_r\epsilon_0 E_0^2/2. \tag{B30}$$

The Poynting flux is

$$S = EH = Ev_p\epsilon_r\epsilon_0 E = v_p\epsilon_r\epsilon_0 E^2 = v_p W_{\text{em}}, \tag{B31}$$

in agreement with the common practice of regarding the time-averaged Poynting flux as being the product of the phase velocity and the sum of the average energy densities of the electric and magnetic fields.

The total energy density is

$$\begin{aligned} W_t &= W_e + W_m + W_{\text{pe}} + W_k \\ &= (\epsilon_0\epsilon_r E^2/2 + \mu_0\mu_r H^2/2 \\ &\quad + [\omega^2\omega_p^2/(\omega_0^2 - \omega^2)^2](\epsilon_0 E_0^2/2)(\sin^2\varphi + \cos^2\varphi) \\ &= \epsilon_0\epsilon_r E^2 + [\omega^2\omega_p^2/(\omega_0^2 - \omega^2)^2](\epsilon_0 E_0^2/2) \\ &= W_{\text{em}} + W_s. \end{aligned} \tag{B32}$$

The first term is the sum of the electric and magnetic field energy densities, which propagates with the phase velocity. The second is mechanical energy—the sum of the kinetic energy and that part of the potential energy that cannot be treated as if it were a part of the electric field energy density; it is constant in space and time so long as E_0 is constant, and it does not propagate (except in the sense that it is continually added to in the growing portion of a wave packet by expenditure of propagating energy, and it is continually removed from the trailing portion of a wave packet by conversion to propagating energy. The average value of the total energy density is

$$\langle W_t\rangle = \epsilon_r\frac{\epsilon_0 E_0^2}{2} + \frac{\omega^2\omega_p^2}{(\omega_0^2 - \omega^2)^2}\frac{\epsilon_0 E_0^2}{2} = (\epsilon_0 E_0^2/2)[\epsilon_r + \omega^2\omega_p^2/(\omega_0^2 - \omega^2)^2]. \tag{B33}$$

Therefore, making use of Eqs. (B31), (B33), and (B19),

$$\langle S\rangle = v_p(\epsilon_0\epsilon_r E_0^2/2)$$
$$= v_p\epsilon_r[\epsilon_r + \omega^2\omega_p^2/(\omega_0^2-\omega^2)^2]^{-1}\langle W_t\rangle$$
$$= v_p\frac{\omega_0^2-\omega^2+\omega_p^2)(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\omega_0^2\omega_p^2}\langle W_t\rangle \quad \text{(B34)}$$
$$= v_g\langle W_t\rangle. \quad \text{(B35)}$$

The last step makes use of Eq. (B25c).

Usage varies as to what is meant by the term electromagnetic energy. Some take it to be synonymous with the total energy W_t. Others consider it to be the sum of the electric and magnetic energy densities expressed in terms of ϵ_r (and μ_r), while in no way denying the presence of kinetic and potential energy in the medium due to the presence of the electromagnetic fields; we use the latter terminology and designate the sum of the electric and magnetic energy densities W_{em}. Jackson[11] introduces two other useful energy densities, E_{field} and E_{mech}. E_{field} is defined as the sum of the electric and magnetic energy densities evaluated with $\epsilon_r=\mu_r=1$, i.e., E_{field} is the sum of the true electric field and magnetic field energy densities not taking into account the energy stored in the medium that will be released each half-cycle as if it were electric or magnetic field energy density. Thus for electromagnetic fields

$$E_{\text{field}} = \frac{\epsilon_0E^2}{2}+\frac{B^2}{2\mu_0} = (1+\epsilon_r)\frac{\epsilon_0E^2}{2}. \quad \text{(B36)}$$

E_{mech} includes the kinetic energy density and all the potential energy density, including that part that is often considered to be electric field energy density, $\chi_e\epsilon_0E^2/2$. Thus, in our terminology,

$$E_{\text{mech}} = W_k + W_{\text{pot}} = W_k + W_{\text{pe}} + \chi_e(\epsilon_0E^2/2)$$
$$= W_s + \chi_e(\epsilon_0E^2/2), \quad \text{(B37)}$$

and the total energy density is

$$W_t = E_{\text{field}} + E_{\text{mech}}$$
$$= (1+\epsilon_r)(\epsilon_0E^2/2) + W_s + \chi_e(\epsilon_0E^2/2)$$
$$= \epsilon_r\epsilon_0E^2 + W_s, \quad \text{(B38)}$$

in agreement with Eq. (B32).

To summarize, the potential energy that is converted into electrical energy each half-cycle propagates with the individual waves at the phase velocity; it is considered to be a part of the electric field energy density when the latter is evaluated as $\epsilon_r\epsilon_0E^2/2$. The remainder of the potential energy is converted into kinetic energy each half-cycle and it does not propagate with the individual waves; it is converted into propagating energy and carried off by the wave field only as the wave field diminishes with the passage of the group. The group velocity is the weighted average of all the energy associated with the wave field in the dielectric.

APPENDIX C: LOSSLESS PLASMAS

For propagation of electromagnetic waves in nonabsorbing plasmas at frequencies above the plasma frequency, the energization of the medium during the passage of the waves includes the kinetic energy of the electrons, just as in dielectrics. However, the restoring forces associated with the oscillations of the electrons are electric. The forced oscillations of the plasma do not propagate energy; they produce no magnetic field because their contribution to the displacement current exactly cancels the conduction current, and hence the Poynting vector associated with these oscillations is zero.

For electromagnetic wave propagation in nonabsorbing plasmas at wave frequencies well above the plasma frequency, where the electric field is described by

$$E = E_x = E_0e^{i\varphi}, \quad \varphi = \omega t - kz, \quad \text{(C1)}$$

the equations of motion for the electrons (acceleration a, velocity v, and displacement x) are

$$a = \frac{-eE}{m}, \quad v = i\frac{eE}{m\omega}, \text{ and } x = \frac{eE}{m\omega^2}. \quad \text{(C2)}$$

Note that x is in phase with E and that v leads E by $\pi/2$. The conduction current is

$$J_f = -Nev, \quad \text{(C3)}$$

which lags E by $\pi/2$. The displacement of the electrons relative to the positive charges produces a component E_f of the total field E, increasing the field over what it would have been if electrons were immobile, and

$$E_f = \frac{Nex}{\epsilon_0} = \frac{Ne^2}{m\epsilon_0}\frac{E}{\omega^2} = \left(\frac{\omega_p^2}{\omega^2}\right)E, \quad \text{(C4)}$$

where $\omega_p^2 = Ne^2/m\epsilon_0$ is the square of the plasma frequency. By comparison with dielectrics, this suggests the use of an apparent electrical susceptibility that is negative,

$$\chi_e' = -\omega_p^2/\omega^2, \quad \text{(C5)}$$

and an apparent relative permittivity,

$$\epsilon_r' = 1 - \omega_p^2/\omega^2 \quad \text{(C6)}$$

that is less than unity. The electric displacement is

$$D = \epsilon_0E - Nex = \epsilon_0E - (\omega_p^2/\omega^2)\epsilon_0E = \epsilon_0\epsilon_r'E. \quad \text{(C7)}$$

The electric field E, part of which (E_f) results from charge separation in the plasma, is related to $\partial\mathbf{B}/\partial t$ by $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$. As

$$\nabla\times E = \hat{\mathbf{j}}(-ik)E \text{ and } \frac{\partial\mathbf{B}}{\partial t} = \hat{\mathbf{j}}i\omega B = \hat{\mathbf{j}}i\omega\mu_0H,$$

therefore,

$$E = (\omega/k)\mu_0H = v_pB, \quad \text{(C8)}$$

where v_p is the phase velocity.

The magnetic field intensity $\mathbf{H}$ is related to the current density by $\nabla\times\mathbf{H} = \mathbf{J}_f + \mathbf{J}_d$, where

$$\mathbf{J}_d = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} = i\omega\epsilon_0\mathbf{E}, \quad \text{(C9)}$$

$$\mathbf{J}_f = -Ne\mathbf{v} = -i(Ne^2/m\omega)\mathbf{E} = -i\omega(\omega_p^2/\omega^2)\epsilon_0\mathbf{E}, \quad \text{(C10)}$$

and

$$J_f/J_d = -\omega_p^2/\omega^2 = \chi_e'. \quad \text{(C11)}$$

Note that J_d leads E by $\pi/2$ and is π out of phase with J_f.[12] These relationships are portrayed in a phasor diagram in Fig. 5.

The conductivity is

$$\sigma = J_f/E = -i\omega\epsilon_0\omega_p^2/\omega^2. \quad \text{(C12)}$$

For conductors in general [see Ref. 12, Eq. (28-11)],

$$k^2 = k_0^2\epsilon_r\mu_r[1 - i(\sigma/\omega\epsilon)], \quad \text{(C13)}$$

where k_0 is the vacuum wave number. For plasmas,

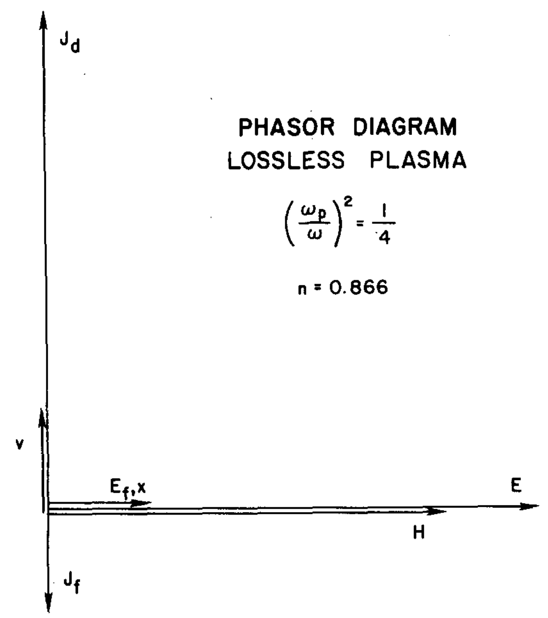

Fig. 5. A phasor diagram illustrating the phase relationships for various quantities involved in electromagnetic wave propagation in a plasma.

$\epsilon_r = \mu_r = 1$, and

$$k^2 = k_0^2(1 - \omega_p^2/\omega^2). \tag{C14}$$

This is the dispersion relation, and λ_0 is smaller than λ by the factor

$$n = \sqrt{1 - \omega_p^2/\omega^2} = \sqrt{\epsilon_r'} < 1, \tag{C15}$$

and $v_p = c/n$ is greater than c. Taking differentials to get the group velocity,

$$v_g = \frac{\partial \omega}{\partial k} = \frac{kc^2}{\omega} = \frac{c^2}{v_p} = nc. \tag{C16}$$

Therefore,

$$v_p v_g = c^2. \tag{C17}$$

The Poynting flux is

$$\begin{aligned} S = EH &= E\frac{E}{v_p\mu_0} = \frac{n}{c}\frac{E^2}{\mu_0} = nc\epsilon_0 E^2 \\ &= n^2 v_p \epsilon_0 E^2 = v_p \epsilon_r' \epsilon_0 E^2, \\ &= v_p[\epsilon_r'(\epsilon_0 E^2/2) + B^2/2\mu_0], \end{aligned} \tag{C18}$$

and

$$\begin{aligned} \langle S \rangle &= nc\frac{\epsilon_0 E_0^2}{2} = v_p n^2 \frac{\epsilon_0 E_0^2}{2} = v_p \epsilon_r' \frac{\epsilon_0 E_0^2}{2} \\ &= v_p[\epsilon_r'(\epsilon_0 E_0^2/4) + B_0^2/4\mu_0]. \end{aligned} \tag{C19}$$

Thus $\epsilon_r'(\epsilon_0 E^2/2)$ is the electric field energy density that propagates with the individual waves; however, it is not the total electric field energy density, as there is additional, nonpropagating, electric field energy density associated with the forced plasma oscillations. The kinetic energy density is

$$\begin{aligned} W_k &= \frac{Nmv^2}{2} = \frac{Nm}{2}\left(\frac{eE_0}{m\omega}(-\sin\varphi)\right)^2 \\ &= \frac{\omega_p^2}{\omega^2}\frac{\epsilon_0 E_0^2}{2}\sin^2\varphi = (1-n^2)\frac{\epsilon_0 E_0^2}{2}\sin^2\varphi \\ &= -\chi_e'(\epsilon_0 E_0^2/2)\sin^2\varphi \end{aligned} \tag{C20}$$

This is associated with an electric field energy density W_{np} that does not propagate with the individual waves (it being associated with the local plasma oscillations) of equal amplitude but opposite phase, where

$$W_{np} = \frac{\omega_p^2}{\omega^2}\frac{\epsilon_0 E_0^2}{2}\cos^2\varphi = -\chi_e'\frac{\epsilon_0 E_0^2}{2}\cos^2\varphi. \tag{C21}$$

Adding this to the propagating electric field energy density, we get for the total electric field energy density

$$W_e = \epsilon_0 E^2/2. \tag{C22}$$

We now list a number of *energy densities of interest in plasmas*:

Electric:

$$W_e = \epsilon_0 E^2/2 \tag{C23}$$

Magnetic:

$$W_m = \frac{\mu_0 H^2}{2} = \frac{\mu_0}{2}\frac{E^2}{v_p^2\mu_0^2} = n^2\frac{\epsilon_0 E^2}{2} \tag{C24}$$

Electromagnetic:

$$W_{\text{em}} = W_e + W_m = (1+n^2)\epsilon_0 E^2/2 \tag{C25}$$

$$W_e - W_m = (1-n^2)\frac{\epsilon_0 E^2}{2} = \frac{\omega_p^2}{\omega^2}\frac{\epsilon_0 E^2}{2} \tag{C26}$$

Kinetic:

$$W_k = -\chi_e'(\epsilon_0 E_0^2/2)\sin^2\varphi. \tag{C27}$$

Total

$$\begin{aligned} W_t &= W_{\text{em}} + W_k \\ &= (\epsilon_0 E_0^2/2)\cos^2\varphi + n^2(\epsilon_0 E_0^2/2)\cos^2\varphi \\ &\quad + (1-n^2)(\epsilon_0 E_0^2/2)\sin^2\varphi \\ &= (1+n^2)(\epsilon_0 E_0^2/2)\cos^2\varphi \\ &\quad + (1-n^2)(\epsilon_0 E_0^2/2)\sin^2\varphi \\ &= 2n^2(\epsilon_0 E_0^2/2)\cos^2\varphi + (1-n^2)(\epsilon_0 E_0^2/2) \end{aligned} \tag{C28}$$

$$= W_p + W_s, \tag{C29}$$

where

$$W_p = n^2\epsilon_0 E_0^2\cos^2\varphi = \epsilon_r' E^2 = 2W_m \tag{C30}$$

is the energy density that propagates with the phase velocity, and

$$W_s = (1-n^2)\frac{\epsilon_0 E_0^2}{2} = \frac{\omega_p^2}{\omega^2}\frac{\epsilon_0 E_0^2}{2} = 2\langle W_k \rangle \tag{C31}$$

is the immobilized energy density.

Note that ED/2 is not equal to the electric-field energy density W_e; ED/2 is equal to $n^2\epsilon_0 E^2/2$, or W_m. It is equal to just that part of the electric-field energy density that propagates with the phase velocity; it does not include the immobilized part that is associated with the forced plasma oscillations and their local exchanges between kinetic and electric field energy. This property of ED/2 has been discussed by Booker[13] in a much more general context.

In terms of values averaged over a wave period,

$$\langle W_p \rangle = \langle W_{em} - W_k \rangle = n^2(\epsilon_0 E_0^2/2), \quad \text{(C32)}$$

$$\langle S \rangle = v_p \langle W_p \rangle ,$$

$$\langle W_t \rangle = \langle W_{em} + W_k \rangle = \epsilon_0 E_0^2/2, \quad \text{(C33)}$$

$$\langle S \rangle = v_g \langle W_t \rangle ,$$

$$\langle W_k \rangle = \langle W_e - W_m \rangle = \frac{1-n^2}{2}\frac{\epsilon_0 E_0^2}{2} = \frac{1}{2}\frac{\omega_p^2}{\omega^2}\frac{\epsilon_0 E_0^2}{2}, \quad \text{(C34)}$$

$$\frac{\langle W_k \rangle}{\langle W_{em} \rangle} = \frac{1-n^2}{1+n^2} = \frac{\omega_p^2/\omega^2}{2-\omega_p^2/\omega^2} = \frac{\omega_p^2}{2\omega^2 - \omega_p^2}, \quad \text{(C35)}$$

$$\begin{aligned}\langle S \rangle &= v_p \langle W_{em} - W_k \rangle = v_g \langle W_{em} + W_k \rangle \\ &= v_p n^2(\epsilon_0 E_0^2/2) = v_p 2 \langle W_m \rangle \\ &= v_g(\epsilon_0 E_0^2/2). \end{aligned} \quad \text{(C36)}$$

Stratton[14] gives an expression for the difference between the average electric and magnetic field densities using the complex Poynting vector, and it can adapted to give the result presented above in Eq. (C34).

Plasma oscillations at the plasma frequency are strictly local and do not propagate.[15] The treatment presented here shows that forced plasma oscillations at frequencies above the plasma frequency are also local and do not propagate. Both cases are associated with the fact that the conduction current exactly cancels that part of the displacement current associated with the plasma oscillations, so the plasma oscillations produce no magnetic field and no electromagnetic energy flow.

Booker[13] has given much consideration to energy densities in electromagnetic waves in plasmas, and he came close to developing the concept put forth here. He recognized that the kinetic energy does not propagate with the individual waves (even though it appears to do so, as the phases of the plasma oscillations are controlled by the passing field of electromagnetic waves), and he discussed the electric field density in a way that is equivalent to considering a portion of it, on average equal to the average kinetic energy density, as nonpropagating. Anyone interested in pursuing this subject in more depth, including the effects of ions, collisions, and ambient magnetic fields, should consult this source.

APPENDIX D: PROPAGATION IN WAVEGUIDES

Consider two coherent fields of plane waves of equal amplitude, with their electric vectors in the y direction, propagating in directions making angles $+\psi$ and $-\psi$ with the z axis and angles $-\theta$ and $\theta - \pi$ with the x axis. The propagation vectors are

$$\mathbf{k}_1 = \hat{\mathbf{i}}k_x + \hat{\mathbf{k}}k_z,$$

$$\mathbf{k}_2 = -\hat{\mathbf{i}}k_x + \hat{\mathbf{k}}k_z,$$

where $\tan\theta = k_z/k_x$. Let the amplitudes of each of the two wave fields be $E_0/2$; then the combined wave field is described by

$$\begin{aligned}E = E_1 + E_2 &= (E_0/2)\{[\cos(\omega t - k_z z - k_x x)] \\ &\quad + \cos(\omega t - k_z z + k_x x)\} \\ &= E_0 \cos(\omega t - k_z z)\cos(k_x x). \end{aligned} \quad \text{(D1)}$$

This pattern is periodic in the x and z directions, and it moves in the $+z$ direction with velocity $\omega/k_z = \omega/(k\sin\theta) = c/\sin\theta$. Figure 6 portrays the wave crests; the intersections (regions of maximum constructive interference) are identified by solid circles and the regions of maximum destructive interference are identified by crosses. The dashed lines identify two planes on which $\cos(k_x x) = 0$, where $E = 0$ at all times; the spacing between them is $\pi/k_x = \lambda/2\cos\theta$, and additional planes with $E = 0$ occur with similar spacing.

A reflecting surface could be placed along any of these planes, for example, at the lower one shown in Fig. 6 at $x = -\lambda/4\cos\theta$, and the wave field above that plane would then be reproduced by a single incident beam, the angle of incidence being θ. Another reflecting surface could be placed at the upper plane shown in the figure at $x = +\lambda/4\cos\theta$, with the wave field between the two planes being maintained by multiple reflections. This corresponds to the TE_1 mode in a waveguide. (If the second reflector were placed at $3\lambda/4\cos\theta$, the pattern would correspond to the TE_2 mode.) The pattern of dots along the z axis, representing regions of maximum constructive interference, propagates along the waveguide with velocity $c/\sin\theta$, and this is the phase velocity so far as propagation along the waveguide is concerned. However, the group velocity reduces the speed at which information can be transmitted to $c\sin\theta$ (corresponding to the zig–zag path length traveled at velocity c). To examine the immobilized energy density associated with the phenomenon of group velocity, we must look at the magnetic vectors.

Since the E vectors are in the y direction, the H vectors are in the x-z plane and have x and z components. The ratio E/H has its conventional value $c\mu_0$ for each of the two constituent waves, but the vector fields add differently for H and for E. The components for H are

$$H_{1x} = -(E_1/c\mu_0)\sin\theta, \quad H_{2x} = -(E_2/c\mu_0)\sin\theta, \quad \text{(D2a)}$$

$$H_{1z} = +(E_1/c\mu_0)\cos\theta, \quad H_{2z} = -(E_2/c\mu_0)\cos\theta. \quad \text{(D2b)}$$

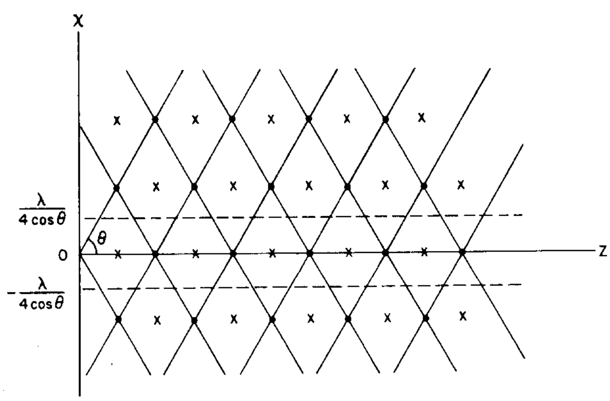

Fig. 6. The interference pattern for two plain wave fields whose wave fronts make angles $\pm\theta$ with the z axis. The portion of the wave field between $x = \lambda/(4\cos\theta)$ and $x = -\lambda/(4\cos\theta)$ is the same as that in a waveguide with propagation in the TE_1 mode. The whole pattern moves in the $+z$ direction with velocity $c/\sin\theta$.

Thus, at $x = 0$, where $E_1 + E_2$ has its maximum amplitude and where $E_1 = E_2$,

$$H_z = H_{1z} + H_{2z} = 0$$

and

$$H = H_x = -[(E_1 + E_2)/c\mu_0]\sin\theta$$
$$= -(2E_1/c\mu_0)\sin\theta.$$

At $x = \pm\lambda/4\cos\theta$, i.e., at the reflecting surfaces, $E_1 + E_2 = 0$. The x components of H_1 and H_2 cancel, while the z components add, yielding

$$H = H_z = \pm(2E_1/c\mu_0)\cos\theta. \quad \text{(D3)}$$

The presence of a z component in the magnetic field is indicative of a standing wave between the mirror surfaces (or in the wave pattern of the superimposed plane-wave fields). To evaluate this, consider the individual waves:

$$\mathbf{E}_1 = \hat{\mathbf{j}}(E_0/2)\cos(\omega t - k_z z - k_x x),$$
$$\mathbf{E}_2 = \hat{\mathbf{j}}(E_0/2)\cos(\omega t - k_z z + k_x x),$$
$$\mathbf{H}_1 = -\hat{\mathbf{i}}(E_0/2c\mu_0)\cos(\omega t - k_z z - k_x x)\sin\theta + \hat{\mathbf{k}}(E_0/2c\mu_0)\cos(\omega t - k_z z - k_x x)\cos\theta,$$
$$\mathbf{H}_2 = -\hat{\mathbf{i}}(E_0/2c\mu_0)\cos(\omega t - k_z z + k_x x)\sin\theta - \hat{\mathbf{k}}(E_0/2c\mu_0)\cos(\omega t - k_z z + k_x x)\cos\theta,$$
$$H_x = (-E_0/2c\mu_0)[\cos(\omega t - k_z z - k_x x) + \cos(\omega t - k_z z + k_x x)]\mathit{sin}\ \theta,$$
$$= (-E_0/2c\mu_0)2\cos(\omega t - k_z z)\cos(k_x x)\mathit{sin}\ \theta$$
$$= (-E/c\mu_0)\sin\theta. \quad \text{(D4a)}$$
$$H_z = (E_0/c\mu_0)\sin(\omega t - k_z z)\sin(k_x x)\cos\theta. \quad \text{(D4b)}$$

Note that H_z is in quadrature with E. The contribution of H_z to the Poynting vector is in the $\pm x$ direction, and it averages zero over a wave period. The contribution of H_x to the Poynting vector is in the $+z$ direction. Of the average electric field energy density, an amount equal to that part of the average magnetic field energy density corresponding to the z component of the magnetic field is associated with the standing wave and does not propagate in the z direction. The average energy density that propagates with the phase velocity is the remaining part of the average electric field energy density plus the average energy density associated with the x component of the magnetic field.

Let $\langle W_{mz}\rangle$ be *the time and space average* of the energy density associated with the z component of the magnetic field, $\langle W_{mx}\rangle$ the average energy density associated with the x component, and $\langle W_e\rangle$ the average energy density of the electric field. The total average electromagnetic energy density is then $\langle W_{em}\rangle = \langle W_e\rangle + \langle W_{mx}\rangle + \langle W_{mz}\rangle$. The average energy density of the standing wave in the transverse direction is $2\langle W_{mz}\rangle$, and the average energy density propagating with the phase velocity is

$$\langle W_{em}\rangle - 2\langle W_{mz}\rangle = \langle W_e\rangle + \langle W_{mx}\rangle - \langle W_{mz}\rangle. \quad \text{(D5)}$$

In this Appendix $\langle\ \rangle$ means average over space and time. Averaging over time gives average values equal to half the peak values, and averaging over the cross-sectional area of the waveguide further reduces the maximum values by another factor of 2, so the average densities are

$$\langle W_e\rangle = \epsilon_0 E_0^2/8, \quad \text{(D6)}$$
$$\langle W_{mz}\rangle = (\epsilon_0 E_0^2/8)\cos^2\theta, \quad \text{(D7)}$$
$$\langle W_{mx}\rangle = (\epsilon_0 E_0^2/8)\sin^2\theta, \quad \text{(D8)}$$
$$\langle W_m\rangle = \langle W_{mz}\rangle + \langle W_{mx}\rangle = \langle W_e\rangle = \epsilon_0 E_0^2/8, \quad \text{(D9)}$$
$$\langle W_t\rangle = \langle W_{em}\rangle = \langle W_e\rangle + \langle W_m\rangle = \epsilon_0 E_0^2/4, \quad \text{(D10)}$$
$$\langle W_s\rangle = 2\langle W_{mz}\rangle = \epsilon_0 E_0^2/4)\cos^2\theta, \quad \text{(D11)}$$

and

$$\langle W_p\rangle = \langle W_e\rangle + \langle W_{mx}\rangle - \langle W_{mz}\rangle$$
$$= (\epsilon_0 E_0^2/8)(1 + \sin^2\theta - \cos^2\theta)$$
$$= (\epsilon_0 E_0^2/4)\sin^2\theta. \quad \text{(D12)}$$

To obtain the average value of the Poynting vector, we need consider only its z component (which is associated with the x component of H). At $x = 0$,

$$S_z = -E_y H_x = (E_1 + E_2)(2E_1 \sin\theta)/c\mu_0$$
$$= 4E_1^2 \sin\theta)/c\mu_0$$
$$= 4c\epsilon_0 E_1^2 \sin\theta = (E^2/c\mu_0)\sin\theta$$
$$= c\epsilon_0 E^2 \sin\theta. \quad \text{(D13)}$$

Taking the time average at $x = 0$ yields $c(\epsilon_0 E_0^2/2)\sin\theta$. Then taking the average over x,

$$\langle S\rangle = c(\epsilon_0 E_0^2/4)\sin\theta = c\sin\theta\langle W_t\rangle = v_g\langle W_t\rangle. \quad \text{(D14)}$$

This can also be expressed as

$$\langle S\rangle = (c/\sin\theta)(\epsilon_0 E_0^2/4)\sin^2\theta = v_p\langle W_p\rangle. \quad \text{(D15)}$$

In this case, the most natural association to make with the average Poynting flux is group velocity times the time- and space-averaged electromagnetic energy density. However, if instantaneous values of the energy flux are desired, the product of the phase velocity and instantaneous values of the space-averaged propagating energy density $(\epsilon_0 E^2/2)\sin^2\theta$ must be used.

[1] F. S. Crawford, "Water-wave machine for demonstrating group velocity," Am. J. Phys. **41**, 1203–1205 (1973). Also see footnote 8 of Ref. 2.

[2] F. S. Crawford, "Elementary derivation of the wake pattern of a boat," Am. J. Phys. **52**, 782–785 (1984).

[3] W. Froude, *On Experiments upon the Effect Produced on the Wave-Making Resistance of Ships by Length of Parallel Middle Body* (Institution of Naval Architects, 1877), or see K. S. M. Davidson, *Principles of Naval Architecture* (Society of Naval Architects and Marine Engineers, New York, 1939), Vol. 2, p. 67.

[4] Courtesy of Dr. Jack A. C. Kaiser, U. S. Naval Research Laboratory, Washington, D.C.

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[8] L. Brillouin, *Wave Propagation and Group Velocity* (Academic, San Diego, 1960), p. 99, Eq. (30).

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[10] E. Hecht, *Optics* (Addison-Wesley, Reading, MA, 1987), p. 60.

[11] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., p. 237.

[12] P. Lorrain, D. R. Corson, and F. Lorrain, *Electromagnetic Waves and Fields* (Freeman, New York, 1988), p. 544 give the above phase relationships among the displacement current, the conduction current, and the electric field in a plasma, using the terminology equivalent relative permittivity in place of the apparent relative permittivity used here; in the more general case that they discuss, this may be positive or negative. They do not discuss velocity of energy flow, but they pose a problem (29-12) in

which they ask that the kinetic energy be evaluated and related to the Poynting vector and the group velocity.

[13] H. G. Booker, *Cold Plasma Waves* (Martinus Nijhoff, Boston, 1984), Chap. 3.

[14] The difference between the average electrical and magnetic energy densities can be evaluated using the complex Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{H}^*)/2$ [J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 137]. Stratton states that the divergence of the imaginary part of div **S** [his Eq. (33)] is equal to 2ω times this difference, implicitly assuming the conductivity σ to be real, although this is not a requirement of the derivation. For the case under consideration here, σ is purely imaginary, as indicated by Eq. (C12), and the rhs of Stratton's equation becomes purely imaginary. Further, for a steady-state wave field, the divergence of the complex Poynting vector is zero. With these adjustments, Stratton's equation yields the result given here for the difference between the average electric and magnetic energy densities.

[15] G. Schmidt, *Physics of High Temperature Plasmas* (Academic, New York, 1966), p. 200.

The random walk method for dc circuit analysis

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A novel method for circuit analysis is presented, which can help in the development of intuition about the current distribution in a complicated circuit with an emf source and resistors, and can be used for analytic and numerical calculations.

I. INTRODUCTION

The connection between random walks and potential theory has been known for some time.[1,2] Application to electrical circuits and the equivalence of random walks to Kirchhoff's laws are discussed briefly in some more recent mathematical treatises.[3,4] A very nice small book that appeared 5 years ago presents in pedagogical detail the identity between Markov chains and resistor networks.[5] These presentations are made by mathematicians and seem to be little known in the physics and electrical engineering community. To my knowledge, the method has not been used to help physics students understand the behavior of complex circuits. In this note the ideas will be presented in the simplest terms and it is hoped that physics teachers may find this interesting and useful in teaching circuit theory.

For beginning students of electromagnetism, the concepts of charge and force are thought to be easy, while field and potential are more difficult to comprehend. Simple key points are the repulsion of like charges and charge conservation. The study of dc currents introduces resistors, wires (zero resistance), and batteries (source of emf), which are put together into circuits. A circuit consists of a set of "nodes" with each node connected to *other* nodes by resistors. There may be several resistors connecting the same pair of nodes. Any set of points of the network, mutually connected by wires, together with those wires themselves, constitutes a single node. The terminals of one or more batteries will be connected to nodes. The resistance of a resistor is usually defined by Ohm's law i.e., by $R = \Delta V/I$, where ΔV is the electric potential difference between the ends of the resistor.

II. THE PROBLEM

It is important for students to understand both quantitatively and qualitatively the way the current will be distributed in a circuit. The simplest example worked out in detail in a first course is the effective resistance of two resistors in parallel, $1/R = 1/R_1 + 1/R_2$. This equation follows from the fact that the current will branch through the two resistors in the ratio of their conductances, i.e., $I_1/I_2 = S_1/S_2$, where the conductance $S = 1/R$.

The usual method to treat such problems is the use of Kirchhoff's node and loop laws. The node law, that there be no net current into a node, follows from charge conservation and the fact that the repulsion of like charges prohibits the accumulation of net charge at any node or resistor, and is easy for students to understand. The loop law is a little more difficult since the concepts of IR drop and emf are required. The law states that the algebraic sum of IR drops around any loop must equal the emf gain of the same loop. The resulting equations are easy to formulate, but the solution of the resulting coupled linear system is not transparent and the qualitative nature of the solution is not easily seen short of doing the calculation.

We present here an alternative method for any case in which a *single* source of emf is producing current in a circuit of any complexity. The circuit will have one node called the "input" node, where the current enters from the positive terminal of the battery, and one called the "output" node, where the current exits to the negative terminal. Between these nodes will be the network of resistors. Extension to circuits with several batteries is easy.

One way to think about such a circuit physically is to note that the emf of the battery will produce **E** fields in the resistors. The fields "push" the charge, producing currents that will follow these fields, branching at a node more or less in the same way as the lines of the field branch out. This gives a nice picture, but is not very helpful to the intuition since the bending of the **E** fields in the wires and resistors near a node is quite complicated. Since the branching *ratio* for currents is independent of the emf for the case we con-

