The problem of spherically symmetric electromagnetic radiation

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Using Maxwell equations for the radiation's transverse fields, it is proved that a genuine spherically symmetric electromagnetic radiation cannot exist. © *2002 American Association of Physics Teachers.*
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The existence of a spherically symmetric wave might be regarded as a self-evident phenomenon having a rather simple mathematical structure. Such a structure is obviously true for the longitudinal sound wave emitted from a breathing sphere that is placed in an isotropic medium. Also, the blackbody radiation emitted from an isothermic sphere is statistically spherical. These kinds of waves are not discussed here. We will discuss waves in the wave zone whose distance from the source is much greater than both the source's linear size and the radiation's wavelength. The problem analyzed here is called ''genuine spherically symmetric electromagnetic radiation.'' This term means that *at the wave zone*, there exists a set of concentric spherical shells, where at all points of each shell, the Poynting vector[1,2]

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \tag{1}$$

is *radial* and has the *same* magnitude. (This magnitude varies as the radius of the shell changes and may vary as a function of time.)

The case analyzed in this paper is not restricted to states whose fields have a well-defined parity. Indeed, the Poynting vector has a well-defined parity $\mathbf{S}(\mathbf{r}) = -\mathbf{S}(-\mathbf{r})$. On the other hand, the transverse electromagnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are not necessarily related to $\mathbf{E}(-\mathbf{r})$ and $\mathbf{B}(-\mathbf{r})$, respectively.

Unlike longitudinal sound waves, a system emitting a genuine spherically symmetric electromagnetic radiation is far from being trivial. Indeed, an elementary radiating object is either a quantum mechanical system emitting appropriate dipole or higher multipole radiation or a loop of a time-dependent classical current. The latter can also be expanded in multipoles.[3] As is well known, each of these objects emits radiation whose intensity varies with direction. However, it is not clear whether or not one can arrange infinitely many multipoles (each of which emits an infinitesimal amount of energy) so that the entire result takes the form of a genuine spherically symmetric electromagnetic radiation.

As is well known, electrodynamics examines two kinds of entities, charges and their currents on one hand and electromagnetic fields on the other. An examination of the possible ways of building a radiating system containing an arbitrary number of dipoles (or higher multipoles) looks very difficult. In the following, the fields of a genuine spherically symmetric radiation are analyzed, and it is shown that this kind of radiation is inconsistent with Maxwell equations. This conclusion enhances the insight into the internal structure of electrodynamics. The proof assumes that a genuine spherically symmetric electromagnetic radiation exists and arrives at a contradiction. The case discussed here provides an example of constraints imposed by Maxwell equations, and indicates that not every arrangement of electromagnetic fields can be realized.

Let us consider a spherical shell S belonging to the set mentioned above and examine it at an instant when $\mathbf{S}$ in Eq. (1) does not vanish. Because the electromagnetic fields are perpendicular to $\mathbf{S}$, they have no radial component. Moreover, because at all points on S, the magnitude of the Poynting vector is the same, the magnitude of the transverse field $\mathbf{E}$ (and $\mathbf{B}$) is the same also. The latter conclusion relies on the orthogonality of $\mathbf{E}$ and $\mathbf{B}$ at the wave zone.[4]

Consider a point P on the spherical shell S. $\mathbf{E}(P)$ denotes the electric field at P. The direction of $\mathbf{E}(P)$ determines a great circle on S that passes through P such that the electric field $\mathbf{E}(P)$ is tangent to it. Henceforth, this great circle is called the equator (see Fig. 1).

Now let us construct a trajectory C on S that is based on the following assumptions. C starts at P. Also for every point p on C, the tangent to C at p is in the direction of $\mathbf{E}(p)$. Because $|\mathbf{E}|$ has the same value at all points of S, $\mathbf{E}$ does not vanish there and the definition of the tangent is a unique function defined at every point belonging to S. This definition of C is unique. Indeed, the tangent unit vector of differential geometry takes the following form:

$$\mathbf{t} \equiv \frac{d\mathbf{r}}{dl} = \frac{\mathbf{E}}{E}, \tag{2}$$

where l denotes the arclength.[5] Hence, we obtain a well-defined ordinary first-order differential equation. Thus, by the existence and uniqueness theorems of differential equations,[6] C is well-defined and unique.

The trajectory C is used below and the following cases are analyzed:

(1) One can find a quantity $\epsilon > 0$ such that the trajectory C passes at the point Q whose distance from the equator is greater than ϵ (see Fig. 1).
(2) Otherwise.

Assume that case (1) holds. Thus after reaching point Q, the trajectory C is closed by adding to it the shorter part C' of the great circle that passes through P and Q (the arc QRP in Fig. 1). Evidently, the length of C' is shorter than that of C. The above assumptions imply that the value of the following path integral is positive,

$$\oint_{C+C'} \mathbf{E} \cdot d\mathbf{l} = \int_C \mathbf{E} \cdot d\mathbf{l} + \int_{C'} \mathbf{E} \cdot d\mathbf{l} > 0. \tag{3}$$

Indeed, the length of C is greater than that of C' and, at every point of C, the electric field $\mathbf{E}$ is tangent to the path.
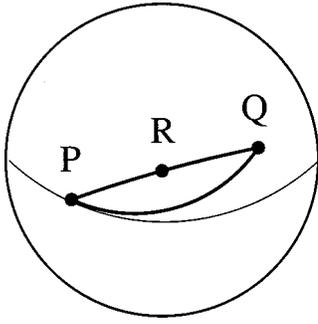
Fig. 1. A sphere at the wave zone. The thin line denotes a great circle called the equator. The path PQ is tangent to the equator at point P. The path QRP is a portion of a great circle (see the text).

Thus, $\cos\theta$ of the scalar product takes the maximal value of unity. Because $|\mathbf{E}|$ is uniform on S, the integral on C is greater than the absolute value of that of C'.

Now, let us turn to the second case. Here C coincides with the equator which is taken as the closed path. Hence, in this case C' reduces to a point, but Eq. (3) still holds. Using vector analysis, we find from Eq. (3) that

$$\oint_{C+C'} \mathbf{E}\cdot d\mathbf{l} = \int_{S'} (\mathbf{\nabla}\times\mathbf{E})\cdot d\mathbf{s} > 0, \tag{4}$$

where S' is any surface whose boundary is the closed path $C+C'$. Let us choose the corresponding part of the spherical shell S as the surface S'. It follows from Eq. (4) that there exists a region on S' where the radial part of the following vector satisfies $(\mathbf{\nabla}\times\mathbf{E})_r > 0$. Hence, Maxwell's equation yields

$$(\mathbf{\nabla}\times\mathbf{E})_r = -\frac{1}{c}\frac{\partial B_r}{\partial t} > 0. \tag{5}$$

Thus, the assumption that genuine spherically symmetric radiation exists yields a contradiction, because for any spherical radiation, the longitudinal field component B_r together with its derivative with respect to the time, vanishes identically.[7] This result completes the proof that a genuine spherically symmetric electromagnetic radiation cannot exist. It can be easily seen that the proof holds not only for an outgoing radiation but for an incoming one as well.

As stated, there is an infinite number of different systems emitting electromagnetic radiation. Each of these systems is a particular arrangement of electric and magnetic dipoles or higher multipoles and appropriate classical currents. Thus, there are infinitely many patterns of electromagnetic radiation. However, in spite of the infinite degrees of freedom available, not every pattern of radiation can be realized.

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[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), p. 76.
[2] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 237.
[3] See Ref. 2, pp. 744–747.
[4] See Ref. 1, p. 162, and Ref. 2, p. 657.
[5] D. E. Rutherford, *Vector Methods* (Oliver and Boyd, Edinburgh, 1962), 9th ed., pp. 16–19.
[6] E. L. Ince, *Ordinary Differential Equation* (Dover, New York, 1956), Chap. III.
[7] See Ref. 1, p. 162, Ref. 2, p. 657.