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[17]Often $j_{\mu}(x)$ is defined as the negative of the expression (8).
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[19]The antisymmetric tensors are called *super-potentials* by Jackiw (Refs. 10 and 11).
[20]See Ref. 8, pp. 229–235.
[21]See Ref. 8, p. 25.
[22]The corresponding expression obtained in Ref. 16 is incorrect.
[23]The fact that one obtains the same equations of motion from $\mathscr{L}$ and $\mathscr{L}_i$ is a special case of the nonuniqueness theorem proven in the Appendix.

A communication on electrical charge relaxation in metals

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(Received 4 April 1988; accepted for publication 13 March 1989)

The correction of an erroneous textbook derivation on electrical charge relaxation in conductors is discussed. The actual decay in a good conductor is damped oscillatory instead of the simple exponential decay that is often claimed, while short wavelength disturbances spread through the medium much like particles of mass $m = \hbar\omega_p/\langle v^2\rangle$, where ω_p is the plasmon frequency and $\langle v^2\rangle$ is the mean-square electron velocity.

A derivation that is found in many prominent electromagnetics textbooks for physicists[1] and engineers,[2] and at least one well-known optics text,[3] concerns the relaxation of electrical charge in a conductor. Its expressed purpose is to demonstrate that the free relaxation of a disturbance away from equilibrium in the charge density is an extremely rapid process, so that on the time scale of most physical events no electrical charge perturbance can prevail inside a good conductor. Even though it has been pointed out before[4,5] that both the proof and the result given in these books are seriously in error, many authors of new texts continue to include it. The error is due to the implicit assumption that the relaxation time of the charge density is a dc phenomenon, even though the time scale of the process is calculated to be as short as 10^{-19} s. This decay rate actually corresponds to the x-ray frequency region! It is desirable that proofs presented in undergraduate textbooks be within the grasp of the average student but educational objectives are poorly served by erroneous proofs, especially when widely adopted. Moreover, for those texts that are written at the graduate level, such "simplification" is even less appropriate.

This issue was apparently first attended to by Ashby,[4] who points out that the actual electrical decay time of a conductor should in fact be of the order of the electron collision time, which is about 2×10^{-14} s for copper. A more recent discussion, by Ohanian,[5] makes an excellent point of the fact that the relaxation actually proceeds in three stages. The first stage is the relaxation of electrical charge density. The second stage is the expulsion of the electric and magnetic fields to the exterior of the conductor and that of currents to the surface. During the third stage, the process terminates with the slower ohmic and radiative damping of the surface currents. Ohanian's discussion centers on the second and third stages of the process, while Ashby treats the first.

Physical consideration shows that a description of the first stage, that of charge relaxation, depends on whether or not the total charge is zero. In the former case, the relaxation of current and charge densities is accompanied by a transport of the surplus charge to the surface of the conductor, which requires special mathematical treatment. The time constant associated with this stage is then also related to the size and geometry of the conductor, as well as to the initial charge distribution. Moreover, the equations describing the transport of the uncompensated charge are nonlinear, the solution of which would be much more difficult. And finally, since there is no definitive decoupling

between fields and charges, the division between the first and second stages is expected to be blurred. We discuss here only the relaxation of the more important case of compensated charge fluctuations, which may be accurately described by linearized equations. Then the relaxation of electrical charge density is largely independent of initial conditions and of the conductor's size and shape.

The diffusion of the electric and magnetic fields and associated currents will endure much longer, being characterized by a time constant $\tau_D = \mu_0 \sigma d^2$,[5] where d is a length of the order of magnitude of the conductor's dimensions. These fields and currents diffuse to the surface of the conductor, and the fields are expelled to the exterior, persisting with the surface currents until they are damped by Ohmic and radiation losses.[5]

The procedure generally followed in textbooks is based on the equation of continuity of the current density $\mathbf{J}(\mathbf{r},t)$ and charge density $\rho(\mathbf{r},t)$:

$$\operatorname{div} \mathbf{J}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0. \tag{1}$$

The assumption of instantaneous, linear response is made through Ohm's law, i.e.,

$$\mathbf{J}(\mathbf{r},t) = \sigma \mathbf{E}(\mathbf{r},t), \tag{2}$$

where σ is the (dc) conductivity. Finally, Gauss' law, i.e.,

$$\operatorname{div} \mathbf{E}(\mathbf{r},t) = \rho(\mathbf{r},t)/\epsilon_0 \tag{3}$$

is substituted. This yields a first-order differential equation describing the time dependence of $\rho(\mathbf{r},t)$:

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \frac{\rho(\mathbf{r},t)}{t_r} = 0, \tag{4}$$

where

$$t_r = \epsilon_0/\sigma \tag{5}$$

is the relaxation time constant. The solution of Eq. (4) is

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0)\exp(-t/t_r). \tag{6}$$

A typical numerical value of t_r as calculated from Eq. (5) is 1.5×10^{-19} s for copper, which is the value reported in Refs. 1–3. One expects that on such a short time scale the assumption of instantaneous response, as implied by Eq. (2), would almost certainly be invalid.

The simplest model describing the relaxation of electrical charge leading to an essentially correct result is derived from the classical equation of motion of the average electron velocity, known as the "Drude model" of electrical conduction.[6] This model is named after the German physicist P. K. L. Drude who in 1900 proposed the existence of a dense electron gas in metals. Even though the model is classical, the Drude model accounts well for the conduction of electricity in metals where the electron distribution is highly degenerate, as well as in nonmetals.

Consider an initially neutral conducting medium containing in equilibrium n_0 conduction electrons per unit volume, which are compensated by an equal concentration of positive lattice ions. At the time $t = 0$ a charge perturbation $\rho(\mathbf{r},0)$ is introduced. The volume integral of this function is required to vanish. Then the mean velocity $\mathbf{v}(\mathbf{r},t)$ of mobile electrons at the point $\mathbf{r}$ satisfies the classical equation of motion of a particle of mass m and charge e that is acted upon by an electric field $\mathbf{E}(\mathbf{r},t)$ and a viscous damping force that is proportional to the electron velocity, i.e.,

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} + \frac{\mathbf{v}}{\tau} = \frac{e\mathbf{E}(\mathbf{r},t)}{m}. \tag{7}$$

The relaxation time constant τ is of the order of magnitude of the collision time. To a first approximation it may be assumed that the damping force acts instantaneously, as expressed in Eq. (7). It is not clear that Eq. (7) should apply to electrons having kinetic energies deep within the Fermi surface of a metal because the exclusion principle prohibits the occupation of any state by more than one electron. The explanation[7] is that under the action of the electric field $\mathbf{E}$ the entire Fermi surface makes a uniform translation of magnitude $\delta\mathbf{q} = e\mathbf{E}\tau/h$, so that each electron acts effectively as if it were independent of the others.

Multiplying Eq. (7) by en_0, we obtain an equation in the mean current density $\mathbf{J}(\mathbf{r},t) = en_0\mathbf{v}(\mathbf{r},t)$ [note that this definition approximates the actual electron number density $n(\mathbf{r},t)$ by its average n_0], i.e.,

$$\frac{\partial \mathbf{J}(\mathbf{r},t)}{\partial t} + \frac{\mathbf{J}(\mathbf{r},t)}{\tau} = \frac{e^2 n_0}{m} E(\mathbf{r},t). \tag{8}$$

Next we take the divergence of both side of this equation, and again substitute Eqs. (1) and (3) to eliminate $\mathbf{J}$ and $\mathbf{E}$. If part of $\mathbf{E}$ is an external field, which is divergenceless, then this part will not couple to the volume charge density although it can induce surface charges. The resulting differential equation is of second order in the charge density:

$$\frac{\partial^2 \rho(\mathbf{r},t)}{\partial t^2} + \tau^{-1}\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \omega_p^2 \rho(\mathbf{r},t) = 0, \tag{9}$$

where

$$\omega_p^2 = e^2 n_0/m\epsilon_0 \tag{10}$$

is the square of the (angular) plasmon frequency[8] of the conduction electrons in the metal. Equation (9) is the usual differential equation of the damped harmonic oscillator in the absence of a driving force.

We solve Eq. (9) by assuming an exponential time dependence of the excess charge density

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0)\exp(-st), \tag{11}$$

which on substitution yields the relaxation rate s:

$$s = (2\tau)^{-1} \pm i\left[\omega_p^2 - (2\tau)^{-2}\right]^{1/2}. \tag{12}$$

Equation (12) shows that for a conductor satisfying $\omega_p\tau > \frac{1}{2}$, the electrical charge density relaxes during time 2τ and is accompanied by oscillations in the charge density at the frequency ν, given by

$$\nu = \left[\omega_p^2 - (2\tau)^{-2}\right]^{1/2}. \tag{13}$$

Conforming to the assumption that the time τ is independent of frequency,[9] its numerical value is estimated from the dc conductivity according to the relation obtained from Eq. (8), i.e.,

$$\sigma = n_0 e^2 \tau/m. \tag{14}$$

Using the values $\sigma = 6\times10^5$ $(\Omega\,\mathrm{cm})^{-1}$ and $n_0 = 8.5\times10^{22}\ \mathrm{cm}^{-3}$ corresponding to those of copper,[6] then Eq. (14) gives $\tau = 2\times10^{-14}$ s and from Eq. (10) $\omega_p = 1.6\times10^{16}$ Hz.

The contrary condition $\omega_p\tau < 0.5$ yields a pair of decay rates, which in the limit

$$\omega_p\tau \ll 0.5 \tag{15}$$

approach the expressions

$$\Gamma_1 = \sigma/\epsilon \tag{16a}$$

and

$$\Gamma_2 = \tau^{-1}, \tag{16b}$$

respectively. By Eq. (10), condition (15) is equivalent to

$$\sigma \ll \epsilon/\tau, \tag{17}$$

which is the limit of a very poor conductor and also the limit in which Eq. (5) applies. The corresponding solution for $\rho(t)$ is

$$\rho(t) = A\exp(-\Gamma_1 t) + B\exp(-\Gamma_2 t), \tag{18}$$

where A and B are determined from initial conditions

$$A + B = \rho(0),$$

$$\Gamma_1 A + \Gamma_2 B = -\frac{d\rho(0)}{dt}.$$

Assuming that any initial time rate of change of ρ on the time scale τ is likely to be small, we take $d\rho(0)/dt \approx 0$; then $B \approx 0$ since $\Gamma_1 \ll \Gamma_2$ and, therefore, most of the charge disturbance relaxes at the slower rate Γ_1.

It is of interest to make a brief estimate of the validity of the approximations made in this calculation. Thus, in the steps leading from Eq. (7) to (8), a term was neglected of the order of the fraction $\delta n/n_0$ as large as the other terms so that this approximation is valid when $\delta n/n_0 \ll 1$. In addition, a convection term equal to $v\,\nabla v$ was neglected on the left-hand side of Eq. (7). This quantity is of magnitude v^2/d, where d is a characteristic distance of the extent of the charge disturbance. Setting $E \approx \rho d/\epsilon_0$, we obtain as a condition

$$v^2/d \ll e\rho d/\epsilon_0.$$

Eliminating v by the estimate $v \approx eE\tau/m$ yields for $\delta n = \rho/e$

$$\delta n \ll m\epsilon_0/(e\tau)^2.$$

Since the right-hand side of the inequality is of the order of $10^{26}\ \mathrm{m}^{-3}$ for a conductor, it is satisfied in almost any physical situation.

The Drude model can be derived within the framework of semiclassical quantum mechanics from an electron velocity distribution function in the state of momentum $\hbar\mathbf{q}$, i.e., $f_q(\mathbf{r},t)$, by solution of the Boltzmann equation.[7] Since the distribution function specifies both the momentum and the position of electrons, this approach is restricted to disturbances of characteristic wavelength greater than the typical deBroglie wavelength of electrons. The quantum mechanically more rigorous approach, based upon the individual particle approximation, is available from linear response theory[10] for the longitudinal dielectric constant.

The longitudinal dielectric constant is defined in terms of the charge density that is induced according to the linear response theory by a "test charge" density at fixed spatial and temporal frequencies, e.g., if $\rho(\omega,\mathbf{q})$ and $\rho_0(\omega,\mathbf{q})$ represent the induced and test charge densities, respectively, then[11]

$$\epsilon(\omega,\mathbf{q}) = \rho_0(\omega,\mathbf{q})/[\rho(\omega,\mathbf{q}) + \rho_0(\omega,\mathbf{q})]. \tag{19}$$

In the absence of a test charge, the partial differential equation satisfied by $\rho(\mathbf{r},t)$ is then obtained from

$$\epsilon_{\mathrm{op}}\rho(\mathbf{r},t) = 0, \tag{20}$$

where ϵ_{op} is the operator obtained from $\epsilon(\omega,\mathbf{q})$ by the substitutions

$$\omega \to i\frac{\partial}{\partial t}, \tag{21a}$$

$$\mathbf{q} \to -i\,\mathrm{grad}. \tag{21b}$$

The result of the local theory, with $\mathbf{q} = 0$, Eq. (9) is accordingly obtained from the long-wavelength form of the dielectric constant of a plasma

$$\epsilon(\omega,0) = 1 - \omega_p^2/\omega^2. \tag{22}$$

For finite q, the appropriate starting formula for the dielectric constant of a metal is the Lindhard expression,[10] given by

$$\epsilon(\omega,\mathbf{q}) = 1 - \left(\frac{e}{\epsilon_0 q}\right)^2 \sum_{\mathbf{k}} \frac{f_{\mathbf{k}+\mathbf{q}} - f_{\mathbf{k}}}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k})}, \tag{23}$$

where f is the Fermi distribution function and $E(\mathbf{k}) = (\hbar k)/2m$ is the energy of an electron in the free-electron approximation. Some simple manipulations put Eq. (23) in the following form:

$$\begin{aligned}\epsilon(\omega,\mathbf{q}) &= 1 - \frac{(e/q)^2}{\epsilon_0}\sum_{\mathbf{k}} \frac{f_{\mathbf{k}}[2E(\mathbf{k}) - E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}-q)]}{[E(\mathbf{k}) - E(\mathbf{k}-\mathbf{q}) + \hbar\omega][E(\mathbf{k}) - E(\mathbf{k}+\mathbf{q}) - \hbar\omega)]} \\ &= 1 - \frac{e^2}{(m\epsilon_0\omega^2)}\sum_{\mathbf{k}} \frac{f_{\mathbf{k}}}{(1 + \hbar\mathbf{k}\cdot\mathbf{q}/m\omega)^2 - (\hbar q^2/2m\omega)^2}.\end{aligned} \tag{24}$$

The summation over momentum states is carried out by expanding Eq. (24) in powers of q up to quadratic terms in q,

$$\epsilon(\omega,\mathbf{q}) = 1 - e^2/(m\epsilon_0\omega)^2 \sum_{\mathbf{k}} f_{\mathbf{k}} \times\left[1 - \frac{2\hbar\,\mathbf{k}\cdot\mathbf{q}}{(m\omega)} + 3\left(\frac{\hbar}{(m\omega)}\right)^2(\mathbf{k}\cdot\mathbf{q})^2 + \cdots\right].$$

Using the properties of the distribution function $f_{\mathbf{k}}$, i.e.,

$$\sum_{\mathbf{k}} f_{\mathbf{k}} = n_0,$$

$$\sum_{\mathbf{k}} f_{\mathbf{k}}\,\mathbf{k}\cdot\mathbf{q} = 0,$$

and

$$\sum_{\mathbf{k}} f_{\mathbf{k}}(\mathbf{k}\cdot\mathbf{q})^2 = n_0\langle k_z^2\rangle q^2,$$

and upon including damping by adding the imaginary con-

stant $\frac{1}{2}i\alpha$ to ω, we have for the dielectric constant of the electron gas of a metal of cubic crystal symmetry

$$\epsilon(\omega,\mathbf{q}) = 1 - e^2 n_0/m\epsilon_0(\omega + \tfrac{1}{2}i\alpha)^2(1 + \bar{v}^2q^2 + \cdots), \quad (25)$$

since by symmetry

$$\langle(\hbar k/m)^2\rangle = \tfrac{1}{3}\langle(\hbar k/m)^2\rangle = 1/3\bar{v}^2,$$

where $\bar{v}^2 = \langle v^2\rangle$ is the average of the square of the velocity of the conduction electrons. Equation (25) gives the dielectric constant of the noninteracting electron gas to second order in q.

Finally, making the substitutions (21) in Eq. (25) and operating on the density of charge $\rho(\mathbf{r},t)$ leads to the following differential equation:

$$\frac{\partial^2\rho}{\partial t^2} + \alpha\frac{\partial\rho}{\partial t} + \omega_p^2\rho = \bar{v}^2\,\nabla^2\rho, \quad (26)$$

where at zero degrees Kelvin:

$$\bar{v}^2 = \tfrac{3}{5}v_F^2 = \tfrac{2}{5}E_F/m,$$

where v_F is the velocity on the Fermi sphere's surface and E_F is the Fermi energy. Equation (26) shows that a sinusoidal charge disturbance is propagated with a phase velocity v_p satisfying

$$\infty > v_p > \bar{v} \approx 0.7v_F,$$

where $v_F = 1.57\times10^8$cm/s for copper. Since the smallest wave vector at which a spatially harmonic disturbance leads to significant dispersion is roughly of magnitude $\omega_p/v_F \approx 1/a$, where a is the lattice parameter, the term on the right-hand side of Eq. (26) is normally negligible except for charge density disturbances of spatial dimensions smaller than a small multiple of a.

As an example, let the charge disturbance at $t = 0$ be $\rho(\mathbf{r},0) = \rho$ when $r < a$ and $\rho(\mathbf{r},0) = 0$ when $r > a$. Then the solution of Eq. (26), for $t > 0$, is

$$\rho(r,t) = \frac{2\rho_0 a}{\pi} \times \int_0^\infty dk\, ka\, j_1(ka)\, j_0(kr)\cos(\omega_k t)e^{-\alpha t/2}, \quad (27)$$

where

$$\omega_k = (\omega^2 + \bar{v}^2k^2 - \tfrac{1}{4}\alpha^2)^{1/2}.$$

Equation (26) also has time-independent solutions, such as the following, describing the static screening of an ion by free electrons

$$\rho(r) = Cr^{-1}\exp(-\omega_p r/\bar{v}). \quad (28)$$

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[9] This assumption is justified insofar as it does not appear to lead to serious disagreement with the experimentally determined conductivity.

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