is the application to a semiconductor. I do not believe a parallel development has been pursued in this case.

I traditionally devote a portion of my course in electricity and magnetism to these issues. The simple form of the term on the right-hand side of (4) makes it possible to generate some interesting examples. Although mine have been a bit contrived to keep the mathematics transparent, there is, nevertheless, satisfying insight here for the student.

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Answer to Question #52. Group velocity and energy propagation

The discussion of group velocities $v_g \equiv \partial\omega/\partial k$, by contrast with phase velocities $v_p \equiv \omega/k$, of propagating waves assumes three things. (i) The wave does not have a singular spectrum with just one carrier frequency ω_0 and one wave number k_0. It is actually a wave packet, hence the group velocity plays a role. (ii) The bandwidth of its spectrum is not too large (v_g well defined). (iii) The dispersion is nonlinear, i.e., ω is not proportional to k for the eigenmodes, $v_g \neq v_p$.

Awati and Howes [Am. J. Phys. **64** (11), 1353 (1996)] ask for a general proof of the relationship between group velocity and the velocity of energy propagation. Velocities are conventionally defined by identifying a characteristic point at some time t and place r, watching as it moves within a time Δt to another place $r+\Delta r$, then setting $v \equiv \Delta r/\Delta t$. If the parameter is the energy density[1] $S(r,t)$, one would, for example, associate the motion $r(t)$ of the characteristic point with a local maximum of the density, $\partial_r S(r,t)=0$. The link between this formula in local space and the dispersion $\omega(k)$ is inevitably given by the Fourier transform of this defining equation,

$$\int d^3\, k d\omega k S(k,\omega) \exp[i(kr-\omega t)]=0. \qquad (1)$$

A relation between Δr and Δt follows, because this equation must hold for some (r,t) as well as for another $(r+\Delta r, t+\Delta t)$. For small Δt and Δr, a part of the integrand may be expanded up to first order[2] in Δr and Δt,

$$ke^{ik(r+\Delta r)-i\omega(t+\Delta t))} \approx ke^{i(kr-\omega t)} + ike^{i(kr-\omega t)} \times (k\Delta r - \omega \Delta t). \qquad (2)$$

The following argument resembles a mathematical proof via "induction" from t to $t+\Delta t$. The integral over the zeroth-order term is assumed to be already zero. Δr and Δt are brought in relation to each other to ensure that the integral over the linear orders vanishes as well. We may introduce the central frequency ω_0 and wave number k_0,

$$k\Delta r - \omega\Delta t = k_0\Delta r - \omega_0 \Delta t + (k-k_0)\Delta r - (\omega-\omega_0)\Delta t.$$

The first two terms on the right-hand side do not depend on k. Their integrals with the kernel (2) are consequently zero by means of (1). This is the crucial reason why we do *not* need to have $k_0\Delta r - \omega_0\Delta t = 0$ and v is *not* primarily connected with v_p. To ensure that the integrals over the third and fourth terms are also zero, it is best to have $(k-k_0)\Delta r = (\omega-\omega_0)\Delta t$, which means $\Delta r/\Delta t = (\omega-\omega_0)/(k-k_0)$, i.e., $v=v_g$.[3–5]

The motion of special wave packet points with the group velocity rather than with the phase velocity is a mathematical feature of the Fourier transformation, independent of the spectral composition S of the wave amplitudes, the particular dispersion, and which physical quantity waves. Even the factor k in the Fourier integral (1), representing the gradient and maximum property in local space, is subordinate and may be replaced by more general functions.

[1]...a product of two local quantities, as in the case of the Poynting vector, or an (auto)correlation function in cases where the energy is a product of wave functions in (k,ω)-space, or something more general.
[2]An inexact justification is that the velocity is to be determined in the limit of $\Delta r, \Delta t \rightarrow 0$.
[3]Provided that the spectral width of $S(k,\omega)$ is small enough that the derivative $\partial\omega/\partial k$ can be well approximated by the quotient of the differences.
[4]If v_g is constant in the region of nonzero $S(k,\omega)$, this is valid for all orders in Δr and Δt, as the analysis can be performed within the argument of the exponential function (principle of the "stationary phase"), even if the factor k in the integrand is replaced be any function of k and ω. This property is useful, if, by some accidental characteristic of $S(k,\omega)$, the integral over the linear term vanishes for any pair of Δr and Δt, and vanishing of the first orders provides no information.
[5]Damped waves are generally described by *complex* valued dispersions $\omega(k)$ for the eigenmodes but *real* valued $v_g \equiv \partial\, \mathrm{Re}\, \omega/\partial k$. The Fourier integral is also defined for paths over the *real* k and ω axes.

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Answer to Question #52. Group velocity and energy propagation

The question by K. M. Awati and T. Howes [Am. J. Phys. **64** (11), 1353 (1996)] seeks a general proof showing that wave energy propagates at the group velocity rather than the phase velocity. No such proof exists because the result is not true. The arguments that connect group velocity and the velocity of a wave packet are all approximations which hold only for wide wave packets. A narrow wave packet will in general change shape as it moves, so the term "velocity of the wave packet" is poorly defined. But in addition, in a dispersive medium the group velocity $d\omega/dk$ is a function of k. A narrow wave packet has a broad spectral extent, i.e., it is a superposition of sine waves with a wide range of values for k. Thus for a narrow wave packet there is no one precise group velocity...the term "group velocity" is also poorly defined.

In this note we first supply a "hand-waving" argument showing why a wave packet should in general be expected to change its shape as it moves. We then give another hand-waving argument connecting the group velocity with the velocity of wave packets, and then show how that argument breaks down for narrow wave packets. Finally we review experimental results, where we bring up the point that dispersion is necessarily connected to dissipation, and where we find situations in which the group velocity is greater than the speed of light, and where it is actually negative! The phase velocity will play a very small role in our discussion. This is not surprising, because the phase velocity is an artificial construct involving the speed of a sine wave, which—unlike any real waveform—has infinite extent in both space and time.

Any finite wave consists of a superposition of sine waves. If the medium is nondispersive then each of these constituent sine waves moves at the same velocity, namely the single phase velocity. If we were to step into a reference frame moving at this velocity, then all of the component waves would be stationary, so naturally the combination wave packet moves without distortion. But if the medium is dispersive, then each constituent sine wave moves at a different phase velocity. Stepping into the reference frame of any given component, we would see all the other components move ahead or fall behind. Only in exceptional circumstances will the sum of all this relative motion add up to a wave packet that retains its shape.

Indeed, we can begin by considering the simplest possible superposition, namely the superposition of two sine waves of equal amplitude.[1] (Admittedly, this superposition is just as infinite as the sine itself, but it gives an indication of what transpires when an infinite number of sines superpose to form a finite waveform.) Suppose the first sine component has wave number $k+\Delta k$ and frequency $\omega+\Delta\omega$, while the second has wave number $k-\Delta k$ and frequency $\omega-\Delta\omega$. The total wave is

$$\eta(x,t)=A\cos[(k+\Delta k)x-(\omega+\Delta\omega)t] + A\cos[(k-\Delta k)x-(\omega-\Delta\omega)t],$$

which, after a little manipulation, can be written as

$$\eta(x,t)=2A\cos(\Delta kx-\Delta\omega t)\cos(kx-\omega t).$$

This is the well-known "beat" wave which at any moment has, if $\Delta k \ll k$, the appearance of a short-wavelength ($\lambda = 2\pi/k$) "hash" sine wave modulated by a long-wavelength ($\lambda_{\text{envelope}}=2\pi/\Delta k$) envelope. As time goes on, both the hash and the envelope move. The period of the envelope at any given point is $T_{\text{envelope}}=2\pi/\Delta\omega$. If we ignore the short-wavelength hash and ask only how fast the envelope moves, the direct answer is

$$v_{\text{envelope}}=\frac{\lambda_{\text{envelope}}}{T_{\text{envelope}}}=\frac{\Delta\omega}{\Delta k}.$$

This beat wave is clearly a very special case, but it renders plausible the notion that the envelope for a more general waveform will move at a speed given by the slope of the dispersion curve $\omega(k)$ in the vicinity of those wave numbers that are prominently represented in the waveform's Fourier spectrum. (Notice that even in this special case, it is only the envelope of the wave that moves at the group velocity...the detailed shape of the waveform does in fact change with time.)

Everything about this argument—the picture of the beat wave as a modulating envelope, the use of a single slope for the dispersion curve appropriate to all of the component wave numbers—relies on a waveform with a narrow spread of wave numbers. [Narrow, of course, relative to the curvature of the dispersion curve, so that $\omega(k)$ as a function of k is substantially linear over the range of wave numbers.] There are a number of other arguments[2] that produce the same result: through Fourier methods,[3] the treatment of a Gaussian wave packet,[4] and the method of stationary phase.[5] All of these rely on a wave packet with a narrow spread of wave numbers and hence (through a generalized "Heisenberg uncertainty principle") a broad range of positions.

If you are willing to go farther afield for a less direct analogy, one is provided by traffic jams. The speed of individual cars on a highway is very different from the speed at which a high density of cars propagates. Indeed, a traffic jam on an eastbound highway tends to creep slowly to the west.

If we wish to compare these ideas to experiment, one substantial obstacle remains. The above discussion ignored the role of dissipation, but no such "dispersive, nonabsorbing" medium can exist. The Kramers–Kronig relations between the real and imaginary parts of the dielectric constant, namely

$$\text{Re } \epsilon(\omega)=1+\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\text{Im } \epsilon(\omega')}{\omega'-\omega}\,d\omega',$$

$$\text{Im } \epsilon(\omega)=-\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\text{Re } \epsilon(\omega')-1}{\omega'-\omega}\,d\omega',$$

show that the only nonabsorbing medium in existence is the vacuum $[\epsilon(\omega)=1]$. One cannot have dispersion without dissipation.

While at Bell Labs (in pre-Lucent days), one of us showed experimentally[6] that, under the highly dispersive (and thus highly dissipative) conditions near an absorption peak, the group velocity of a light pulse could be *ten times* the speed of light in vacuum, c. While this did lead to amusing speculation on the properties of a hyper-relativistic telephone system ("Complete your call before you've even dialed it!"), the reality is that energy never traveled faster than the speed of light. Theoretical work[7,8] showed that under relevant conditions (including a thin media depth), an initially Gaussian pulse would remain Gaussian, and that it would move forward at the group velocity which, for varying wave number k, could be greater than c, pass through $\pm\infty$, or even become negative!

Here is a suggestive (but nonrigorous) way to understand this phenomenon. Imagine a light pulse incident on a medium consisting of dipoles modeled by damped simple harmonic oscillators. Assume that the pulse's frequency spectrum spread is smaller than the spectral width of the damped oscillators or, equivalently, that the time for the pulse to pass over a given point is longer than the response time of the oscillators. Then, when the leading edge of the pulse hits one point in the medium, the medium does not have time to respond and there is minimal polarization. When the peak of the pulse hits that same point, the medium has responded and is polarized strongly. When the trailing edge of the pulse hits that point, the medium remains polarized due to the effects remaining from the peak of the pulse. Thus the leading and trailing edges of the pulse behave quite differently. The polarized material radiates and thus affects the net absorption of the pulse, so there is greater absorption at the trailing edge than at the leading edge. The net effect is to cut off the trailing edge of the pulse, so the peak moves forward at very high velocity. (For an initially Gaussian wave packet, the net effect curiously maintains a substantially Gaussian profile.) For the very thin epilayer samples used in our experiment, calculations showed that the peak of the exiting pulse could leave the sample *before* the peak of the incident pulse entered the sample (negative group velocity). The exiting pulse, however, had a much smaller amplitude than the incident pulse, and if the exiting pulse were superposed over an image of where the incident pulse would have been had it not gone through the medium, the exiting pulse would always be contained within the envelope of the unmodified incident pulse. Thus the speed of energy transfer never exceeded the speed of the incident pulse and was safely less than the speed of light.

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Answer to Question #54. Chapter summaries: Blessing or curse?

Ralph Baierlein [Am. J. Phys. **64** (12), 1448 (1996)] invites comments on the seemingly ubiquitious practice of attaching summaries to the chapters of physics textbooks.

One of the skills that I would like my students to have is that of summarizing the high points of an article, a chapter, a topic from a text, an outline of a proof, and so on—something that might be placed on a 3 in.×5 in. card, for example. I have assigned class group exercises, homework, and even test questions asking for such summaries. For years now this practice has been thwarted by chapter summaries which almost all texts are now including. How can you distinguish where the author's words end and the student's begin? What do you say to the student who claims that his or her summary is identical to the author's?

Students do bypass the text and rely heavily on the summaries for problem solving. The problem, however, is not just with summaries. While summaries are seen as an aggravation, an even greater aggravation to me are those texts that group the problems by chapter sections. If students can't find the formula to solve the problem in the summary, they can then go to the particular referenced section and usually with minimal scanning find a clue.

I would like students to learn to read the text thoroughly and engage the author in conversation by asking questions at every step of the way. I try to do that by requiring group exercises in class with articles copied from elsewhere that deal at least peripherally with the topic at hand. Bolton in his excellent book, *Patterns in Physics*, provides me with lots of suggestions. In particular, he starts off his chapter on "Energy" with four different definitions of energy from four excellent physics books (none of which have summaries at the backs of chapters) that always generate heat and light when students try to reconcile their current notion of energy with those of the experts.

Every review that I do for authors or publishers includes a note that they should drop the summaries. If I get a response, it usually includes something akin to "Everyone else is doing it, so we have to."

If we want back-of-the-chapter-problem-solvers only, this trend is in the right direction, leading to little books with all the possible formulas of physics collected therein with sample applications, as seen in study guides for passing standardized tests. If we want to encourage creative problem solving, we need to get rid of the summaries and insert more probing questions and problems at appropriate places throughout the textual material, followed by collections of sample problems that might require the student to reach back to previous chapters or even previous classes in order to solve them.

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Answer to Question #60. Interference of two independent sources

There seems to be an ongoing debate about the meaning of Dirac's famous remark[1] that "Interference between two different photons never occurs" (see Ref. 2). I definitely agree with Fewell[3] that interference between two independent sources of light can occur and has been experimentally shown to occur. It is, however, in my opinion misleading to confuse *independent sources* with *different photons*. Since a superposition principle (in spacetime) is valid for solutions of Maxwell's equations, classical light waves must always interfere, although this interference may remain without effect for wave packets that do not overlap in spacetime. (This