

Negative group velocity

Kirk T. McDonald^{a)}

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544

(Received 21 August 2000; accepted 21 September 2000)

[DOI: 10.1119/1.1331304]

I. PROBLEM

Consider a variant on the physical situation of “slow light”^{1,2} in which two closely spaced spectral lines are now both optically pumped to show that the group velocity can be negative at the central frequency, which leads to apparent superluminal behavior.

A. Negative group velocity

In more detail, consider a classical model of matter in which spectral lines are associated with oscillators. In particular, consider a gas with two closely spaced spectral lines of angular frequencies $\omega_{1,2} = \omega_0 \pm \Delta/2$, where $\Delta \ll \omega_0$. Each line has the same damping constant (and spectral width) γ .

Ordinarily, the gas would exhibit strong absorption of light in the vicinity of the spectral lines. But suppose that lasers of frequencies ω_1 and ω_2 pump both oscillators into inverted populations. This can be described classically by assigning negative oscillator strengths to these oscillators.³

Deduce an expression for the group velocity $v_g(\omega_0)$ of a pulse of light centered on frequency ω_0 in terms of the (univalent) plasma frequency ω_p of the medium, given by

$$\omega_p^2 = \frac{4\pi N e^2}{m}, \quad (1)$$

where N is the number density of atoms, and e and m are the charge and mass of an electron. Give a condition on the line separation Δ compared to the linewidth γ such that the group velocity $v_g(\omega_0)$ is negative.

In a recent experiment by Wang *et al.*,⁴ a group velocity of $v_g = -c/310$, where c is the speed of light in vacuum, was demonstrated in cesium vapor using a pair of spectral lines with separation $\Delta/2\pi \approx 2$ MHz and linewidth $\gamma/2\pi \approx 0.8$ MHz.

B. Propagation of a monochromatic plane wave

Consider a wave with electric field $E_0 e^{i\omega(z/c - t)}$ that is incident from $z < 0$ on a medium that extends from $z = 0$ to a . Ignore reflection at the boundaries, as is reasonable if the index of refraction $n(\omega)$ is near unity. Particularly simple results can be obtained when you make the (unphysical) assumption that the $\omega n(\omega)$ varies linearly with frequency about a central frequency ω_0 . Deduce a transformation that has a frequency-dependent part and a frequency-independent part between the phase of the wave for $z < 0$ to that of the wave inside the medium, and to that of the wave in the region $a < z$.

C. Fourier analysis

Apply the transformations between an incident monochromatic wave and the wave in and beyond the medium to the Fourier analysis of an incident pulse of form $f(z/c - t)$.

D. Propagation of a sharp wave front

In the approximation that ωn varies linearly with ω , deduce the waveforms in the regions $0 < z < a$ and $a < z$ for an incident pulse $\delta(z/c - t)$, where δ is the Dirac delta function. Show that the pulse emerges out of the gain region at $z = a$ at time $t = a/v_g$, which appears to be earlier than when it enters this region if the group velocity is negative. Show also that inside the negative group velocity medium a pulse propagates backwards from $z = a$ at time $t = a/v_g < 0$ to $z = 0$ at $t = 0$, at which time it appears to annihilate the incident pulse.

E. Propagation of a Gaussian pulse

As a more physical example, deduce the waveforms in the regions $0 < z < a$ and $a < z$ for a Gaussian incident pulse $E_0 e^{-(z/c - t)^2/2\tau^2} e^{i\omega_0(z/c - t)}$. Carry the frequency expansion of $\omega n(\omega)$ to second order to obtain conditions of validity of the analysis such as maximum pulse width τ , maximum length a of the gain region, and maximum time of advance of the emerging pulse. Consider the time required to generate a pulse of rise time τ when assessing whether the time advance in a negative group velocity medium can lead to superluminal signal propagation.

II. SOLUTION

The concept of group velocity appears to have been first enunciated by Hamilton in 1839 in lectures of which only abstracts were published.⁵ The first recorded observation of the group velocity of a (water) wave is due to Russell in 1844.⁶ However, widespread awareness of the group velocity dates from 1876 when Stokes used it as the topic of a Smith's Prize examination paper.⁷ The early history of group velocity has been reviewed by Havelock.⁸

H. Lamb⁹ credits A. Schuster with noting in 1904 that a negative group velocity, i.e., a group velocity of opposite sign to that of the phase velocity, is possible due to anomalous dispersion. Von Laue¹⁰ made a similar comment in 1905. Lamb gave two examples of strings subject to external potentials that exhibit negative group velocities. These early considerations assumed that in case of a wave with positive group and phase velocities incident on the anomalous medium, energy would be transported into the medium with a positive group velocity, and so there would be waves with negative phase velocity inside the medium. Such negative phase velocity waves are formally consistent with Snell's

law¹¹ (since $\theta_i = \sin^{-1}[(n_i/n_t)\sin\theta_i]$ can be in either the first or second quadrant), but they seemed physically implausible and the topic was largely dropped.

Present interest in negative group velocity is based on anomalous dispersion in a gain medium, where the sign of the phase velocity is the same for incident and transmitted waves, and energy flows inside the gain medium in the opposite direction to the incident energy flow in vacuum.

The propagation of electromagnetic waves at frequencies near those of spectral lines of a medium was first extensively discussed by Sommerfeld and Brillouin,¹² with emphasis on the distinction between signal velocity and group velocity when the latter exceeds c . The solution presented here is based on the work of Garrett and McCumber,¹³ as extended by Chiao *et al.*^{14,15} A discussion of negative group velocity in electronic circuits has been given by Mitchell and Chiao.¹⁶

A. Negative group velocity

In a medium of index of refraction $n(\omega)$, the dispersion relation can be written

$$k = \frac{\omega n}{c}, \quad (2)$$

where k is the wave number. The group velocity is then given by

$$\begin{aligned} v_g &= \text{Re} \left[\frac{d\omega}{dk} \right] = \frac{1}{\text{Re} [dk/d\omega]} \\ &= \frac{c}{\text{Re} [d(\omega n)/d\omega]} = \frac{c}{n + \omega \text{Re} [dn/d\omega]}. \end{aligned} \quad (3)$$

We see from Eq. (3) that if the index of refraction decreases rapidly enough with frequency, the group velocity can be negative. It is well known that the index of refraction decreases rapidly with frequency near an absorption line, where ‘‘anomalous’’ wave propagation effects can occur.¹² However, the absorption makes it difficult to study these effects. The insight of Garrett and McCumber¹³ and of Chiao *et al.*^{14,15,17-19} is that demonstrations of negative group velocity are possible in media with inverted populations, so that gain rather than absorption occurs at the frequencies of interest. This was dramatically realized in the experiment of Wang *et al.*⁴ by use of a closely spaced pair of gain lines, as perhaps first suggested by Steinberg and Chiao.¹⁷

We use a classical oscillator model for the index of refraction. The index n is the square root of the dielectric constant ϵ , which is in turn related to the atomic polarizability α according to

$$D = \epsilon E = E + 4\pi P = E(1 + 4\pi N\alpha) \quad (4)$$

(in Gaussian units), where D is the electric displacement, E is the electric field, and P is the polarization density. Then, the index of refraction of a dilute gas is

$$n = \sqrt{\epsilon} \approx 1 + 2\pi N\alpha. \quad (5)$$

The polarizability α is obtained from the electric dipole moment $p = ex = \alpha E$ induced by electric field E . In the case of a single spectral line of frequency ω_j , we say that an electron is bound to the (fixed) nucleus by a spring of constant $K = m\omega_j^2$, and that the motion is subject to the damping

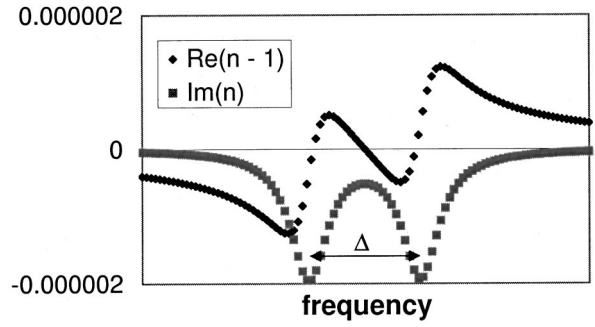


Fig. 1. The real and imaginary parts of the index of refraction in a medium with two spectral lines that have been pumped to inverted populations. The lines are separated by angular frequency Δ and have widths $\gamma = 0.4\Delta$.

force $-m\gamma_j\dot{x}$, where the dot indicates differentiation with respect to time. The equation of motion in the presence of an electromagnetic wave of frequency ω is

$$\ddot{x} + \gamma_j\dot{x} + \omega_j^2 x = \frac{eE}{m} = \frac{eE_0}{m} e^{i\omega t}. \quad (6)$$

Hence,

$$x = \frac{eE}{m} \frac{1}{\omega_j^2 - \omega^2 - i\gamma_j\omega} = \frac{eE}{m} \frac{\omega_j^2 - \omega^2 + i\gamma_j\omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}, \quad (7)$$

and the polarizability is

$$\alpha = \frac{e^2}{m} \frac{\omega_j^2 - \omega^2 + i\gamma_j\omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}. \quad (8)$$

In the present problem we have two spectral lines, $\omega_{1,2} = \omega_0 \pm \Delta/2$, both of oscillator strength -1 to indicate that the populations of both lines are inverted, with damping constants $\gamma_1 = \gamma_2 = \gamma$. In this case, the polarizability is given by

$$\begin{aligned} \alpha &= -\frac{e^2}{m} \frac{(\omega_0 - \Delta/2)^2 - \omega^2 + i\gamma\omega}{((\omega_0 - \Delta/2)^2 - \omega^2)^2 + \gamma^2\omega^2} \\ &\quad - \frac{e^2}{m} \frac{(\omega_0 + \Delta/2)^2 - \omega^2 + i\gamma\omega}{((\omega_0 + \Delta/2)^2 - \omega^2)^2 + \gamma^2\omega^2} \\ &\approx -\frac{e^2}{m} \frac{\omega_0^2 - \Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2} \\ &\quad - \frac{e^2}{m} \frac{\omega_0^2 + 2\Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 + \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2}, \end{aligned} \quad (9)$$

where the approximation is obtained by the neglect of terms in Δ^2 compared to those in $\Delta\omega_0$.

For a probe beam at frequency ω , the index of refraction (5) has the form

$$\begin{aligned} n(\omega) &\approx 1 - \frac{\omega_p^2}{2} \left[\frac{\omega_0^2 - \Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2} \right. \\ &\quad \left. + \frac{\omega_0^2 + \Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 + \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2} \right], \end{aligned} \quad (10)$$

where ω_p is the plasma frequency given by Eq. (1). This is illustrated in Fig. 1.

The index at the central frequency ω_0 is

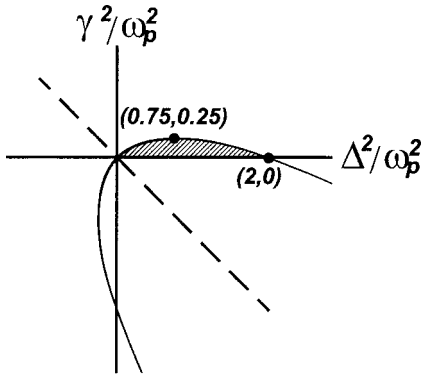


Fig. 2. The allowed region (14) in (Δ^2, γ^2) space such that the group velocity is negative.

$$n(\omega_0) \approx 1 - i \frac{\omega_p^2 \gamma}{(\Delta^2 + \gamma^2) \omega_0} \approx 1 - i \frac{\omega_p^2}{\Delta^2} \frac{\gamma}{\omega_0}, \quad (11)$$

where the second approximation holds when $\gamma \ll \Delta$. The electric field of a continuous probe wave then propagates according to

$$E(z, t) = e^{i(kz - \omega_0 t)} = e^{i\omega(n(\omega_0)z/c - t)} \\ \approx e^{z/[\Delta^2 c / \gamma \omega(2/p)]} e^{i\omega_0(z/c - t)}. \quad (12)$$

From this we see that at frequency ω_0 the phase velocity is c , and the medium has an amplitude gain length $\Delta^2 c / \gamma \omega_p^2$.

To obtain the group velocity (3) at frequency ω_0 , we need the derivative

$$\left. \frac{d(\omega n)}{d\omega} \right|_{\omega_0} \approx 1 - \frac{2\omega_p^2(\Delta^2 - \gamma^2)}{(\Delta^2 + \gamma^2)^2}, \quad (13)$$

where we have neglected terms in Δ and γ compared to ω_0 . From Eq. (3), we see that the group velocity can be negative if

$$\frac{\Delta^2}{\omega_p^2} - \frac{\gamma^2}{\omega_p^2} \geq \frac{1}{2} \left(\frac{\Delta^2}{\omega_p^2} + \frac{\gamma^2}{\omega_p^2} \right)^2. \quad (14)$$

The boundary of the allowed region (14) in (Δ^2, γ^2) space is a parabola whose axis is along the line $\gamma^2 = -\Delta^2$, as shown in Fig. 2. For the physical region $\gamma^2 \geq 0$, the boundary is given by

$$\frac{\gamma^2}{\omega_p^2} = \sqrt{1 + 4 \frac{\Delta^2}{\omega_p^2}} - 1 - \frac{\Delta^2}{\omega_p^2}. \quad (15)$$

Thus, to have a negative group velocity, we must have

$$\Delta \leq \sqrt{2} \omega_p, \quad (16)$$

which limit is achieved when $\gamma = 0$; the maximum value of γ is $0.5\omega_p$ when $\Delta = 0.866\omega_p$.

Near the boundary of the negative group velocity region, $|v_g|$ exceeds c , which alerts us to concerns of superluminal behavior. However, as will be seen in the following sections, the effect of a negative group velocity is more dramatic when $|v_g|$ is small rather than large.

The region of recent experimental interest is $\gamma \ll \Delta \ll \omega_p$, for which Eqs. (3) and (13) predict that

$$v_g \approx -\frac{c}{2} \frac{\Delta^2}{\omega_p^2}. \quad (17)$$

A value of $v_g \approx -c/310$ as in the experiment of Wang corresponds to $\Delta/\omega_p \approx 1/12$. In this case, the gain length $\Delta^2 c / \gamma \omega_p^2$ was approximately 40 cm.

For later use we record the second derivative,

$$\left. \frac{d^2(\omega n)}{d\omega^2} \right|_{\omega_0} \approx 8i \frac{\omega_p^2 \gamma (3\Delta^2 - \gamma^2)}{(\Delta^2 + \gamma^2)^3} \approx 24i \frac{\omega_p^2}{\Delta^2} \frac{\gamma}{\Delta^2}, \quad (18)$$

where the second approximation holds if $\gamma \ll \Delta$.

B. Propagation of a monochromatic plane wave

To illustrate the optical properties of a medium with negative group velocity, we consider the propagation of an electromagnetic wave through it. The medium extends from $z = 0$ to a , and is surrounded by vacuum. Because the index of refraction (10) is near unity in the frequency range of interest, we ignore reflections at the boundaries of the medium.

A monochromatic plane wave of frequency ω and incident from $z < 0$ propagates with phase velocity c in vacuum. Its electric field can be written

$$E_\omega(z, t) = E_0 e^{i\omega z/c} e^{-i\omega t} \quad (z < 0). \quad (19)$$

Inside the medium this wave propagates with phase velocity $c/n(\omega)$ according to

$$E_\omega(z, t) = E_0 e^{i\omega n z/c} e^{-i\omega t} \quad (0 < z < a), \quad (20)$$

where the amplitude is unchanged since we neglect the small reflection at the boundary $z = 0$. When the wave emerges into vacuum at $z = a$, the phase velocity is again c , but it has accumulated a phase lag of $(\omega/c)(n-1)a$, and so appears as

$$E_\omega(z, t) = E_0 e^{i\omega a(n-1)/c} e^{i\omega z/c} e^{-i\omega t} \\ = E_0 e^{i\omega a n/c} e^{-i\omega(t - (z-a)/c)} \quad (a < z). \quad (21)$$

It is noteworthy that a monochromatic wave for $z > a$ has the same form as that inside the medium if we make the frequency-independent substitutions

$$z \rightarrow a, \quad t \rightarrow t - \frac{z-a}{c}. \quad (22)$$

Since an arbitrary waveform can be expressed in terms of monochromatic plane waves via Fourier analysis, we can use these substitutions to convert any wave in the region $0 < z < a$ to its continuation in the region $a < z$.

A general relation can be deduced in the case where the second and higher derivatives of $\omega n(\omega)$ are very small. We can then write

$$\omega n(\omega) \approx \omega_0 n(\omega_0) + \frac{c}{v_g} (\omega - \omega_0), \quad (23)$$

where v_g is the group velocity for a pulse with central frequency ω_0 . Using this in Eq. (20), we have

$$E_\omega(z, t) \approx E_0 e^{i\omega_0 z(n(\omega_0)/c - 1/v_g)} e^{i\omega z/v_g} e^{-i\omega t} \quad (0 < z < a). \quad (24)$$

In this approximation, the Fourier component $E_\omega(z)$ at frequency ω of a wave inside the gain medium is related to that of the incident wave by replacing the frequency dependence

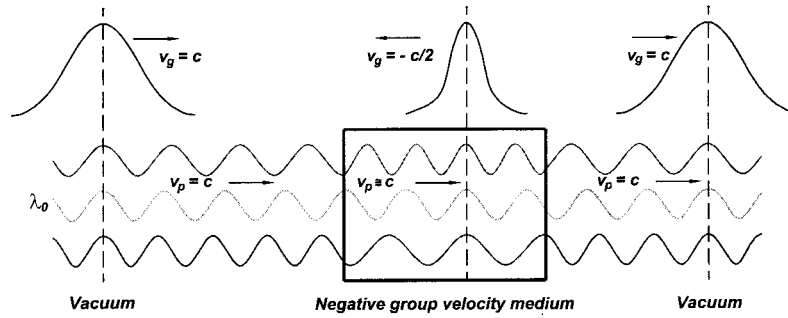


Fig. 3. A snapshot of three Fourier components of a pulse in the vicinity of a negative group velocity medium. The component at the central wavelength λ_0 is unaltered by the medium, but the wavelength of a longer wavelength component is shortened, and that of a shorter wavelength component is lengthened. Then, even when the incident pulse has not yet reached the medium, there can be a point inside the medium at which all components have the same phase, and a peak appears. Simultaneously, there can be a point in the vacuum region beyond the medium at which the Fourier components are again all in phase, and a third peak appears. The peaks in the vacuum regions move with group velocity $v_g = c$, but the peak inside the medium moves with a negative group velocity, shown as $v_g = -c/2$. The phase velocity v_p is c in vacuum, and close to c in the medium.

$e^{i\omega z/c}$ by $e^{i\omega z/v_g}$, i.e., by replacing z/c by z/v_g , and multiplying by the frequency-independent phase factor $e^{i\omega_0 z(n(\omega_0)/c - 1/v_g)}$. Then, using transformation (22), the wave that emerges into vacuum beyond the medium is

$$E_\omega(z, t) \approx E_0 e^{i\omega_0 a(n(\omega_0)/c - 1/v_g)} \times e^{i\omega(z/c - a(1/c - 1/v_g))} e^{-i\omega t} \quad (a < z). \quad (25)$$

The wave beyond the medium is related to the incident wave by multiplying by a frequency-independent phase, and by replacing z/c by $z/c - a(1/c - 1/v_g)$ in the frequency-dependent part of the phase.

The effect of the medium on the wave as described by Eqs. (24) and (25) has been called “rephasing.”⁴

C. Fourier analysis and “rephasing”

The transformations between the monochromatic incident wave (19) and its continuation in and beyond the medium, (24) and (25), imply that an incident wave

$$E(z, t) = f(z/c - t) = \int_{-\infty}^{\infty} E_\omega(z) e^{-i\omega t} d\omega \quad (z < 0), \quad (26)$$

whose Fourier components are given by

$$E_\omega(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(z, t) e^{i\omega t} dt, \quad (27)$$

propagates as

$$E(z, t) \approx \begin{cases} f(z/c - t) & (z < 0) \\ e^{i\omega_0 z(n(\omega_0)/c - 1/v_g)} f(z/v_g - t) & (0 < z < a) \\ e^{i\omega_0 a(n(\omega_0)/c - 1/v_g)} f(z/c - t - a(1/c - 1/v_g)) & (a < z). \end{cases} \quad (28)$$

An interpretation of Eq. (28) in terms of “rephasing” is as follows. Fourier analysis tells us that the maximum amplitude of a pulse made of waves of many frequencies, each of the form $E_\omega(z, t) = E_0(\omega) e^{i\phi(\omega)} = E_0(\omega) e^{i(k(\omega)z - \omega t + \phi_0(\omega))}$ with $E_0 \geq 0$, is determined by adding the amplitudes $E_0(\omega)$. This maximum is achieved only if there exist points (z, t) such that all phases $\phi(\omega)$ have the same value.

For example, we consider a pulse in the region $z < 0$ whose maximum occurs when the phases of all component frequencies vanish, as shown at the left of Fig. 3. Referring

to Eq. (19), we see that the peak occurs when $z = ct$. As usual, we say that the group velocity of this wave is c in vacuum.

Inside the medium, Eq. (24) describes the phases of the components, which all have a common frequency-independent phase $\omega_0 z(n(\omega_0)/c - 1/v_g)$ at a given z , as well as a frequency-dependent part $\omega(z/v_g - t)$. The peak of the pulse occurs when all the frequency-dependent phases vanish; the overall frequency-independent phase does not affect the pulse size. Thus, the peak of the pulse propagates within the medium according to $z = v_g t$. The velocity of the peak is v_g , the group velocity of the medium, which can be negative.

The “rephasing” (24) within the medium changes the wavelengths of the component waves. Typically the wavelength increases, and by greater amounts at longer wavelengths. A longer time is required before the phases of the waves all become the same at some point z inside the medium, so in a normal medium the velocity of the peak appears to be slowed down. But in a negative group velocity medium, wavelengths short compared to λ_0 are lengthened, long waves are shortened, and the velocity of the peak appears to be reversed.

By a similar argument, Eq. (25) tells us that in the vacuum region beyond the medium the peak of the pulse propagates according to $z = ct + a(1/c - 1/v_g)$. The group velocity is again c , but the “rephasing” within the medium results in a shift of the position of the peak by the amount $a(1/c - 1/v_g)$. In a normal medium where $0 < v_g \leq c$ the shift is negative; the pulse appears to have been delayed during its passage through the medium. But after a negative group velocity medium, the pulse appears to have advanced!

This advance is possible because, in the Fourier view, each component wave extends over all space, even if the pulse appears to be restricted. The unusual “rephasing” in a negative group velocity medium shifts the phases of the frequency components of the wave train in the region ahead of the nominal peak such that the phases all coincide, and a peak is observed, at times earlier than expected at points beyond the medium.

As shown in Fig. 3 and further illustrated in the examples in the following, the “rephasing” can result in the simultaneous appearance of peaks in all three regions.

D. Propagation of a sharp wave front

To assess the effect of a medium with negative group velocity on the propagation of a signal, we first consider a waveform with a sharp front, as recommended by Sommerfeld and Brillouin.¹²

As an extreme but convenient example, we take the incident pulse to be a Dirac delta function, $E(z,t) = E_0 \delta(z/c - t)$. Inserting this in Eq. (28), which is based on the linear approximation (23), we find

$$E(z,t) \approx \begin{cases} E_0 \delta(z/c - t) & (z < 0) \\ E_0 e^{i\omega_0 z(n(\omega_0)/c - 1/v_g)} \delta(z/v_g - t) & (0 < z < a) \\ E_0 e^{i\omega_0 a(n(\omega_0)/c - 1/v_g)} \delta(z/c - t - a(1/c - 1/v_g)) & (a < z). \end{cases} \quad (29)$$

According to Eq. (29), the delta-function pulse emerges from the medium at $z = a$ at time $t = a/v_g$. If the group velocity is negative, the pulse emerges from the medium before it enters at $t = 0$!

A sample history of (Gaussian) pulse propagation is illustrated in Fig. 4. Inside the negative group velocity medium, an (anti)pulse propagates backwards in space from $z = a$ at time $t = a/v_g < 0$ to $z = 0$ at time $t = 0$, at which point it appears to annihilate the incident pulse.

This behavior is analogous to barrier penetration by a relativistic electron²⁰ in which an electron can emerge from the far side of the barrier earlier than it hits the near side, if the electron emission at the far side is accompanied by positron emission, and the positron propagates within the barrier so as to annihilate the incident electron at the near side. In the Wheeler–Feynman view, this process involves only a single electron which propagates backwards in time when inside the barrier. In this spirit, we might say that pulses propagate backwards in time (but forward in space) inside a negative group velocity medium.

The Fourier components of the delta function are independent of frequency, so the advanced appearance of the sharp wave front as described by Eq. (29) can occur only for a gain medium such that the index of refraction varies linearly at all frequencies. If such a medium existed with negative slope $dn/d\omega$, then Eq. (29) would constitute superluminal signal propagation.

However, from Fig. 1 we see that a linear approximation to the index of refraction is reasonable in the negative group velocity medium only for $|\omega - \omega_0| \lesssim \Delta/2$. The sharpest wave front that can be supported within this bandwidth has characteristic rise time $\tau \approx 1/\Delta$.

For the experiment of Wang *et al.* where $\Delta/2\pi \approx 10^6$ Hz, an analysis based on Eq. (23) would be valid only for pulses with $\tau \gtrsim 0.1 \mu\text{s}$. Wang *et al.* used a pulse with $\tau \approx 1 \mu\text{s}$, close to the minimum value for which Eq. (23) is a reasonable approximation.

Since a negative group velocity can only be experienced over a limited bandwidth, very sharp wave fronts must be excluded from the discussion of signal propagation. However, it is well known¹² that great care must be taken when discussing the signal velocity if the waveform is not sharp.

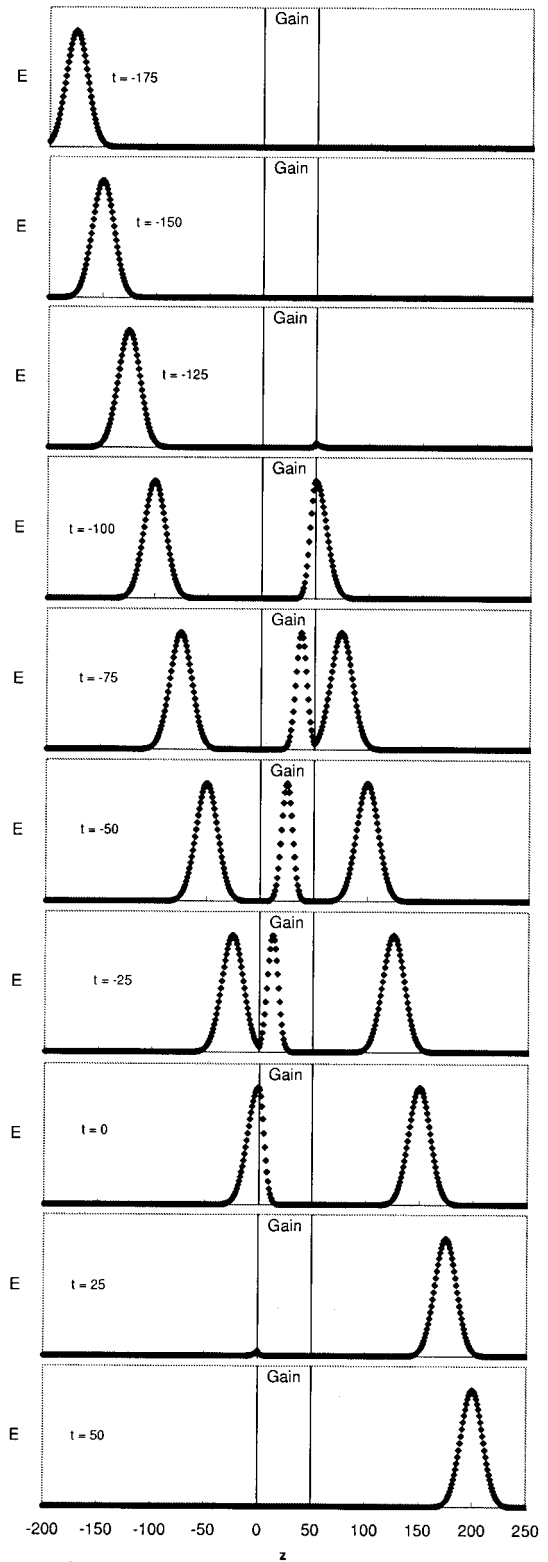


Fig. 4. Ten “snapshots” of a Gaussian pulse as it traverses a negative group velocity region ($0 < z < 50$), according to Eq. (31). The group velocity in the gain medium is $v_g = -c/2$, and c has been set to 1.

E. Propagation of a Gaussian pulse

We now consider a Gaussian pulse of temporal length τ centered on frequency ω_0 (the carrier frequency), for which the incident waveform is

$$E(z,t) = E_0 e^{-(z/c-t)^2/2\tau^2} e^{i\omega_0 z/c} e^{-i\omega_0 t} \quad (z < 0). \quad (30)$$

Inserting this in Eq. (28) we find

$$E(z,t) = \begin{cases} E_0 e^{-(z/c-t)^2/2\tau^2} e^{i\omega_0(z/c-t)} & (z < 0) \\ E_0 e^{-(z/v_g-t)^2/2\tau^2} e^{i\omega_0(n(\omega_0)z/c-t)} & (0 < z < a) \\ E_0 e^{i\omega_0 a(n(\omega_0)-1)/c} e^{-(z/c-a(1/c-1/v_g)-t)^2/2\tau^2} \\ \quad \times e^{i\omega_0(z/c-t)} & (a < z). \end{cases} \quad (31)$$

The factor $e^{i\omega_0 a(n(\omega_0)-1)/c}$ in Eq. (31) for $a < z$ becomes $e^{\omega_p^2 \gamma a / \Delta^2 c}$ using Eq. (11), and represents a small gain due to traversing the negative group velocity medium. In the experiment of Wang *et al.*, this factor was only 1.16.

We have already noted in the previous section that the linear approximation to $\omega n(\omega)$ is only good over a frequency interval about ω_0 of order Δ , and so Eq. (31) for the pulse after the gain medium applies only for pulse widths

$$\tau \gtrsim \frac{1}{\Delta}. \quad (32)$$

Further constraints on the validity of Eq. (31) can be obtained using the expansion of $\omega n(\omega)$ to second order. For this, we repeat the derivation of Eq. (31) in slightly more detail. The incident Gaussian pulse (30) has the Fourier decomposition (27),

$$E_\omega(z) = \frac{\tau}{\sqrt{2\pi}} E_0 e^{-\tau^2(\omega-\omega_0)^2/2} e^{i\omega z/c} \quad (z < 0). \quad (33)$$

We again extrapolate the Fourier component at frequency ω into the region $z > 0$ using Eq. (20), which yields

$$E_\omega(z) = \frac{\tau}{\sqrt{2\pi}} E_0 e^{-\tau^2(\omega-\omega_0)^2/2} e^{i\omega n z/c} \quad (0 < z < a). \quad (34)$$

We now approximate the factor $\omega n(\omega)$ by its Taylor expansion through second order:

$$\omega n(\omega) \approx \omega_0 n(\omega_0) + \frac{c}{v_g} (\omega - \omega_0) + \frac{1}{2} \left. \frac{d^2(\omega n)}{d\omega^2} \right|_{\omega_0} (\omega - \omega_0)^2. \quad (35)$$

With this, we find from Eqs. (26) and (34) that

$$E(z,t) = \frac{E_0}{A} e^{-(z/v_g-t)^2/2A^2\tau^2} e^{i\omega_0 n(\omega_0)z/c} e^{-i\omega_0 t} \quad (0 < z < a). \quad (36)$$

where

$$A^2(z) = 1 - i \left. \frac{z}{c\tau^2} \frac{d^2(\omega n)}{d\omega^2} \right|_{\omega_0}. \quad (37)$$

The waveform for $z > a$ is obtained from that for $0 < z < a$ by the substitutions (22) with the result

$$E(z,t) = \frac{E_0}{A} e^{i\omega_0 a(n(\omega_0)-1)/c} e^{-(z/c-a(1/c-1/v_g)-t)^2/2A^2\tau^2} \times e^{i\omega_0 z/c} e^{-i\omega_0 t} \quad (a < z), \quad (38)$$

where A is evaluated at $z=a$ here. As expected, the forms (36) and (38) revert to those of Eq. (31) when $d^2(\omega n(\omega_0))/d\omega^2 = 0$.

So long as the factor $A(a)$ is not greatly different from unity, the pulse emerges from the medium essentially undistorted, which requires

$$\frac{a}{c\tau} \ll \frac{1}{24} \frac{\Delta^2}{\omega_p^2} \frac{\Delta}{\gamma} \Delta\tau, \quad (39)$$

using Eqs. (18) and (37). In the experiment of Wang *et al.*, this condition is that $a/c\tau \ll 1/120$, which was well satisfied with $a=6$ cm and $c\tau=300$ m.

As in the case of the delta function, the centroid of a Gaussian pulse emerges from a negative group velocity medium at time

$$t = \frac{a}{v_g} < 0, \quad (40)$$

which is earlier than the time $t=0$ when the centroid enters the medium. In the experiment of Wang *et al.*, the time advance of the pulse was $a/|v_g| \approx 300a/c \approx 6 \times 10^{-8}$ s $\approx 0.06\tau$.

If one attempts to observe the negative group velocity pulse inside the medium, the incident wave would be perturbed and the backwards-moving pulse would not be detected. Rather, one must deduce the effect of the negative group velocity medium by observation of the pulse that emerges into the region $z > a$ beyond that medium, where the significance of the time advance (40) is the main issue.

The time advance caused by a negative group velocity medium is larger when $|v_g|$ is smaller. It is possible that $|v_g| > c$, but this gives a smaller time advance than when the negative group velocity is such that $|v_g| < c$. Hence, there is no special concern as to the meaning of negative group velocity when $|v_g| > c$.

The maximum possible time advance t_{\max} by this technique can be estimated from Eqs. (17), (39), and (40) as

$$\frac{t_{\max}}{\tau} \approx \frac{1}{12} \frac{\Delta}{\gamma} \Delta\tau \approx 1. \quad (41)$$

The pulse can advance by at most a few rise times due to passage through the negative group velocity medium.

While this aspect of the pulse propagation appears to be superluminal, it does not imply superluminal signal propagation.

In accounting for signal propagation time, the time needed to generate the signal must be included as well. A pulse with a finite frequency bandwidth Δ takes at least time $\tau \approx 1/\Delta$ to be generated, and so is delayed by a time of order of its rise time τ compared to the case of an idealized sharp wave front. Thus, the advance of a pulse front in a negative group velocity medium by $\lesssim \tau$ can at most compensate for the original delay in generating that pulse. The signal velocity, as defined by the path length between the source and detector divided by the overall time from onset of signal generation to signal detection, remains bounded by c .

As has been emphasized by Garrett and McCumber¹³ and by Chiao,^{18,19} the time advance of a pulse emerging from a gain medium is possible because the forward tail of a smooth pulse gives advance warning of the later arrival of the peak. The leading edge of the pulse can be amplified by the gain medium, which gives the appearance of superluminal pulse

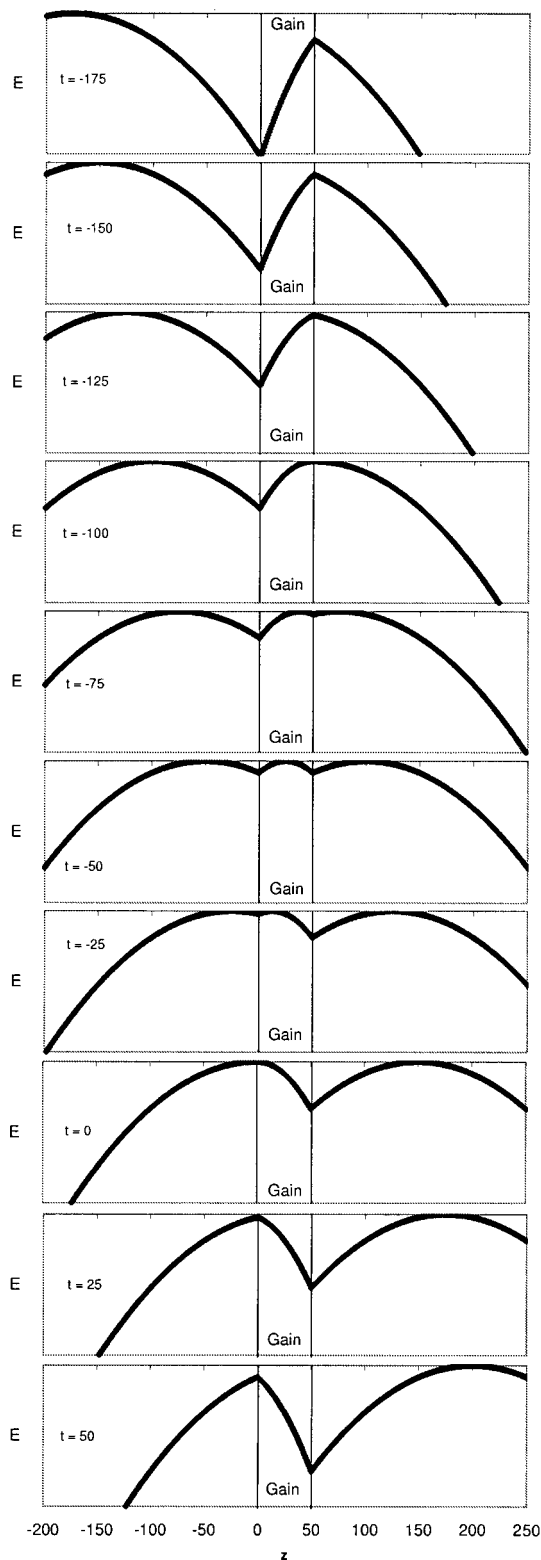


Fig. 5. The same as Fig. 4, but with the electric field plotted on a logarithmic scale from 1 to 10^{-65} .

velocities. However, the medium is merely using information stored in the early part of the pulse during its (lengthy) time of generation to bring the apparent velocity of the pulse closer to c .

The effect of the negative group velocity medium can be dramatized in a calculation based on Eq. (31) in which the

pulse width is narrower than the gain region [in violation of condition (39)], as shown in Fig. 4. Here, the gain region is $0 < z < 50$, the group velocity is taken to be $-c/2$, and c is defined to be unity. The behavior illustrated in Fig. 4 is perhaps less surprising when the pulse amplitude is plotted on a logarithmic scale, as in Fig. 5. Although the overall gain of the system is near unity, the leading edge of the pulse is amplified by about 70 orders of magnitude in this example [the implausibility of which underscores that condition (39) cannot be evaded], while the trailing edge of the pulse is attenuated by the same amount. The gain medium has temporarily loaned some of its energy to the pulse permitting the leading edge of the pulse to appear to advance faster than the speed of light.

Our discussion of the pulse has been based on a classical analysis of interference, but, as remarked by Dirac,²¹ classical optical interference describes the behavior of the wave functions of individual photons, not of interference between photons. Therefore, we expect that the behavior discussed above will soon be demonstrated for a ‘pulse’ consisting of a single photon with a Gaussian wave packet.

ACKNOWLEDGMENTS

The author thanks Lijun Wang for discussions of his experiment, and Alex Granik for references to the early history of negative group velocity and for the analysis contained in Eqs. (14)–(16).

^aElectronic mail: mcdonald@puphed.princeton.edu

¹L. V. Hau *et al.*, ‘‘Light speed reduction to 17 metres per second in an ultracold atomic gas,’’ *Nature (London)* **397**, 594–598 (1999).

²K. T. McDonald, ‘‘Slow light,’’ *Am. J. Phys.* **68**, 293–294 (2000). A figure to be compared with Fig. 1 of the present paper has been added in the version at <http://arxiv.org/abs/physics/0007097>

³This is in contrast to the ‘‘A’’ configuration of the three-level atomic system required for slow light (Ref. 2) where the pump laser does not produce an inverted population, in which case an adequate classical description is simply to reverse the sign of the damping constant for the pumped oscillator.

⁴L. J. Wang, A. Kuzmich, and A. Dogariu, ‘‘Gain-assisted superluminal light propagation,’’ *Nature (London)* **406**, 277–279 (2000). Their website, <http://www.neci.nj.nec.com/homepages/lwan/gas.htm>, contains additional material, including an animation much like Fig. 4 of the present paper.

⁵W. R. Hamilton, ‘‘Researches respecting vibration, connected with the theory of light,’’ *Proc. R. Ir. Acad.* **1**, 267,341 (1839).

⁶J. S. Russell, ‘‘Report on waves,’’ *Br. Assoc. Reports* (1844), pp. 311–390. This report features the first recorded observations of solitary waves (p. 321) and of group velocity (p. 369).

⁷G. G. Stokes, Problem 11 of the Smith’s Prize examination papers (2 February 1876), in *Mathematical and Physical Papers* (Johnson Reprint Co., New York, 1966), Vol. 5, p. 362.

⁸T. H. Havelock, *The Propagation of Disturbances in Dispersive Media* (Cambridge U.P., Cambridge, 1914).

⁹H. Lamb, ‘‘On Group-Velocity,’’ *Proc. London Math. Soc.* **1**, 473–479 (1904).

¹⁰See p. 551 of M. Laue, ‘‘Die Fortpflanzung der Strahlung in Dispergierenden und Absorbierenden Medien,’’ *Ann. Phys. (Leipzig)* **18**, 523–566 (1905).

¹¹L. Mandelstam, *Lectures on Optics, Relativity and Quantum Mechanics* (Nauka, Moscow, 1972); in Russian.

¹²L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960). That the group velocity can be negative is mentioned on p. 122.

¹³C. G. B. Garrett and D. E. McCumber, ‘‘Propagation of a Gaussian Light Pulse through an Anomalous Dispersion Medium,’’ *Phys. Rev. A* **1**, 305–313 (1970).

¹⁴R. Y. Chiao, ‘‘Superluminal (but causal) propagation of wave packets in transparent media with inverted atomic populations,’’ *Phys. Rev. A* **48**, R34–R37 (1993).

- ¹⁵E. L. Bolda, J. C. Garrison, and R. Y. Chiao, "Optical pulse propagation at negative group velocities due to a nearby gain line," *Phys. Rev. A* **49**, 2938–2947 (1994).
- ¹⁶M. W. Mitchell and R. Y. Chiao, "Causality and negative group delays in a simple bandpass amplifier," *Am. J. Phys.* **68**, 14–19 (1998).
- ¹⁷A. M. Steinberg and R. Y. Chiao, "Dispersionless, highly superluminal propagation in a medium with a gain doublet," *Phys. Rev. A* **49**, 2071–2075 (1994).
- ¹⁸R. Y. Chiao, "Population Inversion and Superluminality," in *Amazing*

- Light*, edited by R. Y. Chiao (Springer-Verlag, New York, 1996), pp. 91–108.
- ¹⁹R. Y. Chiao and A. M. Steinberg, "Tunneling Times and Superluminality," in *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 1997), Vol. 37, pp. 347–405.
- ²⁰See p. 943 of R. P. Feynman, "A Relativistic Cut-Off for Classical Electrodynamics," *Phys. Rev.* **74**, 939–946 (1948).
- ²¹P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon, Oxford, 1958), 4th ed., Sec. 4.

Forces in complex fluids

Bruce J. Ackerson^{a)} and Anitra N. Novy^{b)}

Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078-3072

(Received 25 August 2000; accepted 18 December 2000)

[DOI: 10.1119/1.1351151]

I. PROBLEM

Ferrofluids¹ are stable suspensions of magnetic particles having linear dimension on the order of 10 nm. Due to vigorous Brownian motion the magnetic particles assume random orientations rendering the suspension as a whole paramagnetic. These complex fluids show a variety of phenomena and instabilities that amuse and delight students and teachers alike.² Because these fluids are used in a variety of applications including rotary seals, sensors, and actuators,³ they are commercially available.⁴

Figure 1 shows the experimental apparatus for viewing and recording the response of a ferrofluid film trapped at an air–water interface. Figure 2 shows recorded images for a drop ($\sim 40 \mu\text{l}$) of mineral-oil-based ferrofluid⁵ introduced to the surface of clean, filtered, de-ionized (18 M Ω) water. The hydrophobic ferrofluid spreads uniformly over the surface of water contained in a Petri dish. We gently stir the surface to emulsify the film, creating a collection of dark flat circular drops of ferrofluid as recorded in Fig. 2(a). Figure 2(b) shows the film 1 min after a cylindrical magnet having a radius of 1 cm is introduced with the axis of symmetry vertical and the lower end 3.3 cm above the ferrofluid film. The ferrofluid film clears from directly beneath the magnet but moves radially inward at large distances, forming tear-shaped drops with the clearer regions streaming outward. The ferrofluid collects in a ring structure at a finite radius (which is most dense at radius ~ 1.0 cm) from the center of the magnetic field symmetry axis. As the ferrofluid builds up, clumps or cone-shaped structures develop. As the cones grow, they become unstable and migrate one at a time into the central region. Figure 2(c) taken at $3\frac{1}{4}$ min shows the clumping in the ring-shaped structure with one cone at two o'clock escaping to the central region. Finally in Fig. 2(d), taken 21 min after introducing the magnet, a regular "crystalline" array of well-separated ferrofluid cones has formed. Yet there remains a ferrofluid film ring surrounding this crystalline structure.

How is it possible that the ferrofluid is both attracted to (cones) and repelled from (film) the region directly below the cylindrical magnet?

II. SOLUTION

Magnetic body forces, surface tension, viscous drag, and gravity combine to produce the peculiar behavior observed here. We focus on the magnetic body forces as the primary explanation for the question posed above. Figure 3 shows a side view of one of the "cones" of ferrofluid which form the two-dimensional crystalline array beneath the magnet. These clumps have a nearly ellipsoidal shape above the water surface with the symmetry axis parallel to the applied field. Other studies show that ferrofluid droplets submerged in an immiscible fluid deform into ellipsoids and align parallel to the direction of a uniform external magnetic field.⁶ Solutions have long existed for the magnetic (electric) field of an ellipsoid of permeability μ (dielectric constant ϵ) subjected to a uniform external field in a surrounding medium of permeability μ_0 (dielectric constant ϵ_0).⁷ When the symmetry axis of the ellipse is aligned parallel to the external field direction, the field inside the ellipse is uniform and parallel to the applied field (at large distances). Thus, to first order the "ellipsoidal cones" behave like little magnets oriented parallel to the external field and so move along the water surface to the strongest field regions located directly beneath the cylindrical magnet. However, the magnetic field induced in each of these cones is aligned parallel with the neighboring cone fields; therefore, the cones repel one another in the plane of the interface just like parallel oriented permanent magnets. A crystal lattice results.⁸ These results are qualitative, but intuitive, given our experience playing with permanent magnets.

How do we understand the quite different behavior of the film? Rosensweig¹ gives a general derivation of the body force \mathbf{f} or force per unit volume, which reduces for ferrofluid suspensions to

$$\mathbf{f} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} = \mu_0 M \nabla H, \quad (1)$$

where μ_0 is the vacuum permeability, \mathbf{H} is the magnetic field strength, and \mathbf{M} is the magnetization in the film volume element. This functional form suggests the Kelvin force density on an isolated body, except that the local field \mathbf{H} replaces the applied field \mathbf{H}_0 . Intuitively, we understand this body force to be like the force acting on a magnetic dipole.