Does a group velocity larger than c violate relativity?

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It is often assumed that relativity restricts the group velocity to be less than the speed of light. In this article, this assumption is examined in the context of electromagnetic waves propagating through a dilute plasma. The index of refraction is found from the phase shift in a thin slab of plasma. The response of the same slab to a incident δ-function pulse is also found. After passing through the slab, the pulse acquires a constant tail, in the limit of a thin slab, and the response to an oscillatory wave is recovered by superposition. By considering a material in which the tail has the opposite sign, it is shown that the group velocity can be larger than c without having any signal propagate faster than c.

I. INTRODUCTION

For a given medium in which the expression for the dispersion relation $\omega(k)$ is known, both the phase velocity $v_{\rm ph} = \omega/k$ and group velocity $v_{\rm gr} = d\omega/dk$ are mathematically defined[1] for any k. For a wave packet with a spread in wavenumber k, some ambiguity arises in the values of the phase and group velocities because of the spread in k, but, for narrow packets in k space, the uncertainties in these values are small. For such packets, one can conclude unambiguously whether one velocity or the other is larger than or smaller than the speed of light c.

It is also possible to identify physically the phase velocity of a wave packet: The phase velocity is the velocity of a point of constant phase, e.g., the place at which the wave amplitude goes through zero. The corresponding physical description of the group velocity, i.e., the velocity of a point on the envelope of the wave packet, may seem as well defined as the phase velocity, but, in some cases, it is not. If the shape of the wave packet is changing rapidly as the wave propagates, then one is not sure what point on the envelope at a later time corresponds to a given point at an earlier time, removing the possibility of a unique determination of the velocity of the envelope. Even picking a unique point, e.g., the maximum of the envelope, may not lead to a desired description of the group velocity, if, for example, one side of the wave packet is being enhanced and the other side diminished.

Since the phase velocity does not represent the velocity of propagation of energy or transfer of information, $v_{\rm ph}$ can be less than c, equal to c, or greater than c. The group velocity, on the other other hand, is often assumed to be restricted by relativity to $v_{\rm gr} < c$. However, this restriction cannot be a general principle, since there are well-known physical examples in which $v_{\rm gr} > c$. One example is anomalous dispersion, found in some frequency interval near a resonance.[2] However, this interval is also a region of strong absorption, and any wave packet with characteristic wavenumber in this region is changing shape sufficiently rapidly that the meaning of $v_{\rm gr}$ is blurred. In addition, because the wave packet is strongly attenuated as it propagates, one cannot follow the wave packet for a long time in order to make the uncertainty in position within the wave packet small compared to the distance traveled. The lesson to be learned from this example is that if the wave packet is changing shape sufficiently rapidly, relativity does not require that the calculated group velocity be less than the speed of light.

II. INDEX OF REFRACTION FOR A DILUTE PLASMA

Electromagnetic waves propagating in a dilute plasma provide a system in which the relation between group velocity and relativity can be explored. The problem can be attacked in two ways: (1) by a standard analysis of the propagation of waves of a definite frequency; and (2) by a derivation starting with a sharply peaked incident pulse. The latter approach has the advantage that the waves generated by the medium will be seen to be consistent with relativity, and so will be their superposition to form a wave of definite frequency.

The dispersion relation for a plasma, assuming negligible damping, is[3]

$$\omega^2 = k^2c^2 + \omega_p^2, \tag{1}$$

where the plasma frequency ω_p is given by

$$\omega_p^2 = Ne^2/m\epsilon_0, \tag{2}$$

where N is the number density of electrons and $-e$ and m are the charge and mass of an electron. The phase velocity is (for $\omega > \omega_p$)

$$v_{\rm ph} = \omega/k = c(1 - \omega_p^2/\omega^2)^{-1/2} > c, \tag{3}$$

which does not violate relativity for reasons outlined above. However, since $2\omega\, d\omega = k\, dk\, c^2$, $(\omega/k)(d\omega/dk) = v_{\rm ph} v_{\rm gr} = c^2$, and, therefore,

$$v_{\rm gr} = \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} < c\,. \tag{4}$$

At this point in the derivation, a comment is often added along the following lines: "Of course we expect that the group velocity must be less than c because of relativity." Some texts have statements or problems that indicate a similar expectation.[4]

Suppose, however, that results in Eqs. (3) and (4) had come out the other way, i. e., with $v_{\rm ph} < c$ and $v_{\rm gr} > c$. This would have occurred if, for example, the dispersion relation were instead

$$\omega^2 = k^2c^2 - \omega_0^2, \tag{5}$$

in which case $v_{\rm ph}$ and $v_{\rm gr}$ are given by

$$v_{\rm ph} = c(1 + \omega_0^2/\omega^2)^{-1/2} < c,$$
$$v_{\rm gr} = c(1 + \omega_0^2/\omega^2)^{1/2} > c. \tag{6}$$

Would we have concluded that the last result, $v_{\rm gr} > c$, is inconsistent with special relativity?

To understand how a dispersion relation of the kind given by (5) might arise, let us review the results of a standard treatment of the origin of the index of refraction as given, for example, in the Feynman lectures.[5] Here, the number density N is assumed to be sufficiently small (a dilute plasma) that the index of refraction can be approximated as

$$n = c/v_{\text{ph}} = (1 - \omega_p^2/\omega^2)^{1/2} \approx 1 - \omega_p^2/2\omega^2 . \quad (7)$$

In going through a thin slab of plasma of thickness Δx, the presence of a material with an index of refraction $n \neq 1$ implies that an incident sinusoidal wave will be shifted in phase with respect to a similar wave propagating in a vacuum. If the slab is located at $x = 0$, the electric field of the incident wave is,

$$E_{\text{inc}} = E_0 e^{i(kx - \omega t)}, \quad \text{for } x < 0, \quad (8)$$

where the real part is assumed. The wave for $x > \Delta x$ is then

$$\begin{aligned} E_{\text{tot}} &= E_0 e^{i(kx - \omega t)} e^{i(k_n - k)} \Delta x \\ &\approx E_0 e^{i(kx - \omega t)} [1 + i(k_n - k)\Delta x] \\ &= E_{\text{inc}} + i(k_n - k)\Delta x \, E_{\text{inc}} , \end{aligned} \quad (9)$$

where Δx is assumed to be small and we keep terms that are linear in Δx. The wavenumber in the vacuum is $k = \omega/c$ and in the plasma is $k_n = n\omega/c$.

Of the two terms in (9), the first term is the incident wave and the second term is the wave generated by the motion all of the charges in the slab of plasma in response to the incident wave. Writing $E_{\text{tot}} = E_{\text{inc}} + E_s$, the scattered contribution to the wave E_s is

$$E_s = -iNe^2 \Delta x E_{\text{inc}} / (2m\epsilon_0 \omega c). \quad (10)$$

This result is consistent with what is obtained if the radiation fields of each charge are superimposed,[5] assuming the charges move according to the equations of motion

$$m\ddot{y} = -eE_0 e^{-i\omega t} . \quad (11)$$

Note that if the sign of the force on the right-hand side in (11) was reversed, E_s would be reversed, and the index of refraction would be greater than 1 rather than less than 1. If such a case existed, then v_{ph} would be less than c and v_{gr} would be larger than c, as for the dispersion relation (5).

III. RESPONSE OF THE SLAB TO AN INCIDENT δ-FUNCTION PULSE

An alternate way to approach this problem is to consider an incident δ-function wave pulse and to find the response of the thin slab to this pulse. In this manner, we can check causality directly, because there should not be any wave generated by the slab until the pulse has reached the slab. Then, since the incident pulse is moving with the speed of light, no wave signal can be added ahead of the pulse once it has passed the slab. Because the waves generated by the charges all move with the speed of light, but in different directions, the components of velocities in the direction of motion of the incident pulse will be less than or equal to the speed of light. Therefore, the most that can happen is that the charges in the slab produce a resultant wave which, when superimposed on the incident pulse, adds a tail of some sort to that pulse.

To find the form of that tail, we take the incident pulse to be an electric field in the $+y$ direction with waveform

$$E_{\text{inc}} = A \, \delta(x - ct). \quad (12)$$

In response to this field, each charge in the slab, initially at rest, receives a kick at time $t = 0$, changing its velocity suddenly to $v_y = -eA/mc$, which we assume to be much smaller than the speed of light so that magnetic forces can be neglected. The analysis of the fields of such a "kicked" sheet of charge has been given in a previous article.[6] Since we want only the total electric field, and not that generated by the individual charges, the following argument is sufficient to determine the electric field after the δ-function pulse has passed.

Because the charges in the slab at a given time have the same velocity in the $-y$ direction, there will be a volume current density and, therefore, a surface current density, if Δx is small, in the $+y$ direction. From Ampère's law, a surface current density implies a discontinuous magnetic field. By symmetry, the magnitudes of the magnetic field must be the same on both sides of the slab, with the directions opposite, where the field on the forward side of the slab is in the $-z$ direction. In addition, a traveling electromagnetic wave always has an electric field of magnitude $E = cB$ in a direction so that $\mathbf{E} \times \mathbf{B}$ (or the Poynting vector) is in the direction of propagation. This gives an electric field that is in the opposite direction of the incident pulse, but in the same direction on both sides of the slab. Superimposing these fields then gives the electric field for $x > 0$ at time $t > 0$ as

$$E_y = A \, [\delta(x - ct) - (Ne^2 \, \Delta x / 2m\epsilon_0 c^2) \Theta(ct - x)] , \quad (13)$$

where $\Theta(x)$ is the step function with value 0, if $x < 0$, and 1, if $x > 0$, and, for reasons discussed below, the velocity of the charges has been assumed to be constant after the pulse has passed.

The electric field in (13) is, as Maxwell's equations imply, continuous across the slab and, therefore, the charges in the slab experience an electric force in the direction opposite to their velocity. This damping is also implied by the fact that energy must be lost from the sheet to account for the energy being radiated in the tail. In general, consideration of the reaction of the tail of the wave on the slab leads to an exponentially decreasing velocity after the initial pulse has passed the slab. However, if we are interested only in a thin sheet, in the limit $\Delta x \to 0$, and are calculating scattered contributions only to first order in Δx, the effect of damping being proportional to $(\Delta x)^2$, is negligible. This approximation is also consistent with the fact that the energy radiated in the tail, proportional to the square of the fields, is also of order $(\Delta x)^2$. Therefore, to first order in Δx the tail is constant, as represented by (13), and energy conservation holds to this order as well.

Any incident waveform can be considered as a superposition of δ-function pulses and, in particular, so can an oscillatory wave of the form given in (8). Suppose that the electric field of the incident wave is some arbitrary function $f(x - ct)$. Then, from (13), it follows that the total waveform for $x > 0$ is

$$E_y = f(x - ct) - (Ne^2 \, \Delta x / 2m\epsilon_0 c^2) g(x - ct), \quad (14)$$

where $g(u)$ is defined by

$$g(u) = \int_u^\infty f(u') du'. \quad (15)$$

Then, if we want the response to an oscillatory wave $f(x - ct) = e^{i(kx - \omega t)}$, then $f(u) = e^{iku}$ and $g(u) = (i/k) e^{iku}$ [assuming k has a small imaginary part so the integral in (15) is well defined]. Substituting this result into Eq. (14) reproduces the response of the thin slab to an incident oscillatory wave given in Eq. (10). Note that in this way of deriving (10) there is no question about violation of causality or propagation faster than c; in each term in the superposition nothing appears ahead of the δ-function pulse that moves with speed c.

IV. GROUP VELOCITY LARGER THAN c

Suppose that it were possible to change the sign of the second term of Eq. (13), i.e., reverse the sign of the tail, so that the tail had the same sign as the incident pulse. Because again nothing appears ahead of the δ-function pulse, this would be consistent with relativity and causality. However, by superposition, the wave generated by the slab in response to an oscillatory field (8) would be negative that given in (10). In particular, the phase change in going through the slab, given in (9), would be of the opposite sign. This would imply an index of refraction larger than 1, a phase velocity less than c, and a group velocity greater than c. Therefore, having a group velocity larger than c is not ruled out solely by relativity.

Reversal of the sign of the tail would also lead to antidamping, i.e., the electrical force on charges in the slab would be in the direction of their velocity, tending to increase the velocity rather than damping it. If the slab is sufficiently thin and we are interested only in fields generated to first order in Δx, the effects of antidamping, as with damping, will be negligible and the tail will be again constant. However, the presence of antidamping suggests how a material that has the desired properties could, in principle, be constructed. It would have to be active, in the sense that energy must be able to be fed into motion of the charges.

One way of constructing such a system is to suppose that at the position of each charge there was an additional infinitesimal charge that is used to measure the electric field. Then for a given value of the field a device computes and exerts a nonelectrical force on the charge equal to $F_a = +2eE$, i.e., a force twice as large as the electrical force but in the opposite direction. Then the net force on the charge would be reversed in sign and the subsequent velocities would all be reversed. This would reverse the sign of the tail, leading to the case described above for an oscillatory wave, namely, an index of refraction greater than 1, a phase velocity less than c, and a group velocity larger than c. Yet no violation of relativity has occurred, only the introduction of some bizarre, active system.

It should be noted that the system constructed above is inherently unstable. If there is some small fluctuation in the electric field, the net effect is to increase the electric field and the system runs away. Of course, for the slab of infinitesimal thickness, the runaway time goes to infinity. However, for a finite slab, it would be necessary to suppose that the charges are absolutely at rest until the incident pulse arrives in order that the charges in the slab not take off and generate a spurious electromagnetic wave.

Could such a bizarre material occur naturally? One way of changing the sign of the response is to change the sign in the fundamental force law. For example, one way of transforming the equations of electrostatics into the equations of Newtonian gravity is to replace q by m and (in SI units) $1/4\pi\epsilon_0$ by $-G$. If we make this transformation on the dispersion relation (1) and (2), we arrive at a dispersion relation of the form (5), with $\omega_0^2 = -4\pi GmN$, and an index of refraction n for gravitational waves, given in the dilute approximation, by

$$n = 1 + 2\pi GmN/\omega^2 . \tag{16}$$

Of course, the transformation used goes far beyond the static approximations of electrostatics and Newtonian gravity, and so the result (16) can only be considered suggestive and in no sense a derivation.

The question naturally arises as to what is the index of refraction for gravitational waves in a dilute gas and whether the result agrees with (16). The answer is not easy to give. A number of calculations[7–12] employing different techniques and approximations have given different answers. In some of these, the dominant contribution to $n-1$ comes from the pressure/c^2 rather than the much larger mass density, and with different coefficients, including both signs. Others find no deviation of n from 1 to this order. At least one calculation,[13] within the context of linearized general relativity, has given (16). Curiously, it has been argued[9,10] that some expressions can be judged to be incorrect solely on the grounds that they give a group velocity larger than c.

V. CONCLUSIONS

The results of this article can be summarized as follows. While it is true that the group velocity for plasma waves is less than c, it is not correct to state that such a restriction is imposed by relativity. This is demonstrated by giving an example of an active, albeit unstable, system in which the group velocity is larger than c, but which is consistent with relativity and causality. Therefore, relativity alone is not sufficient to rule out group velocities larger than c.

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