$$\langle P\rangle(x_0)=C\gamma\int_{x_0}^{\infty}(e^{-x/d})^2dx=C\,\frac{d\omega}{dk}\,e^{-2x_0/d},$$

where C is a constant that relates the square of the nonoscillatory part of the wave amplitude to its time-averaged energy density. The virial theorem assures us that constant C exists.

The (time-averaged) rate of energy flow across the plane $x=x_0$ is the (time-averaged) energy density there times the desired velocity of energy flow, v_E. This product is just $Cv_E e^{-2x_0/d}$, noting the meaning of constant C.

Hence the velocity of energy flow is

$$v_E=\frac{d\omega}{dk}=v_{\text{group}}.$$

An objection to this argument would be that it doesn't apply if the absorption is too strong (and not if it is too weak as implied in the statement of Question #52). It may be that the heroic efforts of Sommerfeld and Brillouin [Ann. Phys. (1914); see also, for instance, Max Born and Emil Wolf, *Principles of Optics* and Léon Brillouin, *Wave Propagation and Group Velocity*] to clarify signal propagation in the case of highly absorptive anomalous dispersion in optical media (where v_{group} exceeds the speed of light) have left the impression that the more ordinary case is similarly intricate.

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Answer to Question #52. Group velocity and energy propagation

K. M. Awati and T. Howes have asked for a general proof that the energy propagation in a dispersive medium is at the group, and not the phase, velocity. This is an interesting issue because it requires a broader understanding of wave energy than simply the electromagnetic component. It was originally pointed out by Max von Laue in 1905[1] that in a dispersive medium the kinetic energy of the oscillators as well as the field energy must be considered. The discussion was subsequently pursued by numerous authors leading to a general formulation of the total energy of a slightly damped wave in a dispersive medium, particularly a plasma.[2–7] Most of these treatments are based only on Maxwell's equations with a conductivity introduced to account for the particles. Without an explicit identification of the particle energy considered as coherent with the wave, however, the treatment is incomplete. Allis *et al.* do identify this energy for a cold plasma.[3] I carried out a complete description for a fully ionized hot plasma.[8] At that time, considerations of this problem were also being pursued by the group at Göteborg.[9]

If one does not ask about the actual identity of the coherent particle energy, to show that the total wave energy must propagate at the group velocity is fairly straightforward. A dispersive medium is one in which there is a time damping and spatial dispersion of an electromagnetic wave. From Maxwell's equations it is clear that such a medium must contain charges and be capable of producing currents. The presence of these charges is accounted for by introducing a conductivity, σ. The wave disturbances in such a medium are those with time and spatial dependence of the form

$$\mathrm{Re}\{f(\vec{k},\omega)\exp[i\omega t-i\vec{k}\cdot\vec{r}]\},\tag{1}$$

where $f(k,\omega)$ is any of the field or particle quantities of interest. In an isotropic medium the (complex) conductivity, $\sigma(\vec{k},\omega)$, is a scalar. The necessary condition for the existence of such disturbances is

$$\det\{\overleftrightarrow{D}(\vec{k},\omega)\}=0,\tag{2}$$

with

$$\overleftrightarrow{D}(\vec{k},\omega)\equiv\vec{k}\vec{k}+(\omega^2\mu_0\epsilon_0-k^2)\overleftrightarrow{1}-i\omega\mu_0\sigma\overleftrightarrow{1}.\tag{3}$$

To be considered as propagating in the medium, these disturbances, Eq. (1), must exist over a large number of wavelengths. That is, the imaginary parts of k and ω must be small. If one performs an expansion in small imaginary parts, the time average of the field energy equation produces

$$\omega_i U+(-\vec{k}_i)[U\ \mathrm{grad}_{\vec{k}}(\omega)]=-\sigma_r E^2,\tag{4}$$

where

$$U=U_{\text{em}}-\frac{\partial\sigma_i}{\partial\omega}\,E^2\tag{5}$$

is the total energy of the wave disturbance, and σ_r and σ_i are the real and imaginary parts of the conductivity. In (5) U_{em} is the electromagnetic energy density in the wave, and the second term in U is that identified as the coherent particle kinetic energy. The term multiplied by ω_i in (4) is the time derivative of the total energy and that involving the scalar product with $\vec{k}_i$ is the divergence of the flux of total energy. The right-hand side of (4) then represents the loss rate of this total energy to the background dispersive medium. This is the degradation of the coherent particle energy component of U into (noncoherent) thermal energy.

An important step in obtaining (4) is to consider the form of the electrical conductivity for the slightly damped wave. Expanding around the undamped condition, the conductivity is

$$\sigma(\vec{k},\omega)\approx(i\sigma_i)+i\omega_i\,\frac{\partial}{\partial\omega}\,(i\sigma_i)+i\vec{k}_i\cdot\mathrm{grad}_{\vec{k}}(i\sigma_i)+\sigma_r.\tag{6}$$

It is easy to show that the conductivity must be purely imaginary at the propagation condition.

The form of the flux term in (4) provides an answer to the question asked. In a dispersive medium, a general (total) Poynting vector must be considered. This is

$$\vec{S}=U\ \mathrm{grad}_{\vec{k}}(\omega),\tag{7}$$

where, of course, $\mathrm{grad}_{\vec{k}}(\omega)$ is the group velocity of the wave.

As satisfying as this is, the door has only been opened a crack; the real problem is to identify the coherent particle energy. In Ref. 8 this coherent energy is identified as the standard hydrodynamic energy, quadratic in the current, plus a part of the thermal energy related to particle density variations. An equally interesting question is the origin of the damping represented by the real part of the complex conductivity. The form of this term depends on the thermodynamic state of the background medium. Landau's classic treatment of the damping and excitation of Coulomb waves in a Vlasov plasma is an example.[10] Perhaps a more interesting question is the application to a semiconductor. I do not believe a parallel development has been pursued in this case.

I traditionally devote a portion of my course in electricity and magnetism to these issues. The simple form of the term on the right-hand side of (4) makes it possible to generate some interesting examples. Although mine have been a bit contrived to keep the mathematics transparent, there is, nevertheless, satisfying insight here for the student.

[1]M. von Laue, "Die Fortpflanzung der Strahlung in dispergierenden und absorbierenden Medien," Ann. Physik **18** (4), 523–566 (1905).
[2]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley, Reading, MA, 1960).
[3]W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasma* (Technology, Cambridge, MA, 1963).
[4]T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962).
[5]G. Bekefi, *Radiation Processes in Plasmas* (Wiley, New York, 1966).
[6]P. L. Auer, H. Hurwitz, and R. D. Miller, "Collective oscillations in a cold plasma," Phys. Fluids **1**, 501–514 (1958).
[7]T. H Dupree, "Kinetic theory of plasma and the electromagnetic field," Phys. Fluids **6**, 1714–1729 (1963).
[8]C. S. Helrich, "On the Coherent Response of a Plasma to Linear Wave Fields," Tech. Report (Institut für Technische Physik): Jül-895-TP (Kernforschungzentrum Jülich, Germany, 1972).
[9]D. Anderson, "A Generalized Expression for the Energy Density of Electromagnetic Waves in Media with Strong Temporal Dispersion," Zeitschrift für Naturforschung **27a**, 1094–1098 (1972).
[10]L. D. Landau, "On the Vibrations of the Electronic Plasma," J. Phys. USSR **X** (1) (1946).

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Answer to Question #52. Group velocity and energy propagation

The discussion of group velocities $v_g \equiv \partial\omega/\partial k$, by contrast with phase velocities $v_p \equiv \omega/k$, of propagating waves assumes three things. (i) The wave does not have a singular spectrum with just one carrier frequency ω_0 and one wave number k_0. It is actually a wave packet, hence the group velocity plays a role. (ii) The bandwidth of its spectrum is not too large (v_g well defined). (iii) The dispersion is nonlinear, i.e., ω is not proportional to k for the eigenmodes, $v_g \neq v_p$.

Awati and Howes [Am. J. Phys. **64** (11), 1353 (1996)] ask for a general proof of the relationship between group velocity and the velocity of energy propagation. Velocities are conventionally defined by identifying a characteristic point at some time t and place r, watching as it moves within a time Δt to another place $r+\Delta r$, then setting $v \equiv \Delta r/\Delta t$. If the parameter is the energy density[1] $S(r,t)$, one would, for example, associate the motion $r(t)$ of the characteristic point with a local maximum of the density, $\partial_r S(r,t)=0$. The link between this formula in local space and the dispersion $\omega(k)$ is inevitably given by the Fourier transform of this defining equation,

$$\int d^3\, k d\omega k S(k,\omega)\exp[i(kr-\omega t)]=0. \qquad (1)$$

A relation between Δr and Δt follows, because this equation must hold for some (r,t) as well as for another $(r+\Delta r, t+\Delta t)$. For small Δt and Δr, a part of the integrand may be expanded up to first order[2] in Δr and Δt,

$$ke^{ik(r+\Delta r)-i\omega(t+\Delta t))} \approx ke^{i(kr-\omega t)}+ike^{i(kr-\omega t)} \times (k\Delta r-\omega\Delta t). \qquad (2)$$

The following argument resembles a mathematical proof via "induction" from t to $t+\Delta t$. The integral over the zeroth-order term is assumed to be already zero. Δr and Δt are brought in relation to each other to ensure that the integral over the linear orders vanishes as well. We may introduce the central frequency ω_0 and wave number k_0,

$$k\Delta r-\omega\Delta t=k_0\Delta r-\omega_0\Delta t+(k-k_0)\Delta r-(\omega-\omega_0)\Delta t.$$

The first two terms on the right-hand side do not depend on k. Their integrals with the kernel (2) are consequently zero by means of (1). This is the crucial reason why we do *not* need to have $k_0\Delta r-\omega_0\Delta t=0$ and v is *not* primarily connected with v_p. To ensure that the integrals over the third and fourth terms are also zero, it is best to have $(k-k_0)\Delta r = (\omega-\omega_0)\Delta t$, which means $\Delta r/\Delta t=(\omega-\omega_0)/(k-k_0)$, i.e., $v=v_g$.[3-5]

The motion of special wave packet points with the group velocity rather than with the phase velocity is a mathematical feature of the Fourier transformation, independent of the spectral composition S of the wave amplitudes, the particular dispersion, and which physical quantity waves. Even the factor k in the Fourier integral (1), representing the gradient and maximum property in local space, is subordinate and may be replaced by more general functions.

[1]...a product of two local quantities, as in the case of the Poynting vector, or an (auto)correlation function in cases where the energy is a product of wave functions in (k,ω)-space, or something more general.
[2]An inexact justification is that the velocity is to be determined in the limit of $\Delta r, \Delta t \to 0$.
[3]Provided that the spectral width of $S(k,\omega)$ is small enough that the derivative $\partial\omega/\partial k$ can be well approximated by the quotient of the differences.
[4]If v_g is constant in the region of nonzero $S(k,\omega)$, this is valid for all orders in Δr and Δt, as the analysis can be performed within the argument of the exponential function (principle of the "stationary phase"), even if the factor k in the integrand is replaced be any function of k and ω. This property is useful, if, by some accidental characteristic of $S(k,\omega)$, the integral over the linear term vanishes for any pair of Δr and Δt, and vanishing of the first orders provides no information.
[5]Damped waves are generally described by *complex* valued dispersions $\omega(k)$ for the eigenmodes but *real* valued $v_g \equiv \partial\, \mathrm{Re}\, \omega/\partial k$. The Fourier integral is also defined for paths over the *real* k and ω axes.

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Answer to Question #52. Group velocity and energy propagation

The question by K. M. Awati and T. Howes [Am. J. Phys. **64** (11), 1353 (1996)] seeks a general proof showing that wave energy propagates at the group velocity rather than the phase velocity. No such proof exists because the result is not