复旦大学力学与工程科学系

2012～2013学年第一学期期末考试试卷

□A卷　　　□B卷

课程名称：Calculus on Differential Manifolds　课程代码：MATH120008.09

开课院系：力学与工程科学系　　　考试形式：开卷/闭卷/课程论文

姓名：______________　学号：______________　专业：______________

题 号	1/(1)	1/(2)	1/(3)	2 Ⅰ/(1)	2 Ⅰ/(2)	2 Ⅱ/(1)	2 Ⅱ/(2)	3 Ⅰ/(1)	3 Ⅰ/(2)
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题 号	3 Ⅰ/(3)	3 Ⅱ/(1)	3 Ⅱ/(2)	4/(1)	4/(2)	4/(3)	4/(4)	5/(1)	5/(2)
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Problem 1 (General concepts of differential manifolds)

1. *To narrate the general definition of a differential manifold with or without boundary, in the present case the manifold is embedded in a metric space therefor the manifold can be taken as a metric space.*

2. *To prove that the m dimensional smooth surface in $\mathbb{R}^{m+1}$ that can be represented in the implicit form*

$$\mathbf{\Sigma} = \left\{X \in \mathbb{R}^{m+1} \,|\, f(X) = 0 \in \mathbb{R},\, grad\, f(X) \neq 0 \in \mathbb{R}^{m+1}\right\}$$

is a m dimensional smooth manifold.

3. *To interpret the tensor bundle through the following representation*

$$\left\{\mathscr{T}^{p,q}(\mathbf{TM}),\, \mathbf{M},\, GL(m^{p+q},\, \mathbb{R}),\, \mathbb{R}^{m^{p+q}},\, \pi,\, \mathscr{E}\right\}$$

where $\mathbf{M}$ is a smooth manifold with $dim\mathbf{M} = m$. Hereinafter, either the tensor bundle or tensor field will be denoted briefly as $\mathscr{T}^{p,q}(\mathbf{TM})$.

Problem 2 (Homomorphic expansion and Lie derivative) *Considering a tensor field*

$$\mathscr{T}^{p,q}(\mathbf{TM}) \ni \mathbf{\Phi} = \Phi^{i_1\cdots i_p}_{j_1\cdots j_q}(x)\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}(x)$$

Section Ⅰ. General Theory

1. *Let* $\phi_t(\xi) \equiv \phi(t;\xi) \in \mathscr{C}^\infty(\overset{o}{\mathscr{V}}, \overset{t}{\mathscr{V}})$ *is a smooth diffeomorphism, to give the representation of the corresponding homomorphic expansion* $\phi_{t*}\mathbf{\Phi}(x) \equiv \phi_{t*}(\xi)\mathbf{\Phi}(\xi)$*, where* $\{\xi^A\}_{A=1}^m$ *and* $\{x^i\}_{i=1}^m$ *are the coordinates with respect to the domains/configurations* $\overset{\circ}{\mathscr{V}}$ *and* $\overset{t}{\mathscr{V}}$ *respectively.*

2. *Let* $\mathbf{V}(x) = V^i(x)\frac{\partial}{\partial x^i}(x) \in \mathbf{TM}$ *is a vector field that generates a group of diffeomorphism* $\phi_t(\xi) \equiv \phi(t;\xi) \in \mathscr{C}^\infty(\overset{o}{\mathscr{V}}, \overset{t}{\mathscr{V}})$*, namely one has the following dynamical system in the parametric space*

$$\frac{\partial x^i}{\partial t}(t;\xi) = V^i(x(t;\xi),t) \quad \textit{with} \quad x^i(t_0;\xi) = \xi^i, \quad i = 1,\cdots,m$$

The so termed Lie derivative can be defined as follows

$$\mathscr{L}_{\mathbf{V}}\mathbf{\Phi}(\xi) \triangleq \lim_{t\to t_0} \frac{\mathbf{\Phi}(\phi(t;\xi)) - \phi_{t*}(\xi)\mathbf{\Phi}(\xi)}{t - t_0}$$

To deduce the general representation of $\mathscr{L}_{\mathbf{V}}\mathbf{\Phi}(\xi)$*.*

Section Ⅱ. Case Study

Considering the following smooth manifold

$$\mathbf{\Sigma}(x_\Sigma) : \mathscr{D}_{x_\Sigma} \ni \begin{bmatrix} x^1_\Sigma \\ x^2_\Sigma \end{bmatrix} \mapsto \mathbf{\Sigma}(x_\Sigma) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}(x^1_\Sigma, x^2_\Sigma) \triangleq \begin{bmatrix} x^1_\Sigma \cos x^2_\Sigma \\ x^1_\Sigma \sin x^2_\Sigma \\ x^1_\Sigma + x^2_\Sigma \end{bmatrix} \in \mathbb{R}^3$$

that is a two dimensional surface in $\mathbb{R}^3$*, and the smooth diffeomorphism*

$$\phi_t(\xi) : \mathscr{D}_{x_\Sigma} \supset \overset{\circ}{\mathscr{V}} \ni \xi = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \mapsto \phi_t(\xi) = \begin{bmatrix} e^{\xi^1 t} \\ \xi^2 t \end{bmatrix} \in \overset{t}{\mathscr{V}} \subset \mathscr{D}_{x_\Sigma}$$

in the parametric space of the surface.

1. *To sketch the meaning of the homomorphic expansion by the diffeomorphism as mentioned above. The related vectors should be explicitly represented.*

2. *To directly calculate*

$$\mathscr{L}_{\mathbf{V}}\frac{\partial}{\partial x^1_\Sigma}(\xi) = \mathscr{L}_{\mathbf{V}}\left[\delta^s_1 \frac{\partial}{\partial x^s_\Sigma}\right](\xi)$$

based on the original definition of Lie derivative, and to compare the result to the one attained through the theoretical formula of Lie derivative, in the present case it is

$$\mathscr{L}_{\mathbf{V}}\frac{\partial}{\partial x^1_\Sigma}(\xi) = [\mathbf{V}, \frac{\partial}{\partial x^1_\Sigma}](\xi)$$

Problem 3 (Connection on tensor field) *For any tensor field, say as*

$$\mathscr{T}^{p,q}(\mathbf{TM}) \ni \mathbf{\Phi} = \Phi^{i_1\cdots i_p}_{j_1\cdots j_q}(x)\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_q}(x)$$

one can define the connection

$$\mathbf{TM} \ni \mathbf{X} \mapsto \nabla_{\mathbf{X}}\mathbf{\Phi} \in \mathscr{T}^{p,q}(\mathbf{TM})$$

such that

$$\begin{aligned}
&\nabla_{\mathbf{X}}\mathbf{\Phi}(\mathbf{W}_1,\cdots,\mathbf{W}_p;\mathbf{Y}_1,\cdots,\mathbf{Y}_q)\\
&=\mathbf{X}(\mathbf{\Phi}(\mathbf{W}_1,\cdots,\mathbf{W}_p;\mathbf{Y}_1,\cdots,\mathbf{Y}_q)) - \sum_{i=1}^{p}\mathbf{\Phi}(\mathbf{W}_1,\cdots,\nabla_{\mathbf{X}}\mathbf{W}_i,\cdots\mathbf{W}_p;\mathbf{Y}_1,\cdots,\mathbf{Y}_q))\\
&-\sum_{j=1}^{q}\mathbf{\Phi}(\mathbf{W}_1,\cdots,\mathbf{W}_p;\mathbf{Y}_1,\cdots,\nabla_{\mathbf{X}}\mathbf{Y}_j,\cdots,\mathbf{Y}_q)),\quad \forall\{\mathbf{W}_i\}_{i=1}^p\subset\mathbf{T}^*\mathbf{M},\,\forall\{\mathbf{Y}_j\}_{j=1}^q\subset\mathbf{TM}
\end{aligned}$$

Section Ⅰ. General Theory

1. *To prove the following fundamental relation*

$$\nabla_{\frac{\partial}{\partial x^l}}dx^j = -\Gamma^j_{ls}dx^s$$

   where the Christoffel symbol of the second kind is defined as $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} =: \Gamma^s_{ji}\frac{\partial}{\partial x^s}$

2. *To deduce the general representation of* $\nabla_{\mathbf{X}}\mathbf{\Phi}$

3. *For any smooth curve on the manifold, say as*

$$\gamma(t):\ [a,b]\ni t\mapsto\gamma(t)\in\mathbf{M}$$

   the parallel moving of any tensor field $\mathbf{\Phi}$ *along the curve is defined as*

$$\nabla_{\dot\gamma(t)}\mathbf{\Phi}(\gamma(t)) = \mathbf{0}\in\mathscr{T}^{p,q}(\mathbf{TM})$$

Section Ⅱ. Case Study

Considering the sphere in $\mathbb{R}^3$ *with the metric represented by*

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi$$

that is a two dimensional Riemannian surface/manifold in which there exists the so termed Levi-Civita connection that is the christoffel symbol of the first kind is determined by the metric

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

1. *To calculate all of the Christoffel symbols of the second kind, that is* $\Gamma^k_{ij} = g^{kl}\Gamma_{ij,l}$.

2. *To deduce the full representation of the governing equation in the component form of the parallel moving of a affine tensor field along a meridian.*

Problem 4 (Differential operations on manifold) *Let* ϕ *be the smooth mapping between the differential manifolds* M *and* N *with* $\dim M = m$ *and* $\dim N = n$ *respectively. Then, one has*

$$(\phi^*\omega)(x) \equiv \phi^*(y(x))\omega(y(x)) = \phi^*(y(x))\left[\frac{1}{r!}\omega_{\alpha_1,\cdots,\alpha_r}(y(x))dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_r}\right]$$

for any $\omega(y) = \frac{1}{r!}\omega_{\alpha_1,\cdots,\alpha_r}(y)dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_r} \in \wedge^r(\mathbf{N})$.

1. *To prove the general identity*

$$(\phi^*\omega)(x) = \frac{1}{r!}\sum_{1\leq i_1<\cdots<i_r\leq m}\omega_{\alpha_1,\cdots,\alpha_r}(y(x))\left[\frac{\partial(y^{\alpha_1}\cdots y^{\alpha_r})}{\partial(x^{i_1}\cdots x^{i_r})}(x)\right]dx^{i_1}\wedge\cdots\wedge dx^{i_r}$$

2. *To prove*

$$(\phi^*\omega)(x) = \frac{1}{(r!)^2}\omega_{\alpha_1,\cdots,\alpha_r}(y(x))\left[\frac{\partial(y^{\alpha_1}\cdots y^{\alpha_r})}{\partial(x^{i_1}\cdots x^{i_r})}(x)\right]dx^{i_1}\wedge\cdots\wedge dx^{i_r} \in \wedge^r(\mathbf{M})$$

3. *To prove the general identity*

$$\phi^*(d\omega) = d(\phi^*\omega), \quad \forall\omega \in \wedge^r(\mathbf{N})$$

4. *To prove the general identity*

$$(d\circ\mathscr{L}_{\mathbf{V}})\omega = (\mathscr{L}_{\mathbf{V}}\circ d)\omega, \quad \forall\omega \in \wedge^r(\mathbf{M})$$

   where $\mathbf{V} \in \mathbf{TM}$.

Problem 5 (Integral on Differential Manifold) *The integral of one p-form on the smooth manifold with the dimension p is defined as follows:*

$$\int_{\phi(I_p)\subset\mathbf{M}} \omega^p \triangleq \int_{I_p}\phi^*\omega^p, \quad \omega^p \in \wedge^p(\mathbf{M})$$

where $\phi(x) \in \mathscr{C}(I_p, \phi(I_p))$ *is an arbitrary chart.*

The tours in $\mathbb{R}^3$ could be represented as

$$\Sigma(\theta,\phi):\ \mathscr{D}_{\theta\phi} \ni \begin{bmatrix} \theta \\ \phi \end{bmatrix} \mapsto \Sigma(\theta,\phi) = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}(\theta,\phi) \triangleq \begin{bmatrix} (R + r\cos\theta)\cdot\cos\phi \\ (R + r\cos\theta)\cdot\sin\phi \\ R + r\sin\theta \end{bmatrix}$$

where $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi]$.

1. *To calculate the integral*

$$\int_{\Sigma} dX^2 \wedge dX^3$$

2. *To calculate the area of the torus based on the following definition*

$$|\Sigma| \triangleq \int_{\mathscr{D}_{\theta\phi}} \sqrt{g}\, d\theta \wedge d\phi, \quad g \triangleq det[g_{ij}]$$

   where $\{g_{ij}\}$ is the metric on the torus.

Note: To give the deduction and calculation in detail. And as the score is considered, the reflection of the correct methodologies is oriented.