# 曲面形态连续介质有限变形理论—变形刻画

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2015 年 4 月 21 日

## 1　知识要素

基于变形梯度的基本性质, 可类比郭仲衡 (1980)[①]关于体积形态连续介质的一般有限变形理论的处理, 将变形的全部刻画分为 4 类, 对应地结为 4 个性质.

### 1.1　变形梯度基本性质

**性质 1.1** (变形梯度基本性质)**.** 变形梯度具有如下基本性质, 不仅适用于二维曲面理论而且适用于高维曲面理论.

1. $\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F}=(\boldsymbol{V}\otimes\overset{\Sigma}{\square})\cdot\boldsymbol{F}$, 此处 $\overset{\Sigma}{\square}\triangleq\boldsymbol{g}^s\dfrac{\partial}{\partial x^s_\Sigma}(\boldsymbol{x}_\Sigma,t)$;
2. $\dfrac{\mathrm{d}}{\mathrm{d}t}\det\boldsymbol{F}=\theta\det\boldsymbol{F}$, 此处 $\theta\triangleq\boldsymbol{V}\cdot\overset{\Sigma}{\square}=\overset{\Sigma}{\square}\cdot\boldsymbol{V}$.

曲面变形梯度的行列式定义为

$$\det\boldsymbol{F}\triangleq\frac{\sqrt{g_\Sigma}}{\sqrt{G_\Sigma}}\det\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}\right)(\boldsymbol{\xi}_\Sigma,t).$$

### 1.2　各类物质系统的向量值映照刻画

引入, 初始物理构型、当前物理构型中物质曲线的向量值映照刻画

$$\overset{\circ}{\boldsymbol{\Sigma}}(\lambda):[a,b]\ni\lambda\mapsto\overset{\circ}{\boldsymbol{\Sigma}}(\lambda)\triangleq\overset{\circ}{\boldsymbol{\Sigma}}(\boldsymbol{\xi}_\Sigma(\lambda)),$$
$$\overset{t}{\boldsymbol{\Sigma}}(\lambda):[a,b]\ni\lambda\mapsto\overset{t}{\boldsymbol{\Sigma}}(\lambda)\triangleq\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma(\boldsymbol{\xi}_\Sigma(\lambda),t),t).$$

初始物理构型、当前物理构型中物质曲面的向量值映照刻画

$$\overset{\circ}{\boldsymbol{\Sigma}}(\lambda,\mu):D_{\lambda\mu}\ni\{\lambda,\mu\}\mapsto\overset{\circ}{\boldsymbol{\Sigma}}(\lambda,\mu)\triangleq\overset{\circ}{\boldsymbol{\Sigma}}(\boldsymbol{\xi}_\Sigma(\lambda,\mu)),$$
$$\overset{t}{\boldsymbol{\Sigma}}(\lambda,\mu):D_{\lambda\mu}\ni\{\lambda,\mu\}\mapsto\overset{t}{\boldsymbol{\Sigma}}(\lambda,\mu)\triangleq\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma(\boldsymbol{\xi}_\Sigma(\lambda,\mu),t),t).$$

### 1.3　变形刻画

类似于一般有限变形理论以及曲线坐标系显含时间有限变形, 对曲面有限变形理论的变形刻画仍可以归纳为 4 类.

① 郭仲衡. 非线性弹性理论. 北京: 科学出版社, 1980.

### 1.3.1 第一类 初始物理构型与当前物理构型中有向线元面元之间的关系式

**性质 1.2** (初始物理构型－当前物理构型中有向线元、有向面元之间的关系)**.**

1. $\dfrac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)=\boldsymbol{F}\cdot\dfrac{\mathrm{d}\overset{\circ}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)$;
2. $\left(\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu)=\det\boldsymbol{F}\cdot\left|\dfrac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\dfrac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu)\cdot\overset{t}{\boldsymbol{n}}(\lambda,\mu)$.

**证明** 本性质的证明主要利用链式求导法则以及方阵行列式的表达式.

1. 由 $\overset{t}{\boldsymbol{\Sigma}}(\lambda)=\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma(\boldsymbol{\xi}_\Sigma(\lambda),t),t)$, 有

$$\begin{aligned}\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)&=\frac{\partial\boldsymbol{\Sigma}}{\partial x^i_\Sigma}(\boldsymbol{x}_\Sigma,t)\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}(\boldsymbol{\xi}_\Sigma,t)\frac{\mathrm{d}\xi^A_\Sigma}{\mathrm{d}\lambda}(\lambda)\\&=\left(\frac{\partial x^i_\Sigma}{\partial\xi^A_\Sigma}(\boldsymbol{\xi}_\Sigma,t)\boldsymbol{g}_i\otimes\boldsymbol{G}^A\right)\cdot\left(\frac{\mathrm{d}\xi^B_\Sigma}{\mathrm{d}\lambda}(\lambda)\boldsymbol{G}_B\right)\\&=\boldsymbol{F}\cdot\left(\frac{\mathrm{d}\xi^B_\Sigma}{\mathrm{d}\lambda}(\lambda)\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\xi^B_i}(\boldsymbol{\xi}_\Sigma)\right)=\boldsymbol{F}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda).\end{aligned}$$

2. 由 $\overset{t}{\boldsymbol{\Sigma}}(\lambda,\mu)=\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma(\boldsymbol{\xi}_\Sigma(\lambda,\mu),t),t)$, 有

$$\begin{aligned}\left(\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu)&=\left(\boldsymbol{F}\cdot\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}(\lambda,\mu)\right)\times\left(\boldsymbol{F}\cdot\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right)\\&=\left(F^i_{\cdot\,A}\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\boldsymbol{g}_i\right)\times\left(F^j_{\cdot\,B}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\boldsymbol{g}_j\right)\\&=\left(F^i_{\cdot\,A}F^j_{\cdot\,B}\right)\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\varepsilon_{ij3}\overset{t}{\boldsymbol{n}}\\&=\sqrt{g_\Sigma}\left(e_{ij3}F^i_{\cdot\,A}F^j_{\cdot\,B}\right)\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\overset{t}{\boldsymbol{n}}\\&=\sqrt{g_\Sigma}\det\left(F^i_{\cdot\,L}\right)\left(e_{AB3}\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\right)\overset{t}{\boldsymbol{n}}\\&=\frac{\sqrt{g_\Sigma}}{\sqrt{G_\Sigma}}\det\left(F^i_{\cdot\,L}\right)\left(\varepsilon_{AB3}\frac{\partial\overset{\circ}{\Sigma}{}^A}{\partial\lambda}\frac{\partial\overset{\circ}{\Sigma}{}^B}{\partial\mu}\right)\overset{t}{\boldsymbol{n}}\\&=\det\boldsymbol{F}\cdot\left|\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu)\cdot\overset{t}{\boldsymbol{n}}.\end{aligned}$$

□

### 1.3.2 第二类 初始物理构型与当前物理构型中有向线元面元模之间的关系式

**性质 1.3** (初始物理构型－当前物理构型中有向线元、有向面元模之间的关系)**.**

1. $\left|\dfrac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right|_{\mathbb{R}^3} = \left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\cdot\dfrac{\mathrm{d}\overset{\circ}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right|_{\mathbb{R}^3}$;

2. $\left|\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu) = \det\boldsymbol{F}\left|\dfrac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\dfrac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu)$.

**证明** 对第一个等式, 利用性质1.2中相应结论, 有

$$\begin{aligned}\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right|^2_{\mathbb{R}^3} &= \left(\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda),\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right)_{\mathbb{R}^3} = \frac{\mathrm{d}\overset{\circ}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\cdot(\boldsymbol{F}^*\cdot\boldsymbol{F})\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\\ &= \left|(\boldsymbol{F}^*\cdot\boldsymbol{F})^{\frac{1}{2}}\cdot\frac{\mathrm{d}\overset{\circ}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right|^2_{\mathbb{R}^3}.\end{aligned}$$

对第二个等式, 将性质1.2(2) 中结论两端取模, 即得结论. □

### 1.3.3 第三类 当前物理构型中有向线元面元物质导数同其自身之间的关系式

**性质 1.4** (当前物理构型中有向线元、有向面元的物质导数同其自身之间的关系)**.**

1. $\overset{\cdot}{\overline{\dfrac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)}} = \boldsymbol{L}\cdot\dfrac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda),\quad \boldsymbol{L}\triangleq\boldsymbol{V}\otimes\overset{\Sigma}{\square}$;

2. $\overset{\cdot}{\overline{\left(\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu)}} = \boldsymbol{B}\cdot\left(\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\dfrac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu),\quad \boldsymbol{B}\triangleq\theta\boldsymbol{I}-\overset{\Sigma}{\square}\otimes\boldsymbol{V}$.

**证明** 第一个关系式可以容易地获得. 对第二个关系式, 可考虑

$$\begin{aligned}\overset{\cdot}{\overline{\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)}} &= \overset{\cdot}{\overline{\det\boldsymbol{F}}}\cdot\left|\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right|_{\mathbb{R}^3}\cdot\overset{t}{\boldsymbol{n}}(\lambda,\mu)\\ &\quad+\det\boldsymbol{F}\cdot\left|\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right|_{\mathbb{R}^3}\cdot\overset{\cdot}{\overline{\overset{t}{\boldsymbol{n}}(\lambda,\mu)}}\\ &= \boldsymbol{\theta}\cdot\left(\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right)+\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right|_{\mathbb{R}^3}\cdot\overset{\cdot}{\overline{\overset{t}{\boldsymbol{n}}(\lambda,\mu)}}\\ &= \left(\boldsymbol{\theta}\boldsymbol{I}-\overset{\Sigma}{\square}\otimes\boldsymbol{V}\right)\cdot\left(\frac{\partial\overset{t}{\Sigma}}{\partial\lambda}\times\frac{\partial\overset{t}{\Sigma}}{\partial\mu}(\lambda,\mu)\right)\\ &=: \boldsymbol{B}\cdot\left(\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}(\lambda,\mu)\right).\end{aligned}$$

□

此处, 利用了以下引理.

**引理 1.5.**

$$\overset{\cdot}{\overline{\overset{t}{\boldsymbol{n}}(\lambda,\mu)}} = -\left(\overset{\Sigma}{\square}\otimes\boldsymbol{V}\right)\cdot\overset{t}{\boldsymbol{n}}(\lambda,\mu),$$

此处 $\overset{t}{\boldsymbol{n}}(\lambda,\mu)$ 为物质曲面的单位法向量.

**证明** 考虑到 $\overset{t}{\boldsymbol{n}}(\lambda,\mu) := \boldsymbol{n}(x_\Sigma(\xi_\Sigma(\lambda,\mu),t),t)$, 则有

$$\overset{\cdot}{\overline{\overset{t}{\boldsymbol{n}}(\lambda,\mu)}} = \frac{\partial\boldsymbol{n}}{\partial t}(\boldsymbol{x}_\Sigma,t) + \dot{x}^i_\Sigma\frac{\partial\boldsymbol{n}}{\partial x^i_\Sigma}(\boldsymbol{x}_\Sigma,t) = \frac{\partial\boldsymbol{n}}{\partial t}(\boldsymbol{x}_\Sigma,t) - \dot{x}^i_\Sigma\cdot b_{is}\boldsymbol{g}^s,$$

此处

$$\begin{aligned}
\frac{\partial\boldsymbol{n}}{\partial t}(\boldsymbol{x}_\Sigma,t) &= \left(\frac{\partial\boldsymbol{n}}{\partial t}(\boldsymbol{x}_\Sigma,t),\boldsymbol{g}_i\right)_{\mathbb{R}^3}\boldsymbol{g}^i \\
&= -\left(\overset{t}{\boldsymbol{n}},\frac{\partial\boldsymbol{g}_i}{\partial t}(\boldsymbol{x}_\Sigma,t)\right)_{\mathbb{R}^3}\boldsymbol{g}^i = -\left(\overset{t}{\boldsymbol{n}},\frac{\partial}{\partial x^i_\Sigma}(\frac{\partial\boldsymbol{\Sigma}}{\partial t})(\boldsymbol{x}_\Sigma,t)\right)_{\mathbb{R}^3}\boldsymbol{g}^i \\
&= -\left(\overset{t}{\boldsymbol{n}},\frac{\partial}{\partial x^i_\Sigma}(\boldsymbol{V}-\dot{x}^s_\Sigma\boldsymbol{g}_s)(\boldsymbol{x}_\Sigma,t)\right)_{\mathbb{R}^3}\boldsymbol{g}^i \\
&= -\left(\overset{t}{\boldsymbol{n}},\frac{\partial\boldsymbol{V}}{\partial x^i_\Sigma}(\boldsymbol{x}_\Sigma,t)\right)_{\mathbb{R}^3}\boldsymbol{g}^i + \left(\overset{t}{\boldsymbol{n}},\dot{x}^s_\Sigma\frac{\partial\boldsymbol{g}_s}{\partial x^i_\Sigma}(\boldsymbol{x}_\Sigma,t)\right)_{\mathbb{R}^3}\boldsymbol{g}^i \\
&= -\overset{t}{\boldsymbol{n}}\cdot\left(\frac{\partial\boldsymbol{V}}{\partial x^i_\Sigma}(\boldsymbol{x}_\Sigma,t)\otimes\boldsymbol{g}^i\right) + \dot{x}^s_\Sigma b_{is}\boldsymbol{g}^i.
\end{aligned}$$

□

### 1.3.4 第四类 当前物理构型中有向线元与面元模的物质导数同其自身之间的关系式

**性质 1.6** (当前物理构型中有向线元、有向面元模的物质导数同其自身之间的关系)**.**

1. $$\overset{\cdot}{\overline{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}}}(\lambda) = (\boldsymbol{\tau}\cdot\boldsymbol{D}\cdot\boldsymbol{\tau})\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda),\quad \boldsymbol{\tau} = \frac{\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)}{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda)};$$

2. $$\overset{\cdot}{\overline{\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}}}(\lambda,\mu) = \theta\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu),$$

此处 $\boldsymbol{D}\triangleq\dfrac{\boldsymbol{L}+\boldsymbol{L}^*}{2}$ 称为曲面变形理论的**变形率张量**, $\boldsymbol{\tau}$ 表示线元的指向.

**证明** 对第一个等式, 利用性质1.4中相应结论, 有

$$\begin{aligned}
\overset{\cdot}{\overline{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|^2_{\mathbb{R}^3}}}(\lambda) &= 2\left(\overset{\cdot}{\overline{\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)}},\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right)_{\mathbb{R}^3} = 2\left(\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\cdot L^*,\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\right)_{\mathbb{R}^3} \\
&= 2\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\cdot\frac{\boldsymbol{L}+\boldsymbol{L}^*}{2}\cdot\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda) = 2\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda)\cdot\boldsymbol{D}\cdot\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}(\lambda).
\end{aligned}$$

另有

$$\overset{\cdot}{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|^2_{\mathbb{R}^3}}(\lambda)=2\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}(\lambda)\overset{\cdot}{\left|\frac{\mathrm{d}\overset{t}{\boldsymbol{\Sigma}}}{\mathrm{d}\lambda}\right|_{\mathbb{R}^3}}(\lambda),$$

由此即得结论.

对第二个等式, 利用性质1.3(2), 即

$$\left|\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu)=\det\boldsymbol{F}\left|\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}\right|_{\mathbb{R}^3}(\lambda,\mu)$$

以及 $\overset{\cdot}{\det\boldsymbol{F}}=\theta\det\boldsymbol{F}$, 即得结论. □

## 2 应用事例

## 3 建立路径

- 不同于一般的文献, 本讲稿建立的变形刻画关系基于物质线, 物质面对参数的偏导数. 按向量值映照微分学, 这些结论是完全严格的而非差一个一阶无穷小量等.
- 变形刻画关系式的获得原则上仅需依赖于变形梯度的基本性质, 由此也表示变形梯度蕴含了变形的所有信息.

有限变形理论讲稿 谢锡麟