1

写出4阶行列式中包含因子$a_{11}a_{23}$的项, 并指出正负号.

解: 含因子$a_{11}a_{23}$的项的一般形式为

$$(-1)^t a_{11}a_{23}a_{3r}a_{4s},$$

其中rs是2和4构成的排列, 这种排列共有两个, 即24和42. 所以含因子$a_{11}a_{23}$的项分别是

$$(-1)^t a_{11}a_{23}a_{32}a_{44} = -a_{11}a_{23}a_{32}a_{44}$$
$$(-1)^5 a_{11}a_{23}a_{34}a_{42} = a_{11}a_{23}a_{34}a_{42}$$

□

2

证明

$$\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} = 2\begin{vmatrix} a_1+b_1+c_1 & c_1+a_1 & a_1+b_1 \\ a_2+b_2+c_2 & c_2+a_2 & a_2+b_2 \\ a_3+b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}.$$

证明:

$$\begin{aligned} \begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} &= 2\begin{vmatrix} a_1+b_1+c_1 & c_1+a_1 & a_1+b_1 \\ a_2+b_2+c_2 & c_2+a_2 & a_2+b_2 \\ a_3+b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} \\ &= 2\begin{vmatrix} a_1+b_1+c_1 & b_1 & c_1 \\ a_2+b_2+c_2 & b_2 & c_2 \\ a_3+b_3+c_3 & b_3 & c_3 \end{vmatrix} \\ &= 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \end{aligned}$$

□

3

由n阶行列式

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 0$$

试证$n!$个不同的n级排列奇偶各半.

证明:

$$\begin{aligned} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} &= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \\ &= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} \\ &= \sum_{\text{奇排列}} (-1) + \sum_{\text{偶排列}} 1. \end{aligned}$$

所以, 上述行列式等于0可推出n级排列奇偶各半. □

4

数20604, 53227, 25755, 20927和78421 都可被17整除, 证明行列式

$$\begin{vmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{vmatrix}$$

也可被17整除.

证明: 将第1列的10^4倍, 第2列的10^3倍, 第3列的10^2倍, 第4列的10 倍加到第5列上, 得到

$$\begin{vmatrix} 2 & 0 & 6 & 0 & 20604 \\ 5 & 3 & 2 & 2 & 53227 \\ 2 & 5 & 7 & 5 & 25755 \\ 2 & 0 & 9 & 2 & 20927 \\ 7 & 8 & 4 & 2 & 78421 \end{vmatrix},$$

它的第5列有公因子17, 因而行列式可被17整除. □

5

一个n阶行列式$|A|$按反时针或顺时针方向旋转90°后所得行列式$|B|$, 试证$|B| = (-1)^{\frac{n(n-1)}{2}}|A|$.

证明: 设

$$|A_n| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

将其反时针旋转90°, 得到

$$|B_n| = \begin{vmatrix} a_{1n} & \cdots & a_{nn} \\ \cdots & \cdots & \cdots \\ a_{11} & \cdots & a_{n1} \end{vmatrix}.$$

比较$|A_n^T|$与$|B_n|$, 容易看出从$|B_n|$得到$|A_n^T|$需$\frac{n(n-1)}{2}$次行交换, 因而

$$|B_n| = \frac{n(n-1)}{2}|A_n^T| = \frac{n(n-1)}{2}|A_n|.$$

顺时针的情况同理可证. □

6

设$a_i \neq 0, i = 1, 2, \cdots, n$. 试证

$$\begin{vmatrix} 1+a_1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1+a_2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1+a_3 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1+a_{n-1} & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1+a_n \end{vmatrix} = \prod_{i=1}^{n} a_i \left(1 + \sum_{i=1}^{n} \frac{1}{a_i}\right).$$

证明: 从第1列中减去第2列, 从第2列中减去第3列, ..., 从第$(n-1)$列中

减去第n列, 得

$$
\begin{vmatrix}
1+a_1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1+a_2 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1+a_3 & \cdots & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & \cdots & 1+a_{n-1} & 1 \\
1 & 1 & 1 & \cdots & 1 & 1+a_n
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_1 & 0 & 0 & \cdots & 0 & 1 \\
-a_2 & a_2 & 0 & \cdots & 0 & 1 \\
0 & -a_3 & a_3 & \cdots & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & a_{n-1} & 1 \\
0 & 0 & 0 & \cdots & -a_n & 1+a_n
\end{vmatrix}
$$

$$
= \prod_{i=1}^{n} a_i \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 1/a_1 \\
-1 & 1 & 0 & \cdots & 0 & 1/a_2 \\
0 & -1 & 1 & \cdots & 0 & 1/a_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & 1/a_{n-1} \\
0 & 0 & 0 & \cdots & -1 & 1+1/a_n
\end{vmatrix}
$$

$$
= \prod_{i=1}^{n} a_i \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 1/a_1 \\
0 & 1 & 0 & \cdots & 0 & \sum_{i=1}^{2} 1/a_i \\
0 & 0 & 1 & \cdots & 0 & \sum_{i=1}^{3} 1/a_i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & \sum_{i=1}^{n-1} 1/a_i \\
0 & 0 & 0 & \cdots & 0 & 1+\sum_{i=1}^{n} 1/a_i
\end{vmatrix}
$$

$$
= \left(\prod_{i=1}^{n} a_i\right)\left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right).
$$

□

7

设

$$|A_n| = \begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & 0 & \cdots & 0 & 0 \\ 0 & z & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & z & x \end{vmatrix}$$

(1) 求$|A_n|$的递推公式;
(2) 利用递推公式求$|A_n|$.

解: 递推公式为

$$\begin{aligned} &|A_1| = x \\ &|A_n| = x|A_{n-1}| + (-1)^{n+1}yz^{n-1} \qquad n \geq 2. \end{aligned}$$

其特征方程为$t - x = 0$, 解之, 得到特征根$t = x$. 因而, 可以设此递推公式的通解为

$$|A_n| = k_1x^n + k_2z^n.$$

代入初值$|A_1| = x$, $|A_2| = x^2 - yz$,

$$\begin{cases} k_1x + k_2z & = x \\ k_1x^2 + k_2z^2 & = x^2 - yz \end{cases},$$

解得$k_1 = \frac{x^2-xz+yz}{x(x-z)}$, $k_2 = -\frac{xy}{x(x-z)}$. 因此,

$$|A_n| = \frac{(x^2 - xz + yz)x^n - xyz^n}{x(x-z)}.$$

□

8

用数学归纳法证明n阶行列式

$$\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

证明: 将题目中的n阶行列式记做$|A_n|$.

(1) $n = 1, 2$

$|A_1| = \alpha + \beta = \frac{\alpha^2-\beta^2}{\alpha-\beta}$, $|A_2| = (\alpha+\beta)^2 - \alpha\beta = \frac{\alpha^3-\beta^3}{\alpha-\beta}$. 命题成立.

(2) 假设$n = k$时命题成立, 则$n = k + 1$时

$$\begin{aligned}|A_{k+1}| &= (\alpha+\beta)|A_k| - \alpha\beta|A_{k-1}| \\ &= (\alpha+\beta)\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} - \alpha\beta\frac{\alpha^k-\beta^k}{\alpha-\beta} \\ &= \frac{\alpha^{k+2}-\beta^{k+2}}{\alpha-\beta}.\end{aligned}$$

命题成立. □

9

试用克莱姆法则解线性方程组

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 1 \\ 3x_1 - x_2 - x_3 - 2x_4 = -4 \\ 2x_1 + 3x_2 - x_3 - x_4 = -6 \\ x_1 + 2x_2 + 3x_3 - x_4 = -4 \end{cases}.$$

解: 方程组的系数行列式为

$$|A| = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 3 & -1 & -1 & -2 \\ 2 & 3 & -1 & -1 \\ 1 & 2 & 3 & -1 \end{vmatrix} = -153.$$

$|A| \neq 0$, 由克莱姆法则, 方程组有如下唯一解:

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, x_3 = \frac{|A_3|}{|A|}, x_4 = \frac{|A_4|}{|A|},$$

其中

$$|A_1| = \begin{vmatrix} 1 & 1 & 2 & 3 \\ -4 & -1 & -1 & -2 \\ -6 & 3 & -1 & -1 \\ -4 & 2 & 3 & -1 \end{vmatrix} = 153,$$

$$|A_2| = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 3 & -4 & -1 & -2 \\ 2 & -6 & -1 & 1 \\ 1 & -4 & 3 & -1 \end{vmatrix} = 153,$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 3 & -1 & -4 & -2 \\ 2 & 3 & -6 & -1 \\ 1 & 2 & -4 & -1 \end{vmatrix} = 0,$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & -1 & -1 & -4 \\ 2 & 3 & -1 & -6 \\ 1 & 2 & 3 & -4 \end{vmatrix} = -153.$$

代入上式, 得到

$$x_1 = -1, x_2 = -1, x_3 = 0, x_4 = 1.$$

□

10

计算行列式

$$\begin{vmatrix} a_{11} & 0 & a12 & 0 & \cdots & a_{1n} & 0 \\ 0 & b_{11} & 0 & b_{12} & \cdots & 0 & b_{1n} \\ a_{21} & 0 & a_{22} & 0 & \cdots & a_{2n} & 0 \\ 0 & b_{21} & 0 & b_{22} & \cdots & 0 & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & a_{n2} & 0 & \cdots & a_{nn} & 0 \\ 0 & b_{n1} & 0 & b_{n2} & \cdots & 0 & b_{nn} \end{vmatrix}$$

的值.

解:

$$\begin{vmatrix} a_{11} & 0 & a12 & 0 & \cdots & a_{1n} & 0 \\ 0 & b_{11} & 0 & b_{12} & \cdots & 0 & b_{1n} \\ a_{21} & 0 & a_{22} & 0 & \cdots & a_{2n} & 0 \\ 0 & b_{21} & 0 & b_{22} & \cdots & 0 & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & a_{n2} & 0 & \cdots & a_{nn} & 0 \\ 0 & b_{n1} & 0 & b_{n2} & \cdots & 0 & b_{nn} \end{vmatrix}$$

$$=(-1)^{1+2+\cdots+(n-1)} \begin{vmatrix} a_{11} & 0 & a12 & 0 & \cdots & a_{1n} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & a_{2n} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & a_{n2} & 0 & \cdots & a_{nn} & 0 \\ 0 & b_{11} & 0 & b_{12} & \cdots & 0 & b_{1n} \\ 0 & b_{21} & 0 & b_{22} & \cdots & 0 & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_{n1} & 0 & b_{n2} & \cdots & 0 & b_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a12 & \cdots & a_{1n} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 0 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a12 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}.$$

□