电动力学 note by 陈苏迪

5.4 转移矩阵

（1）

$$设\mu(r)=\begin{cases}\mu_1 & r>R_1\\ \mu_2 & R_2<r<R_1\\ & \cdots\\ \mu_i & R_i<r<R_{i-1}\\ & \cdots\\ \mu_n & r<R_{n-1}\end{cases}$$

则磁标势

$$\varphi(r,\theta,\varphi)=\begin{cases}\varphi_1(r,\theta,\varphi) & r>R_1\\ \varphi_2(r,\theta,\varphi) & R_2<r<R_1\\ & \cdots\\ \varphi_i(r,\theta,\varphi) & R_i<r<R_{i-1}\\ & \cdots\\ \varphi_n(r,\theta,\varphi) & r<R_{n-1}\end{cases}$$

对任意 1 到n之间的整数i 均有 $\nabla^2\varphi_i=0$

在$r=R_i$面上的边界条件为$\begin{cases}\varphi_i(R_i)=\varphi_{i+1}(R_i)\\ \mu_i\frac{\partial\varphi_i}{\partial r}|_{r=R_i}=\mu_{i+1}\frac{\partial\varphi_{i+1}}{\partial r}|_{r=R_i}\end{cases}$

当在这样的空间中外加均匀磁场H_0时，可取试解$\varphi_i(r,\theta,\varphi)=\left(c_ir+\frac{d_i}{r^2}\right)\cos\theta$

代入边界条件得

$$\begin{cases}c_iR_i+\frac{d_i}{{R_i}^2}=c_{i+1}R_i+\frac{d_{i+1}}{{R_i}^2}\\ \mu_ic_i-\mu_i\frac{2d_i}{{R_i}^3}=\mu_{i+1}c_{i+1}R_i-\mu_{i+1}\frac{2d_{i+1}}{{R_i}^3}\end{cases}$$

由上式可得

$$\begin{pmatrix}\frac{\mu_i}{3\mu_{i+1}}+\frac{2}{3} & \frac{2}{3{R_i}^3}-\frac{2\mu_i}{3{R_i}^3\mu_{i+1}}\\ \frac{{R_i}^3}{3}-\frac{{R_i}^3\mu_i}{3\mu_{i+1}} & \frac{2\mu_i}{3\mu_{i+1}}+\frac{1}{3}\end{pmatrix}\begin{pmatrix}c_i\\ d_i\end{pmatrix}=\begin{pmatrix}c_{i+1}\\ d_{i+1}\end{pmatrix}$$

或者

$$\begin{pmatrix}c_i\\ d_i\end{pmatrix}=\begin{pmatrix}\frac{\mu_{i+1}}{3\mu_i}+\frac{2}{3} & \frac{2}{3{R_i}^3}-\frac{2\mu_{i+1}}{3{R_i}^3\mu_i}\\ \frac{{R_i}^3}{3}-\frac{{R_i}^3\mu_{i+1}}{3\mu_i} & \frac{2\mu_{i+1}}{3\mu_i}+\frac{1}{3}\end{pmatrix}\begin{pmatrix}c_{i+1}\\ d_{i+1}\end{pmatrix}$$

令$T_i=\begin{pmatrix}\frac{\mu_{i+1}}{3\mu_i}+\frac{2}{3} & \frac{2}{3{R_i}^3}-\frac{2\mu_{i+1}}{3{R_i}^3\mu_i}\\ \frac{{R_i}^3}{3}-\frac{{R_i}^3\mu_{i+1}}{3\mu_i} & \frac{2\mu_{i+1}}{3\mu_i}+\frac{1}{3}\end{pmatrix}$

$$T=T_1T_2\dots T_{n-1}$$

则$\begin{pmatrix}c_1\\ d_1\end{pmatrix}=T\begin{pmatrix}c_n\\ d_n\end{pmatrix}$

（这样写下面比较好算，课件中写的是$T^{-1}\begin{pmatrix}c_1\\ d_1\end{pmatrix}=\begin{pmatrix}c_n\\ d_n\end{pmatrix}$，取逆就可以了）

由无穷远处的边界条件$r\to\infty$时 $\varphi_1=-H_0r\cos\theta$可得$c_1=-H_0$

由原点处的边界条件$r=0$ 时 φ_n有限可得 $d_n=0$

故

$$\begin{pmatrix}-H_0\\ d_1\end{pmatrix}=T\begin{pmatrix}c_n\\ 0\end{pmatrix}$$

解得

$d_1=-\mathrm{H}_0\frac{T_{21}}{T_{11}}$

$\mathrm{c}_n=\frac{-\mathrm{H}_0}{T_{11}}$

对于 n=3 的情况，

$$T=T_1T_2=\begin{pmatrix}\frac{\mu_2}{3\mu_1}+\frac{2}{3} & \frac{2}{3{R_1}^3}-\frac{2\mu_2}{3{R_1}^3\mu_1}\\ \frac{{R_1}^3}{3}-\frac{{R_1}^3\mu_2}{3\mu_1} & \frac{2\mu_2}{3\mu_1}+\frac{1}{3}\end{pmatrix}\begin{pmatrix}\frac{\mu_3}{3\mu_2}+\frac{2}{3} & \frac{2}{3{R_2}^3}-\frac{2\mu_3}{3{R_2}^3\mu_2}\\ \frac{{R_2}^3}{3}-\frac{{R_2}^3\mu_3}{3\mu_2} & \frac{2\mu_3}{3\mu_2}+\frac{1}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\left(\frac{\mu_2}{3\mu_1}+\frac{2}{3}\right)\left(\frac{\mu_3}{3\mu_2}+\frac{2}{3}\right)+\left(\frac{2}{3{R_1}^3}-\frac{2\mu_2}{3{R_1}^3\mu_1}\right)\left(\frac{{R_2}^3}{3}-\frac{{R_2}^3\mu_3}{3\mu_2}\right) & \left(\frac{\mu_2}{3\mu_1}+\frac{2}{3}\right)\left(\frac{2}{3{R_2}^3}-\frac{2\mu_3}{3{R_2}^3\mu_2}\right)+\left(\frac{2}{3{R_1}^3}-\frac{2\mu_2}{3{R_1}^3\mu_1}\right)\left(\frac{2\mu_3}{3\mu_2}+\frac{1}{3}\right)\\ \left(\frac{{R_1}^3}{3}-\frac{{R_1}^3\mu_2}{3\mu_1}\right)\left(\frac{\mu_3}{3\mu_2}+\frac{2}{3}\right)+\left(\frac{2\mu_2}{3\mu_1}+\frac{1}{3}\right)\left(\frac{{R_2}^3}{3}-\frac{{R_2}^3\mu_3}{3\mu_2}\right) & \left(\frac{{R_1}^3}{3}-\frac{{R_1}^3\mu_2}{3\mu_1}\right)\left(\frac{2}{3{R_2}^3}-\frac{2\mu_3}{3{R_2}^3\mu_2}\right)+\left(\frac{2\mu_2}{3\mu_1}+\frac{1}{3}\right)\left(\frac{2\mu_3}{3\mu_2}+\frac{1}{3}\right)\end{pmatrix}$$

$$d_1=-\mathrm{H}_0\frac{T_{21}}{T_{11}}=-\mathrm{H}_0\frac{\left(\frac{{R_1}^3}{3}-\frac{{R_1}^3\mu_2}{3\mu_1}\right)\left(\frac{\mu_3}{3\mu_2}+\frac{2}{3}\right)+\left(\frac{2\mu_2}{3\mu_1}+\frac{1}{3}\right)\left(\frac{{R_2}^3}{3}-\frac{{R_2}^3\mu_3}{3\mu_2}\right)}{\left(\frac{\mu_2}{3\mu_1}+\frac{2}{3}\right)\left(\frac{\mu_3}{3\mu_2}+\frac{2}{3}\right)+\left(\frac{2}{3{R_1}^3}-\frac{2\mu_2}{3{R_1}^3\mu_1}\right)\left(\frac{{R_2}^3}{3}-\frac{{R_2}^3\mu_3}{3\mu_2}\right)}$$

$$=-\mathrm{H}_0\frac{{R_1}^3(\mu_1-\mu_2)(\mu_3+2\mu_2)+{R_2}^3(2\mu_2+\mu_1)(\mu_2-\mu_3)}{(\mu_2+2\mu_1)(\mu_3+2\mu_2)+2\frac{{R_2}^3}{{R_1}^3}(\mu_1-\mu_2)(\mu_2-\mu_3)}$$

$$\mathrm{c}_3=\frac{-\mathrm{H}_0}{T_{11}}=\frac{-9\mathrm{H}_0\mu_1\mu_2}{(\mu_2+2\mu_1)(\mu_3+2\mu_2)+\frac{2{R_2}^3}{{R_1}^3}(\mu_1-\mu_2)(\mu_2-\mu_3)}$$

等效偶极矩

$$m=4\pi d_1=-4\pi\mathrm{H}_0\frac{{R_1}^3(\mu_1-\mu_2)(\mu_3+2\mu_2)+{R_2}^3(2\mu_2+\mu_1)(\mu_2-\mu_3)}{(\mu_2+2\mu_1)(\mu_3+2\mu_2)+2\frac{{R_2}^3}{{R_1}^3}(\mu_1-\mu_2)(\mu_2-\mu_3)}$$

(2)在此题中，取题设的变量名可得

$$m=-4\pi\mathrm{H}_0\frac{\mathrm{R}'^3(\mu-\mu_1)(\mu_2+2\mu_1)+\mathrm{R}^3(2\mu_1+\mu)(\mu_1-\mu_2)}{(\mu_1+2\mu)(\mu_2+2\mu_1)+2\frac{\mathrm{R}^3}{\mathrm{R}'^3}(\mu-\mu_1)(\mu_1-\mu_2)}$$

若视μ为未知数，其余各量均为常数，要使等效偶极矩消失，须有

$$\mu=\mu_1-\frac{3\mathrm{R}^3\mu_1(\mu_1-\mu_2)}{\mathrm{R}'^3(\mu_2+2\mu_1)+\mathrm{R}^3(\mu_1-\mu_2)}$$