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CAPITAL ASSET PRICES: A THEORY OF MARKET EQUILIBRIUM UNDER CONDITIONS OF RISK*

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I. INTRODUCTION

ONE OF THE PROBLEMS which has plagued those attempting to predict the behavior of capital markets is the absence of a body of positive microeconomic theory dealing with conditions of risk. Although many useful insights can be obtained from the traditional models of investment under conditions of certainty, the pervasive influence of risk in financial transactions has forced those working in this area to adopt models of price behavior which are little more than assertions. A typical classroom explanation of the determination of capital asset prices, for example, usually begins with a careful and relatively rigorous description of the process through which individual preferences and physical relationships interact to determine an equilibrium pure interest rate. This is generally followed by the assertion that somehow a market risk-premium is also determined, with the prices of assets adjusting accordingly to account for differences in their risk.

A useful representation of the view of the capital market implied in such discussions is illustrated in Figure 1. In equilibrium, capital asset prices have adjusted so that the investor, if he follows rational procedures (primarily diversification), is able to attain any desired point along a *capital market line*.[1] He may obtain a higher expected rate of return on his holdings only by incurring additional risk. In effect, the market presents him with two prices: the *price of time*, or the pure interest rate (shown by the intersection of the line with the horizontal axis) and the *price of risk*, the additional expected return per unit of risk borne (the reciprocal of the slope of the line).

* A great many people provided comments on early versions of this paper which led to major improvements in the exposition. In addition to the referees, who were most helpful, the author wishes to express his appreciation to Dr. Harry Markowitz of the RAND Corporation, Professor Jack Hirshleifer of the University of California at Los Angeles, and to Professors Yoram Barzel, George Brabb, Bruce Johnson, Walter Oi and R. Haney Scott of the University of Washington.

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1. Although some discussions are also consistent with a non-linear (but monotonic) curve.

At present there is no theory describing the manner in which the price of risk results from the basic influences of investor preferences, the physical attributes of capital assets, etc. Moreover, lacking such a theory, it is difficult to give any real meaning to the relationship between the price of a single asset and its risk. Through diversification, some of the risk inherent in an asset can be avoided so that its total risk is obviously not the relevant influence on its price; unfortunately little has been said concerning the particular risk component which is relevant.

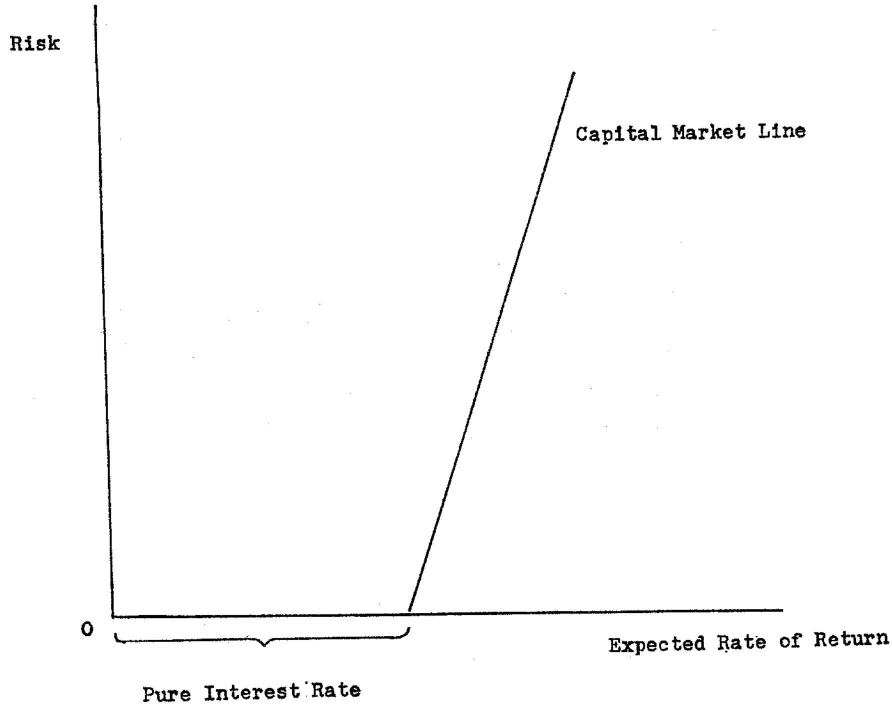

FIGURE 1

In the last ten years a number of economists have developed *normative* models dealing with asset choice under conditions of risk. Markowitz,[2] following Von Neumann and Morgenstern, developed an analysis based on the expected utility maxim and proposed a general solution for the portfolio selection problem. Tobin[3] showed that under certain conditions Markowitz's model implies that the process of investment choice can be broken down into two phases: first, the choice of a unique optimum combination of risky assets; and second, a separate choice concerning the allocation of funds between such a combination and a single riskless

2. Harry M. Markowitz, *Portfolio Selection, Efficient Diversification of Investments* (New York: John Wiley and Sons, Inc., 1959). The major elements of the theory first appeared in his article "Portfolio Selection," *The Journal of Finance*, XII (March 1952), 77-91.

3. James Tobin, "Liquidity Preference as Behavior Towards Risk," *The Review of Economic Studies*, XXV (February, 1958), 65-86.

asset. Recently, Hicks[4] has used a model similar to that proposed by Tobin to derive corresponding conclusions about individual investor behavior, dealing somewhat more explicitly with the nature of the conditions under which the process of investment choice can be dichotomized. An even more detailed discussion of this process, including a rigorous proof in the context of a choice among lotteries has been presented by Gordon and Gangolli.[5]

Although all the authors cited use virtually the same model of investor behavior,[6] none has yet attempted to extend it to construct a *market* equilibrium theory of asset prices under conditions of risk.[7] We will show that such an extension provides a theory with implications consistent with the assertions of traditional financial theory described above. Moreover, it sheds considerable light on the relationship between the price of an asset and the various components of its overall risk. For these reasons it warrants consideration as a model of the determination of capital asset prices.

Part II provides the model of individual investor behavior under conditions of risk. In Part III the equilibrium conditions for the capital market are considered and the capital market line derived. The implications for the relationship between the prices of individual capital assets and the various components of risk are described in Part IV.

II. Optimal Investment Policy for the Individual

The Investor's Preference Function

Assume that an individual views the outcome of any investment in probabilistic terms; that is, he thinks of the possible results in terms of some probability distribution. In assessing the desirability of a particular investment, however, he is willing to act on the basis of only two para-

4. John R. Hicks, "Liquidity," *The Economic Journal*, LXXII (December, 1962), 787-802.

5. M. J. Gordon and Ramesh Gangolli, "Choice Among and Scale of Play on Lottery Type Alternatives," College of Business Administration, University of Rochester, 1962. For another discussion of this relationship see W. F. Sharpe, "A Simplified Model for Portfolio Analysis," *Management Science*, Vol. 9, No. 2 (January 1963), 277-293. A related discussion can be found in F. Modigliani and M. H. Miller, "The Cost of Capital, Corporation Finance, and the Theory of Investment," *The American Economic Review*, XLVIII (June 1958), 261-297.

6. Recently Hirshleifer has suggested that the mean-variance approach used in the articles cited is best regarded as a special case of a more general formulation due to Arrow. See Hirshleifer's "Investment Decision Under Uncertainty," *Papers and Proceedings of the Seventy-Sixth Annual Meeting of the American Economic Association*, Dec. 1963, or Arrow's "Le Role des Valeurs Boursieres pour la Repartition la Meilleure des Risques," *International Colloquium on Econometrics*, 1952.

7. After preparing this paper the author learned that Mr. Jack L. Treynor, of Arthur D. Little, Inc., had independently developed a model similar in many respects to the one described here. Unfortunately Mr. Treynor's excellent work on this subject is, at present, unpublished.

meters of this distribution—its expected value and standard deviation.[8] This can be represented by a total utility function of the form:

$$U = f(E_w, \sigma_w)$$

where E_w indicates expected future wealth and σ_w the predicted standard deviation of the possible divergence of actual future wealth from E_w.

Investors are assumed to prefer a higher expected future wealth to a lower value, ceteris paribus ($dU/dE_w > 0$). Moreover, they exhibit risk-aversion, choosing an investment offering a lower value of σ_w to one with a greater level, given the level of E_w ($dU/d\sigma_w < 0$). These assumptions imply that indifference curves relating E_w and σ_w will be upward-sloping.[9]

To simplify the analysis, we assume that an investor has decided to commit a given amount (W_i) of his present wealth to investment. Letting W_t be his terminal wealth and R the rate of return on his investment:

$$R \equiv \frac{W_t - W_i}{W_i},$$

we have

$$W_t = R\,W_i + W_i.$$

This relationship makes it possible to express the investor's utility in terms of R, since terminal wealth is directly related to the rate of return:

$$U = g(E_R, \sigma_R).$$

Figure 2 summarizes the model of investor preferences in a family of indifference curves; successive curves indicate higher levels of utility as one moves down and/or to the right.[10]

8. Under certain conditions the mean-variance approach can be shown to lead to unsatisfactory predictions of behavior. Markowitz suggests that a model based on the semi-variance (the average of the squared deviations below the mean) would be preferable; in light of the formidable computational problems, however, he bases his analysis on the variance and standard deviation.

9. While only these characteristics are required for the analysis, it is generally assumed that the curves have the property of diminishing marginal rates of substitution between E_w and σ_w, as do those in our diagrams.

10. Such indifference curves can also be derived by assuming that the investor wishes to maximize expected utility and that his total utility can be represented by a quadratic function of R with decreasing marginal utility. Both Markowitz and Tobin present such a derivation. A similar approach is used by Donald E. Farrar in *The Investment Decision Under Uncertainty* (Prentice-Hall, 1962). Unfortunately Farrar makes an error in his derivation; he appeals to the Von-Neumann-Morgenstern cardinal utility axioms to transform a function of the form:

$$E(U) = a + bE_R - cE_R^2 - c\sigma_R^2$$

into one of the form:

$$E(U) = k_1E_R - k_2\sigma_R^2.$$

That such a transformation is not consistent with the axioms can readily be seen in this form, since the first equation implies non-linear indifference curves in the E_R, σ_R^2 plane while the second implies a linear relationship. Obviously no three (different) points can lie on both a line and a non-linear curve (with a monotonic derivative). Thus the two functions must imply different orderings among alternative choices in at least some instance.

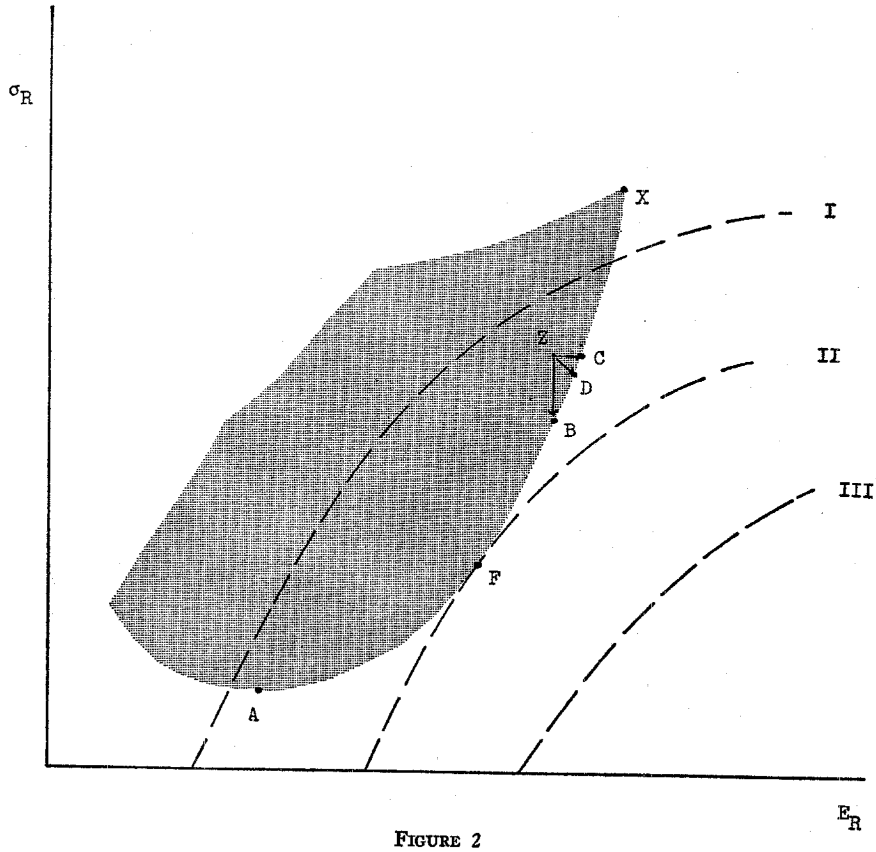

FIGURE 2

The Investment Opportunity Curve

The model of investor behavior considers the investor as choosing from a set of investment opportunities that one which maximizes his utility. Every investment plan available to him may be represented by a point in the E_R, σ_R plane. If all such plans involve some risk, the area composed of such points will have an appearance similar to that shown in Figure 2. The investor will choose from among all possible plans the one placing him on the indifference curve representing the highest level of utility (point F). The decision can be made in two stages: first, find the set of efficient investment plans and, second choose one from among this set. A plan is said to be efficient if (and only if) there is no alternative with either (1) the same E_R and a lower σ_R, (2) the same σ_R and a higher E_R or (3) a higher E_R and a lower σ_R. Thus investment Z is inefficient since investments B, C, and D (among others) dominate it. The only plans which would be chosen must lie along the lower right-hand boundary (AFBDCX)—the *investment opportunity curve.*

To understand the nature of this curve, consider two investment plans —A and B, each including one or more assets. Their predicted expected values and standard deviations of rate of return are shown in Figure 3.

If the proportion α of the individual's wealth is placed in plan A and the remainder (1-α) in B, the expected rate of return of the combination will lie between the expected returns of the two plans:

$$E_{Rc} = \alpha E_{Ra} + (1 - \alpha) E_{Rb}$$

The predicted standard deviation of return of the combination is:

$$\sigma_{Rc} = \sqrt{\alpha^2 \sigma_{Ra}{}^2 + (1 - \alpha)^2 \sigma_{Rb}{}^2 + 2r_{ab}\, \alpha(1 - \alpha)\, \sigma_{Ra}\sigma_{Rb}}$$

Note that this relationship includes r_{ab}, the correlation coefficient between the predicted rates of return of the two investment plans. A value of +1 would indicate an investor's belief that there is a precise positive relationship between the outcomes of the two investments. A zero value would indicate a belief that the outcomes of the two investments are completely independent and —1 that the investor feels that there is a precise inverse relationship between them. In the usual case r_{ab} will have a value between 0 and +1.

Figure 3 shows the possible values of E_{Rc} and σ_{Rc} obtainable with different combinations of A and B under two different assumptions about

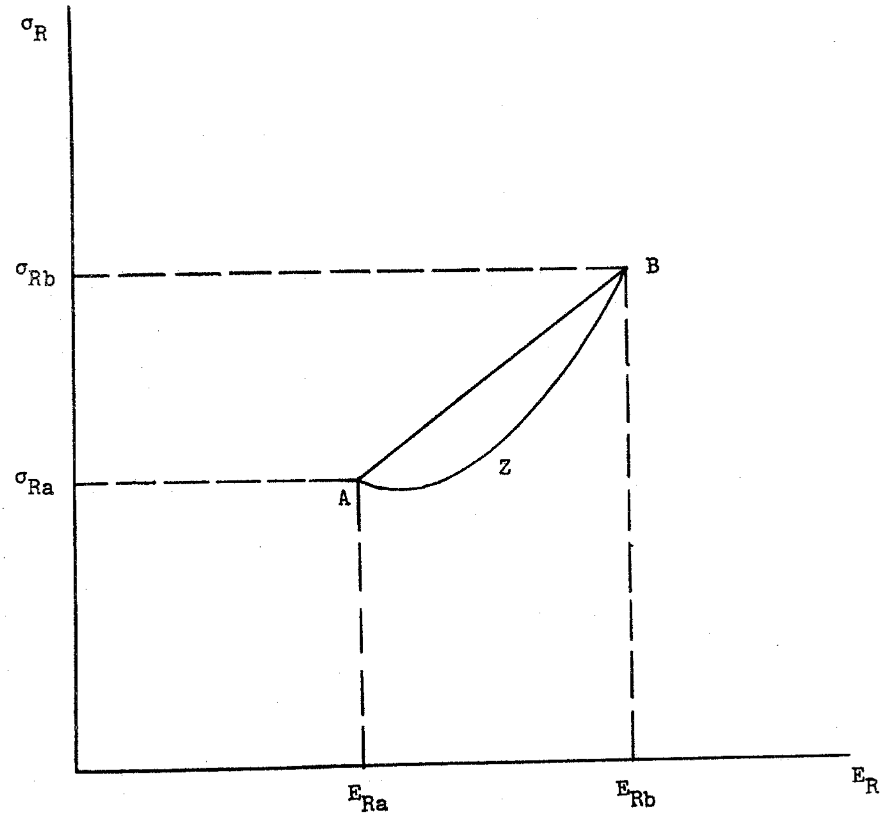

FIGURE 3

the value of r_{ab}. If the two investments are perfectly correlated, the combinations will lie along a straight line between the two points, since in this case both E_{Rc} and σ_{Rc} will be linearly related to the proportions invested in the two plans.[11] If they are less than perfectly positively correlated, the standard deviation of any combination must be less than that obtained with perfect correlation (since r_{ab} will be less); thus the combinations must lie along a curve below the line AB.[12] AZB shows such a curve for the case of complete independence ($r_{ab} = 0$); with negative correlation the locus is even more U-shaped.[13]

The manner in which the investment opportunity curve is formed is relatively simple conceptually, although exact solutions are usually quite difficult.[14] One first traces curves indicating E_R, σ_R values available with simple combinations of individual assets, then considers combinations of combinations of assets. The lower right-hand boundary must be either linear or increasing at an increasing rate ($d^2\, \sigma_R/dE^2_R > 0$). As suggested earlier, the complexity of the relationship between the characteristics of individual assets and the location of the investment opportunity curve makes it difficult to provide a simple rule for assessing the desirability of individual assets, since the effect of an asset on an investor's over-all investment opportunity curve depends not only on its expected rate of return (E_{Ri}) and risk (σ_{Ri}), but also on its correlations with the other available opportunities ($r_{i1}, r_{i2}, \ldots, r_{in}$). However, such a rule is implied by the equilibrium conditions for the model, as we will show in part IV.

The Pure Rate of Interest

We have not yet dealt with riskless assets. Let P be such an asset; its risk is zero ($\sigma_{Rp} = 0$) and its expected rate of return, E_{Rp}, is equal (by definition) to the pure interest rate. If an investor places α of his wealth

11.
$$E_{Rc} = \alpha E_{Ra} + (1-\alpha)\, E_{R_b} = E_{Rb} + (E_{R_a} - E_{R_b})\, \alpha$$
$$\sigma_{R_c} = \sqrt{\alpha^2 \sigma_{R_a}{}^2 + (1-\alpha)^2\, \sigma_{Rb}{}^2 + 2r_{ab}\, \alpha(1-\alpha)\, \sigma_{Ra}\, \sigma_{Rb}}$$
but $r_{ab} = 1$, therefore the expression under the square root sign can be factored:
$$\begin{aligned}\sigma_{Rc} &= \sqrt{[\alpha\sigma_{Ra} + (1-\alpha)\, \sigma_{R_b}]^2} \\ &= \alpha\, \sigma_{Ra} + (1-\alpha)\, \sigma_{Rb} \\ &= \sigma_{Rb} + (\sigma_{Ra} - \sigma_{Rb})\, \alpha\end{aligned}$$

12. This curvature is, in essence, the rationale for diversification.

13. When $r_{ab} = 0$, the slope of the curve at point A is $-\dfrac{\sigma_{Ra}}{E_{Rb} - E_{Ra}}$, at point B it is $\dfrac{\sigma_{Rb}}{E_{Rb} - E_{Ra}}$. When $r_{ab} = -1$, the curve degenerates to two straight lines to a point on the horizontal axis.

14. Markowitz has shown that this is a problem in parametric quadratic programming. An efficient solution technique is described in his article, "The Optimization of a Quadratic Function Subject to Linear Constraints," *Naval Research Logistics Quarterly*, Vol. 3 (March and June, 1956), 111-133. A solution method for a special case is given in the author's "A Simplified Model for Portfolio Analysis," *op. cit.*