教案：关于微分同胚的应用

课程：《数学分析（Ⅱ）》（一年制，面对力学类等）

1. 知识点（教学内容及其目标概述）

本知识点：基于微分同胚，将几何形态不规则的物理区域上的控制方程转化至几何形态规则的参数域上的控制方程。

2. 知识要素（教学内容细致目录）

① 微分同胚的充分必要性条件。实际应用中常用充分性。

$f \;\; x \in C^p \;\; \mathcal{D}_x(\subset \mathbb{R}^m);\mathbb{R}^m$

s. t. 1) $f(x)$ 在 $\mathcal{D}_x$ 上为单射，则有 $f(x) \in C^p \;\; \mathcal{D}_x; f(\mathcal{D}_x)$

2) $Df(x) \in \mathbb{R}^{m\times m}$ 非奇异

注：此时，$f(\mathcal{D}_x) \subset \mathbb{R}^m$ 为开集；$f(x)$ 实现 $\mathcal{D}_x$ 同 $f(\mathcal{D}_x)$ 之间的双射；$f^{-1}(y) \in C^1 \;\; f(\mathcal{D}_x);\mathbb{R}^m$

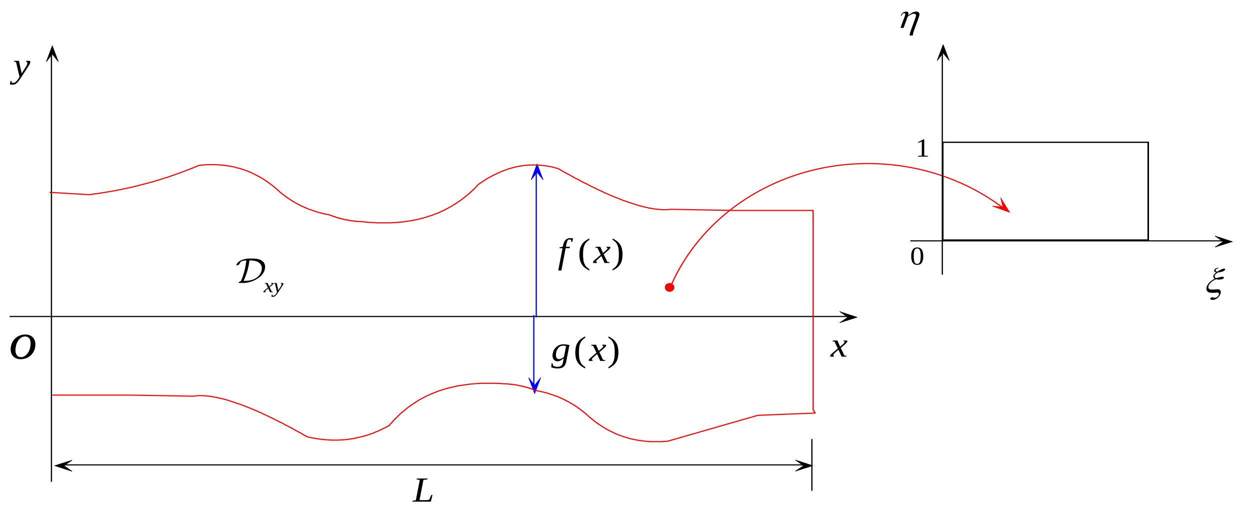
② 由 $f^{-1} \circ f(x) = x, \forall x \in \mathcal{D}_x$，由于 $f^{-1}(y)$ 以及 $f(x)$ 均可微，按复合映照的可微性定理，有：

$Df^{-1} \;\; f(x) \;\; Df(x) = I_m$，故有：$Df^{-1} \;\; f(x) \;\; = \;\; Df(x)^{\;-1}$，

亦即：$\dfrac{D \;\; x^1,\ldots,x^m}{D \;\; y^1,\ldots,y^m} \;\; f(x) \;\; = \left[\dfrac{D \;\; y^1,\ldots,y^m}{D \;\; x^1,\ldots,x^m}\right]^{-1}$

3. 应用事例

事例 1：非规则平面管流



Step1：建立C^p微分同胚，将几何“不规则”的物理区域变换为几何“规则”的参数域。

作： $\begin{bmatrix} x \\ y \end{bmatrix}(\xi,\eta):\mathcal{D}_{\xi,\eta}\ni\begin{bmatrix} \xi \\ \eta \end{bmatrix}\mapsto\begin{bmatrix} x \\ y \end{bmatrix}(\xi,\eta)=\begin{bmatrix} \xi \\ g(\xi)+\eta\left(f(\xi)-g(\xi)\right) \end{bmatrix}$，需验证 $\begin{bmatrix} x \\ y \end{bmatrix}(\xi,\eta)\in C^p\left(\mathcal{D}_{\xi\eta};\mathcal{D}_{xy}\right)$。

当 $f(\xi),g(\xi)\in C^p(0,L)$，则有$\begin{bmatrix} x \\ y \end{bmatrix}(\xi,\eta)\in C^p\left(\mathcal{D}_{\xi\eta};\mathbb{R}^2\right)$

① 易见$\begin{bmatrix} x \\ y \end{bmatrix}$ ξ,η 在$\mathcal{D}_{\xi\eta}$上为单射（结合几何特点）

② $D\begin{bmatrix} x \\ y \end{bmatrix}(\xi,\eta)=\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}(\xi,\eta)=\begin{bmatrix} 1 & 0 \\ \dot{g}(\xi)+\eta\left(\dot{f}(\xi)-\dot{g}(\xi)\right) & f(\xi)-g(\xi) \end{bmatrix}$

有$\det D\begin{bmatrix} x \\ y \end{bmatrix}$ ξ,η $=f(\xi)-g(\xi)\neq 0,\forall\begin{bmatrix} \xi \\ \eta \end{bmatrix}\in\mathcal{D}_{\xi\eta}$，故有：$\begin{bmatrix} x \\ y \end{bmatrix}$ ξ,η $\in C^p\left(\mathcal{D}_{\xi\eta};\mathcal{D}_{xy}=\begin{bmatrix} x \\ y \end{bmatrix}\ \mathcal{D}_{\xi\eta}\right)$。

Step2：获得参数域上的控制方程

设有$\psi(x,y)$定义于 $\mathcal{D}_{xy}$，s. t. $\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}(x,y)+\frac{\partial^2\psi}{\partial x\partial y}(x,y)=0$

① 由于$\begin{bmatrix} \xi \\ \eta \end{bmatrix}\leftrightarrow\begin{bmatrix} x \\ y \end{bmatrix}$之间为双射，则有$\psi(\begin{bmatrix} x \\ y \end{bmatrix})=\psi\left(\begin{bmatrix} x \\ y \end{bmatrix}(\begin{bmatrix} \xi \\ \eta \end{bmatrix})\right)=:\hat{\psi}(\begin{bmatrix} \xi \\ \eta \end{bmatrix})=\hat{\psi}\left(\begin{bmatrix} \xi \\ \eta \end{bmatrix}(\begin{bmatrix} x \\ y \end{bmatrix})\right)$。由此

我们对应有参数域$\mathcal{D}_{\xi\eta}$上的函数$\hat{\psi}(\xi,\eta)$

② 获得$\hat{\psi}(\xi,\eta),\forall\begin{bmatrix}\xi\\ \eta\end{bmatrix}\in\mathcal{D}_{\xi\eta}$的控制方程——利用链式求导法则

利用关系式$\psi(x,y)=\hat{\psi}\left(\xi(x,y),\eta(x,y)\right)$，则有

$$\begin{cases}\dfrac{\partial\psi}{\partial x}(x,y)=\dfrac{\partial\hat{\psi}}{\partial\xi}(\xi,\eta)\dfrac{\partial\xi}{\partial x}(x,y)+\dfrac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\dfrac{\partial\eta}{\partial x}(x,y)\\ \dfrac{\partial\psi}{\partial y}(x,y)=\dfrac{\partial\hat{\psi}}{\partial\xi}(\xi,\eta)\dfrac{\partial\xi}{\partial y}(x,y)+\dfrac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\dfrac{\partial\eta}{\partial y}(x,y)\end{cases}$$

由

$$\begin{bmatrix}\dfrac{\partial\xi}{\partial x} & \dfrac{\partial\xi}{\partial y}\\ \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\eta}{\partial y}\end{bmatrix}(x,y)=\begin{bmatrix}\dfrac{\partial x}{\partial\xi} & \dfrac{\partial x}{\partial\eta}\\ \dfrac{\partial y}{\partial\xi} & \dfrac{\partial y}{\partial\eta}\end{bmatrix}^{-1}(\xi,\eta)=\begin{bmatrix}1 & 0\\ \dot{g}(\xi)+\eta\left(\dot{f}(\xi)-\dot{g}(\xi)\right) & f(\xi)-g(\xi)\end{bmatrix}^{-1}$$

$$=\frac{1}{f-g}\begin{bmatrix}f-g & -\dot{g}-\eta\left(\dot{f}-\dot{g}\right)\\ 0 & 1\end{bmatrix}^{\mathrm{T}}=\frac{1}{f-g}\begin{bmatrix}f-g & 0\\ \eta\left(\dot{g}-\dot{f}\right)-\dot{g} & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ \dfrac{\eta\left(\dot{g}-\dot{f}\right)-\dot{g}}{f-g} & \dfrac{1}{f-g}\end{bmatrix}$$

故有：

$$\begin{cases}\dfrac{\partial\psi}{\partial x}(x,y)=\dfrac{\partial\hat{\psi}}{\partial\xi}(\xi,\eta)+\dfrac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\dfrac{\eta\left(\dot{g}-\dot{f}\right)(\xi)-\dot{g}(\xi)}{(f-g)(\xi)}\\ \dfrac{\partial\psi}{\partial y}(x,y)=\dfrac{\partial\hat{\psi}}{\partial\xi}(\xi,\eta)\cdot 0+\dfrac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\dfrac{1}{(f-g)(\xi)}=\dfrac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\dfrac{1}{(f-g)(\xi)}\end{cases}$$

进一步计算

$$\frac{\partial^2\psi}{\partial x\partial y}(x,y)\equiv\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)(x,y)=\overline{\frac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)}^{x}\cdot\frac{1}{(f-g)(\xi)}+\frac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\overline{\frac{1}{(f-g)(\xi)}}^{x}$$

$$=\left[\frac{\partial^2\hat{\psi}}{\partial\xi\partial\eta}(\xi,\eta)\frac{\partial\xi}{\partial x}(x,y)+\frac{\partial^2\hat{\psi}}{\partial\eta^2}(\xi,\eta)\frac{\partial\eta}{\partial x}(x,y)\right]\frac{1}{(f-g)(\xi)}$$

$$-\frac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\frac{1}{(f-g)^2(\xi)}\left(\frac{df}{d\xi}-\frac{dg}{d\xi}\right)(\xi)\frac{\partial\xi}{\partial x}(x,y)$$

$$=\left[\frac{\partial^2\hat{\psi}}{\partial\xi\partial\eta}(\xi,\eta)+\frac{\partial^2\hat{\psi}}{\partial\eta^2}(\xi,\eta)\frac{\eta\left(\dot{g}-\dot{f}\right)(\xi)-\dot{g}(\xi)}{(f-g)(\xi)}\right]\frac{1}{(f-g)(\xi)}$$

$$-\frac{\partial\hat{\psi}}{\partial\eta}(\xi,\eta)\frac{\left(\dot{f}-\dot{g}\right)(\xi)}{(f-g)^2(\xi)}$$

将上述表达式代入$\psi(x,y)$在$\mathcal{D}_{xy}$的 PDE：$\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}(x,y)+\frac{\partial^2\psi}{\partial x\partial y}(x,y)=0$，即可得$\hat{\psi}(\xi,\eta)$在$\mathcal{D}_{\xi\eta}$的 PDE。

$\mathcal{D}_{\xi\eta}\subset\mathbb{R}^2$几何形态规则，则便于数值求解$\hat{\psi}(\xi,\eta)$，当获得$\hat{\psi}(\xi,\eta)$，则有$\psi(x,y)=\hat{\psi}\left(\begin{bmatrix}\xi\\\eta\end{bmatrix}(x,y)\right)$。

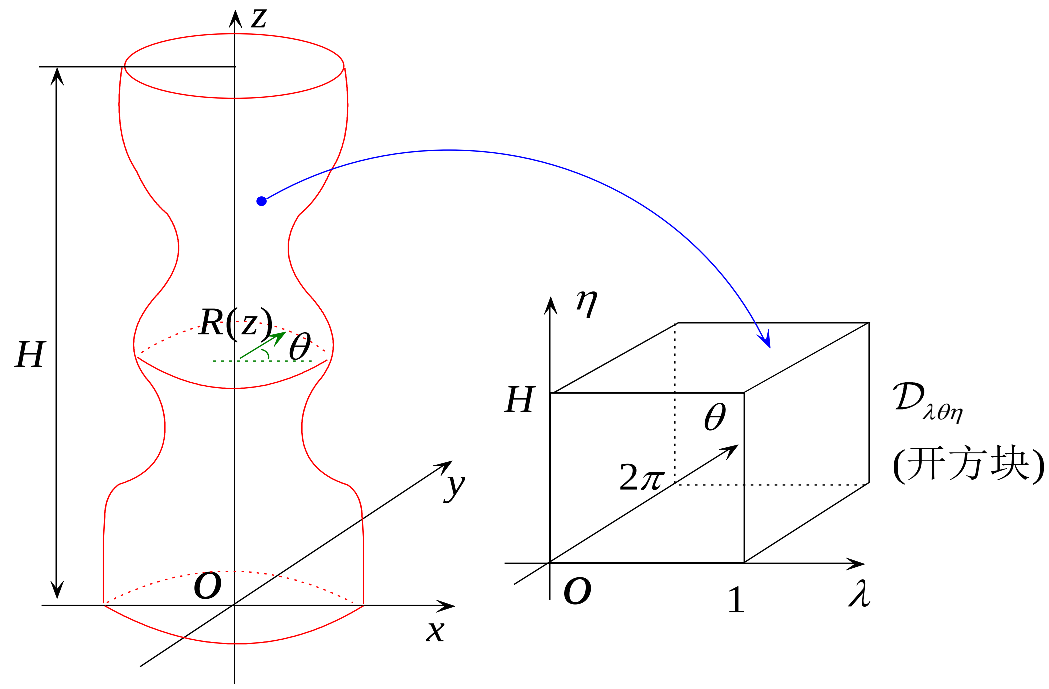
注：一般我们认为$\psi(x,y)\in C^p(\mathcal{D}_{xy};\mathbb{R}^2)$，故可有$\frac{\partial^2\psi}{\partial x\partial y}(x,y)=\frac{\partial^2\psi}{\partial y\partial x}(x,y),\forall\begin{bmatrix}x\\y\end{bmatrix}\in\mathcal{D}_{xy}$，故可对$\frac{\partial\psi}{\partial x}(x,y)$或$\frac{\partial\psi}{\partial y}(x,y)$可选择形式简单的一个，而计算$\frac{\partial^2\psi}{\partial y\partial x}(x,y)$或$\frac{\partial^2\psi}{\partial x\partial y}(x,y)$。

事例 2：轴对称非规则圆管内的流动

Step1. 建立C^p-diffemorphism

$$\begin{bmatrix}x\\y\\z\end{bmatrix}\left(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}\right):\mathcal{D}_{\lambda\theta\eta}\ni\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}\mapsto\begin{bmatrix}x\\y\\z\end{bmatrix}\left(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}\right)\triangleq\begin{bmatrix}\lambda R(\eta)\cos\theta\\\lambda R(\eta)\sin\theta\\\eta\end{bmatrix}\in\mathbb{R}^3$$

此处$\mathcal{D}_{\lambda\theta\eta}$为开方块。



① 显见 $\begin{bmatrix}x\\y\\z\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})$ 为 $\mathcal{D}_{\lambda\theta\eta}\subset\mathbb{R}^3$ 的单射。

② $$D\begin{bmatrix}x\\y\\z\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})=\begin{bmatrix}\frac{\partial x}{\partial\lambda}&\frac{\partial x}{\partial\theta}&\frac{\partial x}{\partial\eta}\\\frac{\partial y}{\partial\lambda}&\frac{\partial y}{\partial\theta}&\frac{\partial y}{\partial\eta}\\\frac{\partial z}{\partial\lambda}&\frac{\partial z}{\partial\theta}&\frac{\partial z}{\partial\eta}\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})=\begin{bmatrix}R(\eta)\cos\theta&-\lambda R(\eta)\sin\theta&\lambda\dot{R}(\eta)\cos\theta\\R(\eta)\sin\theta&\lambda R(\eta)\cos\theta&\lambda\dot{R}(\eta)\sin\theta\\0&0&1\end{bmatrix}$$

有 $\det D\begin{bmatrix}x\\y\\y\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})=\det\begin{bmatrix}R(\eta)\cos\theta&-\lambda R(\eta)\sin\theta\\R(\eta)\sin\theta&\lambda R(\eta)\cos\theta\end{bmatrix}=\lambda R^2(\eta)\neq0,\ \forall\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}\in\mathcal{D}_{\lambda\theta\eta}$

故有 $\begin{bmatrix}x\\y\\y\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})\in C^p\left(\mathcal{D}_{\lambda\theta\eta};\mathcal{D}_{xyz}\triangleq\begin{bmatrix}x\\y\\y\end{bmatrix}(\mathcal{D}_{\lambda\theta\eta})\right)$，此处需具体澄清 $\mathcal{D}_{xyz}$ 的区域，未包含整个管道内部。

Step2. 获得参数域上的PDE

① 由 $f(\begin{bmatrix}x\\y\\y\end{bmatrix})=f(\begin{bmatrix}x\\y\\y\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}))=\hat{f}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}(\begin{bmatrix}x\\y\\y\end{bmatrix}))$

② 由关系式 $f(\begin{bmatrix}x\\y\\y\end{bmatrix})=\hat{f}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}(\begin{bmatrix}x\\y\\y\end{bmatrix}))$ 按复合映照可微性定理，有 $Df(\begin{bmatrix}x\\y\\y\end{bmatrix})=D\hat{f}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})\cdot D\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}(\begin{bmatrix}x\\y\\y\end{bmatrix})$

考虑到 $D\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix}(\begin{bmatrix}x\\y\\y\end{bmatrix})=\left[D\begin{bmatrix}x\\y\\y\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})\right]^{-}$，则有 $Df(\begin{bmatrix}x\\y\\y\end{bmatrix})=D\hat{f}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})\cdot\left[D\begin{bmatrix}x\\y\\y\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})\right]^{-}$

亦即 $\begin{bmatrix}\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}&\frac{\partial f}{\partial z}\end{bmatrix}(x,y,z)=\begin{bmatrix}\frac{\partial\hat{f}}{\partial\lambda}&\frac{\partial\hat{f}}{\partial\theta}&\frac{\partial\hat{f}}{\partial\eta}\end{bmatrix}(\lambda,\theta,\eta)\cdot\begin{bmatrix}R(\eta)\cos\theta&-\lambda R(\eta)\sin\theta&\lambda\dot{R}(\eta)\cos\theta\\R(\eta)\sin\theta&\lambda R(\eta)\cos\theta&\lambda\dot{R}(\eta)\sin\theta\\0&0&1\end{bmatrix}^{-}$

可有 $$\begin{bmatrix}R(\eta)\cos\theta&-\lambda R(\eta)\sin\theta&\lambda\dot{R}(\eta)\cos\theta\\R(\eta)\sin\theta&\lambda R(\eta)\cos\theta&\lambda\dot{R}(\eta)\sin\theta\\0&0&1\end{bmatrix}^{-}=\frac{1}{\lambda R^2}\cdot\begin{bmatrix}\lambda R(\eta)\cos\theta&-R(\eta)\sin\theta&0\\\lambda R(\eta)\sin\theta&R(\eta)\cos\theta&\\-\lambda^2R\dot{R}&0&\lambda R^2\end{bmatrix}^T$$

$$=\frac{1}{\lambda R^2}\cdot\begin{bmatrix}\lambda R(\eta)\cos\theta&\lambda R(\eta)\sin\theta&-\lambda^2R\dot{R}\\-R(\eta)\sin\theta&R(\eta)\cos\theta&0\\0&0&\lambda R^2\end{bmatrix}$$

即有 $\begin{bmatrix} \frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & \frac{\partial\lambda}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} & \frac{\partial\eta}{\partial z} \end{bmatrix}(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \frac{1}{\lambda R^2}?\begin{bmatrix} \lambda R(\eta)\cos\theta & \lambda R(\eta)\sin\theta & -\lambda^2 R\dot{R} \\ -R\ \eta \quad \theta & R\ \eta \quad \theta & \\ 0 & 0 & \lambda R^2 \end{bmatrix}$,

以及

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}(x,y,z) = \begin{bmatrix} \frac{\partial \hat{f}}{\partial \lambda} & \frac{\partial \hat{f}}{\partial \theta} & \frac{\partial \hat{f}}{\partial \eta} \end{bmatrix}(\lambda,\theta,\eta)\cdot\begin{bmatrix} \frac{\cos\theta}{R(\eta)} & \frac{\sin\theta}{R(\eta)} & \frac{-\lambda\dot{R}(\eta)}{R(\eta)} \\ -\frac{\sin\theta}{\lambda R(\eta)} & \frac{\cos\theta}{\lambda R(\eta)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

即 $\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = \frac{\partial \hat{f}}{\partial \lambda}(\lambda,\theta,\eta)\frac{\cos\theta}{R(\eta)} - \frac{\partial \hat{f}}{\partial \theta}\frac{\sin\theta}{\lambda R(\eta)} \\ \frac{\partial f}{\partial y}(x,y,z) = \frac{\partial \hat{f}}{\partial \lambda}(\lambda,\theta,\eta)\frac{\sin\theta}{R(\eta)} + \frac{\partial \hat{f}}{\partial \theta}\frac{\cos\theta}{\lambda R(\eta)} \\ \frac{\partial f}{\partial z}(x,y,z) = \frac{\partial \hat{f}}{\partial \lambda}(\lambda,\theta,\eta)\frac{-\lambda\dot{R}(\eta)}{R(\eta)} + \frac{\partial \hat{f}}{\partial \theta} \end{cases}$

可再计算

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x,y,z) &= \left[\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\partial\lambda}{\partial x} + \frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\partial\theta}{\partial x} + \frac{\partial^2 \hat{f}}{\partial\eta\partial\lambda}\frac{\partial\eta}{\partial x}\right]?\frac{\cos\theta}{R} + \frac{\partial \hat{f}}{\partial\lambda}\left[\cdot - \frac{\sin\theta}{R}\frac{\partial\theta}{\partial x} - \frac{\sin\theta}{R^2}\dot{R}(\eta)\frac{\partial\eta}{\partial x}\right] \\ &\quad - \left[\frac{\partial^2 \hat{f}}{\partial\lambda\partial\theta}\frac{\partial\lambda}{\partial x} + \frac{\partial^2 \hat{f}}{\partial\theta^2}\frac{\partial\theta}{\partial x} + \frac{\partial^2 \hat{f}}{\partial\eta\partial\theta}\frac{\partial\eta}{\partial x}\right]\cdot\frac{\sin\theta}{\lambda R} - \frac{\partial \hat{f}}{\partial\theta}\left[\cdot - \frac{\sin\theta}{\lambda^2 R}\frac{\partial\lambda}{\partial x} - \frac{\cos\theta}{\lambda R}\frac{\partial\theta}{\partial x} - \frac{\sin\theta}{\lambda R^2}\dot{R}(\eta)\frac{\partial\eta}{\partial x}\right] \\ &= \left[\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\cos\theta}{R} - \frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\sin\theta}{\lambda R}\right]\cdot\frac{\cos\theta}{R} + \frac{\partial \hat{f}}{\partial\lambda}\frac{\sin^2\theta}{\lambda R^2} - \left[\frac{\partial^2 \hat{f}}{\partial\lambda\partial\theta}\frac{\cos\theta}{R} - \frac{\partial^2 \hat{f}}{\partial\theta^2}\frac{\sin\theta}{\lambda R}\right]\cdot\frac{\sin\theta}{\lambda R} - \frac{\partial \hat{f}}{\partial\theta}\left[\cdot -2\frac{\sin\theta\cos\theta}{\lambda^2 R^2}\right] \end{aligned}$$

即有： $\frac{\partial^2 f}{\partial x^2}(x,y,z) = \frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\cos^2\theta}{R^2} + \frac{\partial^2 \hat{f}}{\partial\theta^2}\frac{\sin^2\theta}{\lambda^2 R^2} - \frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\sin 2\theta}{\lambda R^2} + \frac{\partial \hat{f}}{\partial\lambda}\frac{\sin^2\theta}{\lambda R^2} + \frac{\partial \hat{f}}{\partial\theta}\frac{\sin 2\theta}{\lambda^2 R^2}$

同样计算

$$\frac{\partial^2 f}{\partial y^2}(x,y,z)=\left[\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\partial\lambda}{\partial y}+\frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\partial\theta}{\partial y}+\frac{\partial^2 \hat{f}}{\partial\eta\partial\lambda}\frac{\partial\eta}{\partial y}\right]\cdot\frac{\sin\theta}{R}+\frac{\partial\hat{f}}{\partial\lambda}\cdot\left[\frac{\cos\theta}{R}\frac{\partial\theta}{\partial y}-\frac{\sin\theta}{R^2}\dot{R}(\eta)\frac{\partial\eta}{\partial y}\right]$$

$$+\left[\frac{\partial^2 \hat{f}}{\partial\lambda\partial\theta}\frac{\partial\lambda}{\partial y}+\frac{\partial^2 \hat{f}}{\partial\theta^2}\frac{\partial\theta}{\partial y}+\frac{\partial^2 \hat{f}}{\partial\eta\partial\theta}\frac{\partial\eta}{\partial y}\right]\cdot\frac{\cos\theta}{\lambda R}+\frac{\partial\hat{f}}{\partial\theta}\left[-\frac{\sin\theta}{\lambda^2 R}\frac{\partial\lambda}{\partial y}-\frac{\sin\theta}{\lambda R}\frac{\partial\theta}{\partial y}-\frac{\sin\theta}{\lambda R^2}\dot{R}(\eta)\frac{\partial\eta}{\partial y}\right]$$

$$=\left[\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\sin\theta}{R}+\frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\cos\theta}{\lambda R}\right]\cdot\frac{\sin\theta}{R}+\frac{\partial\hat{f}}{\partial\lambda}\frac{\cos^2\theta}{\lambda R^2}+\left[\frac{\partial^2 \hat{f}}{\partial\lambda\partial\theta}\frac{\sin\theta}{R}+\frac{\partial^2 \hat{f}}{\partial\theta^2}\frac{\cos\theta}{\lambda R}\right]\cdot\frac{\cos\theta}{\lambda R}+\frac{\partial\hat{f}}{\partial\theta}\left[-2\frac{\sin\theta\cos\theta}{\lambda^2 R^2}\right]$$

$$=\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\sin^2\theta}{R^2}+\frac{\partial^2 \hat{f}}{\partial\theta^2}\frac{\cos^2\theta}{\lambda^2 R^2}+\frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\sin 2\theta}{\lambda R^2}+\frac{\partial\hat{f}}{\partial\lambda}\frac{\cos^2\theta}{\lambda R^2}-\frac{\partial\hat{f}}{\partial\theta}\frac{\sin 2\theta}{\lambda^2 R^2}$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z)=-\left[\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\partial\lambda}{\partial z}+\frac{\partial^2 \hat{f}}{\partial\theta\partial\lambda}\frac{\partial\theta}{\partial z}+\frac{\partial^2 \hat{f}}{\partial\eta\partial\lambda}\frac{\partial\eta}{\partial z}\right]\frac{\lambda\dot{R}(\eta)}{R(\eta)}-\frac{\partial\hat{f}}{\partial\lambda}\cdot\left[\frac{\dot{R}}{R}\frac{\partial\lambda}{\partial z}+\lambda\frac{\ddot{R}R-\dot{R}^2}{R^2}\frac{\partial\eta}{\partial z}\right]$$

$$+\left[\frac{\partial^2 \hat{f}}{\partial\lambda\partial\eta}\frac{\partial\lambda}{\partial z}+\frac{\partial^2 \hat{f}}{\partial\theta\partial\eta}\frac{\partial\theta}{\partial z}+\frac{\partial^2 \hat{f}}{\partial\eta^2}\frac{\partial\eta}{\partial y}\right]$$

$$=-\left[-\frac{\partial^2 \hat{f}}{\partial\lambda^2}\frac{\lambda\dot{R}}{R}+\frac{\partial^2 \hat{f}}{\partial\eta\partial\lambda}\right]\frac{\lambda\dot{R}}{R}-\frac{\partial\hat{f}}{\partial\lambda}\cdot\left[-\frac{\lambda\dot{R}^2}{R^2}+\lambda\frac{\ddot{R}R-\dot{R}^2}{R^2}\right]+\left[-\frac{\partial^2 \hat{f}}{\partial\lambda\partial\eta}\frac{\lambda\dot{R}}{R}+\frac{\partial^2 \hat{f}}{\partial\eta^2}\right]$$

$$=\frac{\lambda^2\dot{R}^2}{R^2}\frac{\partial^2 \hat{f}}{\partial\lambda^2}+\frac{\partial^2 \hat{f}}{\partial\eta^2}-\frac{2\lambda\dot{R}}{R}\frac{\partial^2 \hat{f}}{\partial\eta\partial\lambda}-\frac{\partial\hat{f}}{\partial\lambda}\frac{\lambda\ddot{R}R-2\lambda\dot{R}^2}{R^2}$$

综上有：对 $\Delta f(x,y,z)\equiv\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)(x,y,z)=0$

$$\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)(x,y,z)=\frac{1+\lambda\dot{R}^2(\eta)}{R^2(\eta)}\frac{\partial^2 \hat{f}}{\partial\lambda^2}(\lambda,\theta,\eta)+\frac{1}{\lambda^2 R^2(\eta)}\frac{\partial^2 \hat{f}}{\partial\theta^2}(\lambda,\theta,\eta)$$

$$+\frac{\partial^2 \hat{f}}{\partial\eta^2}-\frac{2\lambda\dot{R}^2(\eta)}{R(\eta)}\frac{\partial^2 \hat{f}}{\partial\eta\partial\lambda}(\lambda,\theta,\eta)+\left(\frac{1}{\lambda R^2(\eta)}+\frac{2\lambda\dot{R}^2-\lambda\ddot{R}R}{R^2}\right)\frac{\partial\hat{f}}{\partial\lambda}(\lambda,\theta,\eta)=0$$

注：再考虑 $\begin{bmatrix}x\\y\\z\end{bmatrix}(\begin{bmatrix}\lambda\\\theta\\\eta\end{bmatrix})=\begin{bmatrix}\lambda R(\eta)\cos\theta\\\lambda R(\eta)\sin\theta\\\eta\end{bmatrix}\in\mathbb{R}^3$，当 $R(\eta)=1,\lambda\in(0,R),R=const$ 则

$\Delta f(x,y,z)=\frac{\partial^2 \hat{f}}{\partial\lambda^2}(\lambda,\theta,\eta)+\frac{1}{\lambda^2}\frac{\partial^2 \hat{f}}{\partial\theta^2}(\lambda,\theta,\eta)+\frac{\partial^2 \hat{f}}{\partial\eta^2}=0$，即对应一般柱坐标系。

事例 3：方程变换*

注：此种问题基本上一致，不过现已知微分同胚（或相应变换）

* 引自《数学分析习题集》林，方等

设有 $\begin{cases} x=\sqrt{vw} \\ y=\sqrt{uv} \\ z=\sqrt{uw} \end{cases}$，有 $f(x,y,z)=F(u,v,w)$，证：$xf_x'+yf_y'+zf_z'=uF_u'+vF_v'+wF_w'$

Step1 验证微分同胚

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}(\begin{bmatrix} u \\ v \\ w \end{bmatrix}) \triangleq \begin{bmatrix} \sqrt{vw} \\ \sqrt{uv} \\ \sqrt{wu} \end{bmatrix} \Rightarrow D\begin{bmatrix} x \\ y \\ z \end{bmatrix}(\begin{bmatrix} u \\ v \\ w \end{bmatrix}) = \begin{bmatrix} 0 & \frac{\sqrt{w}}{2\sqrt{v}} & \frac{\sqrt{v}}{2\sqrt{w}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} & 0 \\ \frac{\sqrt{w}}{2\sqrt{u}} & 0 & \frac{\sqrt{u}}{2\sqrt{w}} \end{bmatrix}$$

$$\Rightarrow \det D\begin{bmatrix} x \\ y \\ z \end{bmatrix}(\begin{bmatrix} u \\ v \\ w \end{bmatrix}) = -\frac{1}{4} \neq 0$$，故认为存在微分同胚

Step2 变换方程

有关系式 $f(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}(\begin{bmatrix} u \\ v \\ w \end{bmatrix})\right) =: \hat{f}\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}(\begin{bmatrix} x \\ y \\ z \end{bmatrix})\right)$

$$\Rightarrow Df(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \left[f_x, f_y, f_z \right](x,y,z) = \left[\hat{f}_u, \hat{f}_v, \hat{f}_w \right](u,v,w)\frac{D(u,v,w)}{D(x,y,z)}(x,y,z)$$

$$= \left[\hat{f}_u, \hat{f}_v, \hat{f}_w \right](u,v,w)\cdot(-4)\cdot\begin{bmatrix} \frac{u}{4\sqrt{vw}} & -\frac{\sqrt{v}}{4\sqrt{w}} & -\frac{\sqrt{w}}{4\sqrt{v}} \\ -\frac{\sqrt{u}}{4\sqrt{v}} & -\frac{\sqrt{v}}{4\sqrt{u}} & \frac{w}{4\sqrt{uv}} \\ -\frac{\sqrt{u}}{4\sqrt{w}} & \frac{v}{4\sqrt{uw}} & -\frac{\sqrt{w}}{4\sqrt{u}} \end{bmatrix}^{\mathrm{T}}$$

亦即有

$$\left[f_x, f_y, f_z \right](x,y,z) = \left[\hat{f}_u, \hat{f}_v, \hat{f}_w \right](u,v,w)\begin{bmatrix} -\frac{u}{\sqrt{vw}} & \frac{\sqrt{v}}{\sqrt{w}} & \frac{\sqrt{w}}{\sqrt{v}} \\ \frac{\sqrt{u}}{\sqrt{v}} & \frac{\sqrt{v}}{\sqrt{u}} & -\frac{w}{\sqrt{uv}} \\ \frac{\sqrt{u}}{\sqrt{w}} & -\frac{v}{\sqrt{uw}} & \frac{\sqrt{w}}{\sqrt{u}} \end{bmatrix}^{\mathrm{T}}$$

$$\Rightarrow\begin{cases} xf_x = -\hat{f}_u \cdot u + \hat{f}_v \cdot v + \hat{f}_w \cdot w \\ yf_y = \hat{f}_u \cdot u + \hat{f}_v \cdot v - \hat{f}_w \cdot w \\ zf_z = \hat{f}_u \cdot u - \hat{f}_v \cdot v + \hat{f}_w \cdot w \end{cases} \Rightarrow xf_x + yf_y + zf_z = u\hat{f}_u + v\hat{f}_v + w\hat{f}_w$$

事例 4：引入 $x = e^{\xi}, y = e^{\eta}$ 变换方程：$\left(ax^2 \frac{\partial^2 u}{\partial x^2} + 2bxy \frac{\partial^2 u}{\partial x \partial y} + cy^2 \frac{\partial^2 u}{\partial y^2}\right)(x, y) = 0$

① 验证微分同胚：$\begin{bmatrix} x \\ y \end{bmatrix}(\begin{bmatrix} \xi \\ \eta \end{bmatrix}) = \begin{bmatrix} e^{\xi} \\ e^{\eta} \end{bmatrix} \Rightarrow \frac{D(x,y)}{D(\xi,\eta)}(\xi,\eta) = \begin{bmatrix} e^{\xi} & 0 \\ 0 & e^{\eta} \end{bmatrix}$

有 $\det \frac{D(x,y)}{D(\xi,\eta)}(\xi,\eta) = e^{\xi+\eta} \neq 0$

② $u(\begin{bmatrix} x \\ y \end{bmatrix}) = u\left(\begin{bmatrix} x \\ y \end{bmatrix}(\begin{bmatrix} \xi \\ \eta \end{bmatrix})\right) =: \hat{u}\left(\begin{bmatrix} \xi \\ \eta \end{bmatrix}(\begin{bmatrix} x \\ y \end{bmatrix})\right)$

有 $Du(x,y) = D\hat{u}(\xi,\eta)\frac{D(\xi,\eta)}{D(x,y)}(x,y) = D\hat{u}(\xi,\eta)\left[\frac{D(x,y)}{D(\xi,\eta)}\right]^{-1}(\xi,\eta)$

亦即有 $\left[u_x, u_y\right] = \left[\hat{u}_{\xi}, \hat{u}_{\eta}\right] \cdot \begin{bmatrix} e^{\xi} & 0 \\ 0 & e^{\eta} \end{bmatrix}^{-}$，$\left[u_x, u_y\right] = \left[\hat{u}_{\xi}, \hat{u}_{\eta}\right] \cdot \begin{bmatrix} e^{-\xi} & 0 \\ 0 & e^{-\eta} \end{bmatrix}$，即 $\begin{cases} u_x = \hat{u}_{\xi} \cdot e^{-\xi} \\ u_y = \hat{u}_{\eta} \cdot e^{-\eta} \end{cases}$。

注：$\begin{bmatrix} e^{-\xi} & 0 \\ 0 & e^{-\eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}(x,y)$

$$\begin{aligned} \Rightarrow u_{xx} &= \left(\hat{u}_{\xi\xi}\frac{\partial \xi}{\partial x} + \hat{u}_{\xi\eta}\frac{\partial \eta}{\partial x}\right)e^{-\xi} + \hat{u}_{\xi}e^{-\xi}(-1)\frac{\partial \xi}{\partial x} \\ &= \hat{u}_{\xi\xi}e^{-2\xi} - \hat{u}_{\xi}e^{-2\xi} \\ u_{yy} &= \left(\hat{u}_{\eta\xi}\frac{\partial \xi}{\partial y} + \hat{u}_{\eta\eta}\frac{\partial \eta}{\partial y}\right)e^{-\eta} + \hat{u}_{\eta}e^{-\eta}(-1)\frac{\partial \eta}{\partial y} \\ &= \hat{u}_{\eta\eta}e^{-2\eta} - \hat{u}_{\eta}e^{-2\eta} \\ u_{xy} &= \left(\hat{u}_{\xi\xi}\frac{\partial \xi}{\partial y} + \hat{u}_{\xi\eta}\frac{\partial \eta}{\partial y}\right)e^{-\xi} + \hat{u}_{\xi}e^{-\xi}(-1)\frac{\partial \xi}{\partial y} \\ &= \hat{u}_{\xi\eta}e^{-(\xi+\eta)} \end{aligned}$$

综上有：

$$\begin{aligned}&ax^2u_{xx}+2bxyu_{xy}+cy^2u_{yy}\\&=ae^{2\xi}\left(\hat{u}_{\xi\xi}-\hat{u}_{\xi}\right)e^{-2\xi}+2be^{\xi+\eta}\hat{u}_{\xi\eta}e^{-(\xi+\eta)}+ce^{2\eta}\left(\hat{u}_{\eta\eta}-\hat{u}_{\eta}\right)e^{-2\eta}\\&=a\left(\hat{u}_{\xi\xi}-\hat{u}_{\xi}\right)+2b\hat{u}_{\xi\eta}+c\left(\hat{u}_{\eta\eta}-\hat{u}_{\eta}\right)\\&=\left(a\hat{u}_{\xi\xi}+2b\hat{u}_{\xi\eta}+c\hat{u}_{\eta\eta}\right)-\left(a\hat{u}_{\xi}+c\hat{u}_{\eta}\right)=0\end{aligned}$$

注：现有方程区别于原方程已经变换为线性方程。

进一步考虑二阶 PDE： $a_{ij}\dfrac{\partial^2 u}{\partial x^i\partial x^j}(x)+c_i\dfrac{\partial u}{\partial x^i}(x)=0$, 此处 $a_{ij}=a_{ji}=const$

引入 $y^\alpha=\theta^{\alpha\beta}x^\beta,\theta^{\alpha\beta}=const$, 则有 $\dfrac{\partial u}{\partial x^i}(x)=\dfrac{\partial\hat{u}}{\partial y^\alpha}(y)\dfrac{\partial y^\alpha}{\partial x^i}(x)=\dfrac{\partial\hat{u}}{\partial y^\alpha}(y)\theta^{\alpha i}$

$$\Rightarrow\frac{\partial^2 u}{\partial x^i\partial x^j}(x)=\frac{\partial^2\hat{u}}{\partial y^\alpha\partial y^\beta}(y)\frac{\partial y^\beta}{\partial x^j}(x)\theta^{\alpha i}=\theta^{\alpha i}\theta^{\beta j}\frac{\partial^2\hat{u}}{\partial y^\alpha\partial y^\beta}(y)$$

故有 $a_{ij}\theta^{\alpha i}\theta^{\beta j}\dfrac{\partial^2\hat{u}}{\partial y^\alpha\partial y^\beta}(y)+c_i\theta^{\alpha i}\dfrac{\partial\hat{u}}{\partial y^\alpha}(y)=0$, 亦即 $\left(\theta^{\alpha i}a_{ij}\theta^{T_{\beta j}}\right)\dfrac{\partial^2\hat{u}}{\partial y^\alpha\partial y^\beta}(y)+\left(\theta^{\alpha i}c_i\right)\dfrac{\partial\hat{u}}{\partial y^\alpha}(y)=0$

考虑到 $A=\left[a_{ij}\right]\in Sym$, 故 $\exists Q\in orth,s.t\ \ Q?A\,Q^T=\Lambda=diag\left[\lambda_1,\dots\ \lambda_m\right],\lambda_i\in\mathbb{R}$,

亦即有 $\theta^{\alpha i}a_{ij}\theta^{T_{j\beta}}=\lambda_\alpha\delta_{\alpha\beta}$

由此， $\displaystyle\sum_{\alpha=1}^{m}\lambda_\alpha\frac{\partial^2\hat{u}}{\partial\left(y^\alpha\right)^2}+c_i\theta^{\alpha i}\frac{\partial\hat{u}}{\partial y^\alpha}(y)=0$ 为分量化形式。

考虑化简方程， $3x^2\dfrac{\partial^2 u}{\partial x^2}-4xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}+3x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$

按后叙一般理论（处理方法）

① 引入 $\begin{bmatrix}x\\y\end{bmatrix}(\begin{bmatrix}\xi\\\eta\end{bmatrix})=\begin{bmatrix}e^\xi\\e^\eta\end{bmatrix}$

$$\begin{aligned}\Rightarrow\mathcal{L}u&=\left(3x^2\frac{\partial^2 u}{\partial x^2}-4xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+3x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)(x,y)\\&=(3\frac{\partial^2\hat{u}}{\partial\xi^2}-4\frac{\partial^2\hat{u}}{\partial\xi\partial\eta}+\frac{\partial^2\hat{u}}{\partial\eta^2})=\mathcal{L}\hat{u}(\xi\eta)=0\end{aligned}$$

② 由 $\left[a_{ij}\right]=\begin{bmatrix}3&-2\\-2&1\end{bmatrix}$， 有 $\exists Q\in orth,s.t\ \ Q\cdot\left[a_{ij}\right]Q^T=diag\left[\lambda_1,\lambda_2\right]$，

$$\det\begin{bmatrix}3-\lambda&-2\\-2&1-\lambda\end{bmatrix}=(\lambda-3)(\lambda-1)-4=\lambda^2-4\lambda-1=0,\ \ \lambda_{1,2}=\frac{4\pm\sqrt{20}}{2}=2\pm\sqrt{5}$$

注：设计上数据选取过于复杂

就微分同胚在变换 PDE 中的应用，可作以下归纳：

考虑以下一般形式的二阶 PDE： $\sum_{i,j=1}^{m} C_{ij} x^i x^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{m} D_i x^i \frac{\partial f}{\partial x^i}(x) = 0$

可有以下程序性的处理：

① 引入 $x^i = e^{\xi^i}, (i = 1, \ldots, m)$，亦即有 $x(\xi) = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix}(\xi) \triangleq \begin{bmatrix} e^{\xi^1} \\ \vdots \\ e^{\xi^m} \end{bmatrix}$，

有 $Dx(\xi) = \begin{pmatrix} e^{\xi^1} & & \\ & \ddots & \\ & & e^{\xi^m} \end{pmatrix}$， $D\xi(x) = \begin{pmatrix} e^{-\xi^1} & & \\ & \ddots & \\ & & e^{-\xi^1} \end{pmatrix}$

由此， $\frac{\partial f}{\partial x^i}(x) = \sum_{\alpha=1}^{m} \frac{\partial \hat{f}}{\partial \xi^\alpha}(\xi) \frac{\partial \xi^\alpha}{\partial x^i}(x) = \sum_{\alpha=1}^{m} \frac{\partial \hat{f}}{\partial \xi^\alpha}(\xi) \delta_{\alpha i} e^{-\xi^\alpha} = \frac{\partial \hat{f}}{\partial \xi^i}(\xi) e^{-\xi^i}$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^i \partial x^j}(x) &= \sum_{\beta=1}^{m} \left[\frac{\partial^2 \hat{f}}{\partial \xi^\beta \partial x^i}(\xi) \frac{\partial \xi^\beta}{\partial x^j}(x) \right] e^{\xi^i} + \frac{\partial \hat{f}}{\partial \xi^i}(\xi) e^{-\xi^i} (-1) \frac{\partial \xi^i}{\partial x^j}(x) \\ &= \frac{\partial^2 \hat{f}}{\partial \xi^j \partial \xi^i} e^{-\left(\xi^i + \xi^j\right)} - \delta_{ji} \frac{\partial \hat{f}}{\partial \xi^i}(\xi) e^{-\left(\xi^i + \xi^j\right)}\end{aligned}$$

即有 $\frac{\partial^2 f}{\partial x^i \partial x^j}(x) = e^{-\left(\xi^i + \xi^j\right)} \frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi) - \delta_{ij} e^{-\left(\xi^i + \xi^j\right)} \frac{\partial \hat{f}}{\partial \xi^i}(\xi)$

$\Rightarrow x^i x^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x) = \frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi) - \delta_{ij} \frac{\partial \hat{f}}{\partial \xi^i}(\xi)$

故有： $\sum_{i,j=1}^{m} C_{ij} x^i x^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x) = \sum_{i,j=1}^{m} C_{ij} \frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi) - \sum_{i=1}^{m} C_{ii} \frac{\partial \hat{f}}{\partial \xi^i}(\xi)$，

$$\sum_{i=1}^{m} D_i x^i \frac{\partial f}{\partial x^i}(x) = \sum_{i=1}^{m} D_i \frac{\partial \hat{f}}{\partial \xi^i}(\xi)$$

亦即有：

$$\begin{aligned}\mathcal{L} f(a) &\triangleq \sum_{i,j=1}^{m} C_{ij} x^i x^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{m} D_i x^i \frac{\partial f}{\partial x^i}(x) \\ &= \sum_{i,j=1}^{m} C_{ij} \frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi) + \sum_{i=1}^{m} \left(D_i - C_{ii}\right) \frac{\partial \hat{f}}{\partial \xi^i}(\xi) =: \mathcal{L} \hat{f}(\xi) = 0\end{aligned}$$

② 引入 $\xi^i = \theta_{ij} \eta^j$，亦即 $\xi = Q\eta,\ Q \in orth, Q_{ij} = Q_{ji}$，故有 $D\xi(\eta) = Q,\ D\eta(\xi) = Q^{-} = Q^T$

亦即：$\dfrac{\partial \eta^i}{\partial \xi^j}(\xi)=Q^T{}_{ij}=Q_{ji}$，另有 $\hat{f}(\xi)=\hat{f}(\xi(\eta))=f(\eta(\xi))$

故有

$$\frac{\partial \hat{f}}{\partial \xi^j}(\xi)=\sum_{\beta=1}^{m}\frac{\partial f}{\partial \eta^\beta}(\eta)\frac{\partial \eta^\beta}{\partial \xi^j}(\xi)=\sum_{\beta=1}^{m}Q_{j\beta}\frac{\partial f}{\partial \eta^\beta}(\eta)$$

$$\Rightarrow \frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi)=\sum_{\alpha=1}^{m}Q_{j\beta}\left[\sum_{\beta=1}^{m}\frac{\partial^2 f}{\partial \eta^\alpha \partial \eta^\beta}\frac{\partial \eta^\alpha}{\partial \xi^i}(\xi)\right]=\sum_{\alpha=1}^{m}Q_{j\beta}Q_{i\alpha}\frac{\partial^2 f}{\partial \eta^\alpha \partial \eta^\beta}(\eta)$$

故有 $\displaystyle\sum_{i,j=1}^{m}C_{ij}\frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi)=\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{m}C_{ij}Q_{j\beta}Q_{i\alpha}\frac{\partial^2 f}{\partial \eta^\alpha \partial \eta^\beta}(\eta)=\sum_{\alpha,\beta=1}^{m}\left[\sum_{i,j=1}^{m}\left(Q^T_{\alpha i}C_{ij}Q_{j\beta}\right)\frac{\partial^2 f}{\partial \eta^\alpha \partial \eta^\beta}(\eta)\right]$

由于 $\left[C_{ij}\right]\in Sym$，故 $\exists Q\in orth, s.t\ \ Q^T\left[C_{ij}\right]Q=\Lambda=diag\left[\lambda_1,\ldots,\lambda_m\right],\lambda_i\in\mathbb{R}$

故有 $\displaystyle\sum_{i,j=1}^{m}\left(Q^T_{\alpha i}C_{ij}Q_{j\beta}\right)=\delta_{\alpha\beta}\lambda_\alpha$，

则有 $\displaystyle\sum_{i,j=1}^{m}C_{ij}\frac{\partial^2 \hat{f}}{\partial \xi^i \partial \xi^j}(\xi)=\sum_{\alpha,\beta=1}^{m}\left[\delta_{\alpha\beta}\lambda_\alpha\frac{\partial^2 f}{\partial \eta^\alpha \partial \eta^\beta}(\eta)\right]=\sum_{\alpha=1}^{m}\left[\lambda_\alpha\frac{\partial^2 f}{\partial\left(\eta^\alpha\right)^2}(\eta)\right]$

综上有：

$$\begin{aligned}\mathcal{L}\hat{f}(\xi)&=\sum_{\alpha=1}^{m}\left[\lambda_\alpha\frac{\partial^2 f}{\partial\left(\eta^\alpha\right)^2}(\eta)\right]+\sum_{i=1}^{m}\sum_{\beta=1}^{m}\left(D_i-C_{ii}\right)Q_{i\beta}\frac{\partial f}{\partial \eta^\beta}(\eta)\\&=\sum_{\alpha=1}^{m}\left[\lambda_\alpha\frac{\partial^2 f}{\partial\left(\eta^\alpha\right)^2}(\eta)\right]+\sum_{\beta=1}^{m}\left[\sum_{i=1}^{m}Q^T_{\beta i}\left(D_i-C_{ii}\right)\right]\frac{\partial f}{\partial \eta^\beta}(\eta)=0\end{aligned}$$

3. 课时安排

本知识点，共计安排 2 课时：

第 1 课时：①

第 2 课时：②

4. 讲述特点及追求效果

✧ 基于对于复旦的学生，研究与实践“从抽象至具体”的教学路径是具有深远意义的；应该尽量鼓励和帮助我们的学生尽量掌握高层次的知识体系，由此将具有更为宽广的实践

范围。

✧

5. 教学方式

全程脱稿板书。