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Option Pricing: A Simplified Approach

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A B S T R A C T

The option pricing problem is examined in its simplest nontrivial setting--the uncertainty of the underlying stock price reduced to discrete binomial movements. Using only elementary mathematics, this leads to a binomial option pricing formula, which contains the Black-Scholes and jump-process formulas as special limiting cases. The discrete binomial formula illuminates the economic intuition behind option pricing theory, without any sacrifice of generality compared to the Black-Scholes analysis. Moreover, the binomial formulation, by its very construction, gives rise to a simple and efficient numerical procedure for valuing options for which premature exercise may be desirable.

I. INTRODUCTION

The original derivation of a satisfactory exact option pricing formula was first published in the *Journal of Policital Economy*, May 1973, in an article written by Fischer Black and Myron Scholes entitled "The Pricing Of Options and Corporate Liabilities." They specifically examined ordinary put and call options. A call is an option to *buy* a fixed number of shares of a specified common stock at a fixed price at any time until a fixed date.[1] A put is similar, except it is an option to *sell* shares. The fixed price is termed the "striking price," and the fixed date, the "expiration date." It is common to distinguish options which can be exercised at any time prior to expiration (American options) from those which can only be exercised at expiration (European options). Although the Black-Scholes paper dealt directly only with European options, it has been possible to extend their approach to value American options as well. Unless otherwise indicated, this paper will likewise pertain to American options, of the type now traded on organized option markets in the United States.

If option pricing theory were confined to the valuation of ordinary puts and calls, despite recent instituonal developments, it would not have attracted widespread academic attention. As Black and Scholes themselves mentioned, virtually all corporate securities can be fruitfully

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[1]A call should not be confused with a futures contract. The latter represents a *commitment* to buy or sell (i.e., on its expiration date, a future must be "exercised").

interpreted as portfolios of puts and calls.[1] Moreover, option pricing theory applies to a very general class of economic problems—the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event.

Unfortunately, the mathematical tools employed in the article are quite advanced, and have served to obscure the underlying economics. However, thanks to a suggestion of William Sharpe, it is possible to derive the Black-Scholes formula using only elementary mathematics. This is done by treating the option pricing problem in its simplest nontrivial setting--the uncertainty of the underlying stock price reduced to discrete binomial movements.[2] One of the conclusions of our paper is that the resulting discrete binomial option pricing formula illuminates the economic intuition behind option pricing theory, without any sacrifice in generality compared to the Black-Scholes analysis.

Not only is the binomial formula of interest in itself, but the formula lies at the fork of two significant limiting cases--the Black-Scholes continuous stochastic process formula, and the Cox-Ross [1976] jump stochastic process formula. As we shall show, each can be derived from the binomial formula by taking the appropriate limits.

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[1] To take an elementary case, consider a firm with a single liability of a homogeneous class of pure discount bonds. The stockholders then have a "call" on the assets of the firm which they can choose to exercise at the maturity date of the debt by paying its principal to the bondholders. In turn, the bonds can be interpreted as a portfolio of a written put on the assets of the firm and a default-free loan with the same face value as the bonds.

[2] Sharpe has partially developed the binomial approach to option pricing in his recent book *Investments*, Prentice-Hall, 1978. Rendleman and Bartter [1977] have also independently discovered the binomial formulation of the option pricing problem.

Other more general option pricing problems seem immune to reduction to a simple formula. Instead, numerical procedures must often be employed to price these more complex options. Following the approach of Michael Brennan and Edwardo Schwartz [1977], the Black-Scholes differential hedging equation is first reduced to a discrete-time difference equation and then the option price is obtained by somewhat elaborate numerical procedures. In contrast, the binomial formulation, by its very construction, gives rise to an alternative numerical procedure which is both far simpler and, for many purposes, computationally more efficient.

II. THE BASIC IDEA

Suppose the current price of an underlying stock is $S = \$50$, and at the end of a period of time, its price must either be $S = \$25$ or $S^* = \$100$. A call[1] on the stock is available with a striking price of $K = \$50$, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest. The one piece of information left unfurnished is the current price C of the call. However, if riskless profitable arbitrage is not possible, we can deduce from the given information *alone* what the price of the call *must* be!

Consider forming the following levered hedge:

> (1) write 3 calls at C each,
> (2) buy 2 shares at \$50 each, and
> (3) borrow \$40 at 25%, to be paid back at end of period.

Table 1 gives the return from this hedge, for each possible level of the stock price at expiration. Regardless of the outcome, the hedge exactly breaks even on the expiration date. Therefore, to prevent profitable riskless arbitrage, its current value must be zero; that is,

$$-3C + 100 - 40 = 0 .$$

The current value of the call must then be $C = \$20$.

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[1] To keep matters simple, assume for now that the call is protected against cash dividends. We also ignore transactions costs, margin, and taxes.

Table 1. Arbitrage Table Illustrating the Formation of a Riskless Hedge

	Present Date	Expiration Date	
		$S^* = 25$	$S^* = 100$
Write 3 calls	-3C	---	-150
Buy 2 shares	100	50	200
Borrow	-40	-50	-50
Total		---	---

If the call were not priced at \$20, a sure profit would be possible. In particular, if $C = \$25$, the above hedge would yield a current cash inflow of \$15 and would experience no further gain or loss in the future. On the other hand, if $C = \$15$, then the same thing could be accomplished by buying 3 calls, selling short 2 shares, and lending \$40.

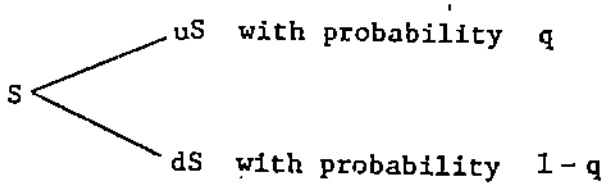
Table 1 can be interpreted as demonstrating that *an appropriately levered position in stock will replicate the future returns of a call.* That is, if we buy shares and borrow against them in the right proportion, we can, in effect, duplicate a pure position in calls. In view of this, it should seem less surprising that all we needed to determine the *exact* value of the call was its *striking price, underlying stock price, range of movement in the underlying stock price, and the rate of interest.* What may seem more incredible is what we don't need to know: among other things, *we don't need to know the probability that*

the stock price will rise or fall. Bulls and bears must agree on the value of the call, relative to its underlying stock price!

Clearly, our numerical example has been chosen for simplicity, not realism. Among other things, it gives no consideration to the existing liquid secondary market, which permits closing transactions any time prior to expiration, and it posits very unrealistic stock price movements. As it turns out, correcting the former deficiency supplies the key to correcting the latter.

III. BINOMIAL OPTION PRICING FORMULA

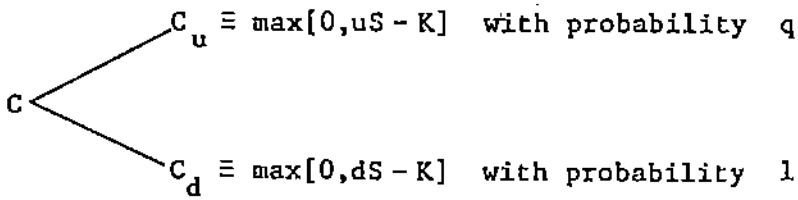
To model the value of a call prior to expiration, we start with the simplest nontrivial situation. Suppose the expiration date is just one "period" away. Denoting the current stock price as S, we assume it follows a binomial process so that at the end of the period at the expiration of the call, its price is either uS or dS, with probability q and $1-q$, respectively. Therefore,



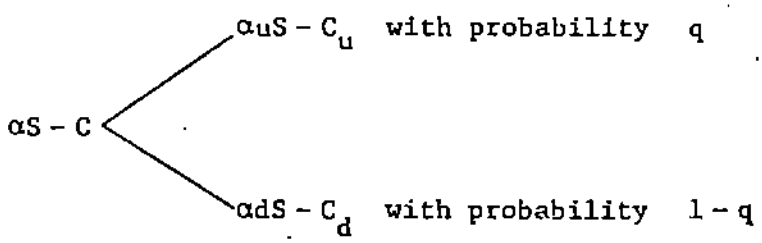
Letting r denote one plus the interest rate over the period, we require $u > r > d$. If these inequalities did not hold, there would be profitable riskless arbitrage opportunities involving only the stock and riskless borrowing and lending. Although we could simplify the example

further by setting $d = u^{-1}$ and $q = .5$, we choose to retain this greater level of generality.

When the call expires, we know that its contract and a rational exercise policy imply that its value must either be $C_u \equiv \max[0, uS - K]$ or $C_d \equiv \max[0, dS - K]$. Therefore,



Suppose we form a hedge at the beginning of the period by writing one call against α shares of stock. This would cost $\alpha S - C$. The buyer of the call will either retain it until expiration or exercise it immediately. This will depend on which is higher, the retention value or the exercise value, $\max[0, S - K]$. To find out, we will first calculate the value of the call if he retains it. If the call is unexercised, then our hedge will return



Now, since we can choose α any way we wish, suppose we select the "neutral" hedge ratio, that is, the α that makes the hedge riskless. We accomplish this by selecting the α which equates the dollar returns in the two possibilities:

$$\alpha dS - C_d = \alpha uS - C_u \ .$$

Solving this equation, the hedge ratio α which eliminates all risk is:

$$\alpha = \frac{C_u - C_d}{(u-d)S} \ .$$

With this hedge ratio, since the return from the hedge, $\alpha dS - C_d$, is riskless, to prevent riskless profitable arbitrage, it must have the same return as an investment of $\alpha S - C$ dollars in riskless borrowing or lending. Therefore,

$$\alpha dS - C_d = r(\alpha S - C) \ .$$

Rearranging this equality and substituting for α,

$$rC = \left(\frac{r-d}{u-d}\right) C_u + \left(\frac{u-r}{u-d}\right) C_d \ .$$

To state this more simply, observe that defining $p \equiv \frac{r-d}{u-d}$, then $1 - p = \frac{u-r}{u-d}$. Therefore,

(1) $$C = [pC_u + (1-p)C_d] \div r \ .$$

This is the exact formula for the value of a call one period prior to expiration in terms of S, K, u, d, and r.

The formula gives the value of the call if, as we assumed, the buyer does not immediately exercise it. However, it is easy to see that

as long as the interest rate is not negative (i.e., $r \geq 1$), premature exercise is not optimal. To see this, suppose the worst, that $uS > K$ and $dS < K$. Then, if he does not exercise immediately,

$$C = \{p \max[0,uS-K] + (1-p)\max[0,dS-K]\} \div r = \left[\frac{p}{r}\right](uS-K) .$$

Our problem is to prove:

$$\left[\frac{p}{r}\right](uS-K) > S - K ,$$

the money he would receive if he exercised immediately. A little algebra easily confirms this inequality. Since it is then not optimal to exercise the call, formula (1) is unambiguously its correct value one period prior to expiration.

This formula has a number of notable features. First, the only assumption imposed on investor behavior is the motivation to eliminate all opportunities for profitable riskless arbitrage. For example, investors can be risk-averse or risk-preferring, and we would derive the same formula. The assumption that no profitable riskless arbitrage opportunities exist is particularly acceptable from a practical point of view, since we gain whether it is true or false. If it is true, we have explained call prices; if it is false, we will be the first to take advantage of its falsity by transacting in the market.

Second, probability q does not appear in the formula. This means, surprisingly, that even if different investors have different subjective probabilities about an upward or downward movement in the stock

price, they would still agree on the relationship of C to S and r. This can be understood if it is remembered that the formula is only a *relative* pricing relationship between C on one side and S and r on the other. The probability q will affect the values of S and r and, only transmitted through them, will *indirectly* affect the value of C.

Third, $p \equiv (r-d)/(u-d)$ has all the properties of a probability measure: that is, $0 < p < 1$. However, p is generally not a subjective probability (i.e., a probability in an investor's mind). Indeed, to require p to be a subjective probability would unnecessarily restrict the context of the formula. The current call value would then be equal to its expected future value discounted at the riskless rate with no adjustment for risk. Only if investors were risk-neutral or the risk of the call costlessly eliminated by diversification would such an interpretation be reasonable. Nonetheless, and this will be important subsequently, the same formula for C in terms of S, K, u, d, and r holds even if we interpret p as a subjective probability. That is, had we made this interpretation initially, we would not have been misled, since we would have derived the same formula. Even in a risk-averse or risk-preference environment, the relationship of C to S, K, u, d, and r would be determined as if investors were risk-neutral. Incidentally, if investors were risk-neutral, then $q = p$. To see this, since then

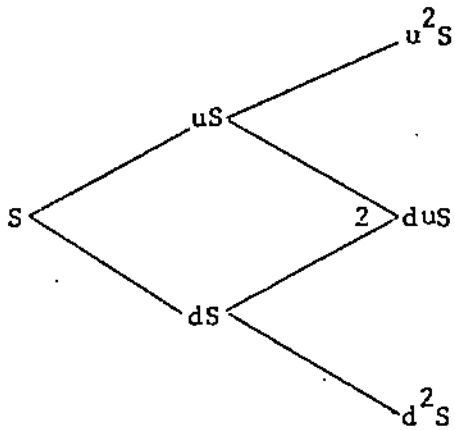
$$q(uS) + (1-q)(dS) = rS$$

then

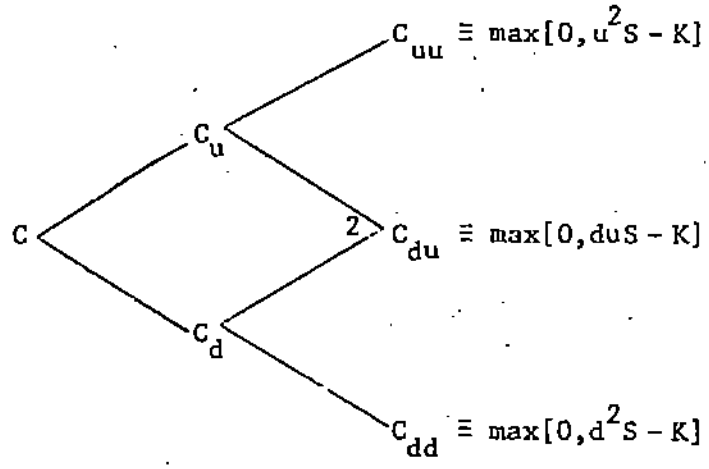
$$q = (r-d)/(u-d) = p\,.$$

Finally, the only random variable the call value depends upon is the stock price itself. In particular, it does not depend, in addition, on the random prices of other securities or portfolios, such as the "market" portfolio containing all securities in the economy. Indeed, if C depended separately on some other random variable, since its price would then be different than formula (1), we know from the derivation of this formula that profitable riskless arbitrage would be possible.

In reality, we can sell or exercise a call at many dates prior to its expiration. Stepping backward one more period, we will now examine what happens to the call value two periods before expiration. In keeping with our binomial stochastic process, for the stock:



In other words, S follows a stationary random binomial process with step size u or d with probability q or 1 - q, respectively. Additionally, we assume one plus the interest rate r is stationary period to period. Then, for the call:



From our previous analysis with one period left, applying formula (1),

$$C_u = [pC_{uu} + (1-p)C_{ud}]/r \quad \text{and} \quad C_d = [pC_{du} + (1-p)C_{dd}]/r \ .$$

Again we construct a riskless hedge at the beginning of the first period by choosing α so that, investing $\alpha S - C$, we are certain to receive $\alpha dS - C_d = \alpha uS - C_u$. Following the same reasoning as before, to prevent profitable riskless arbitrage, again $C = [pC_u + (1-p)C_d]/r$. Substituting in the above equations for C_u and C_d,

$$C = [p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}] \div r^2 \ . \tag{2}$$

Note that $C_{du} = C_{ud}$. Again, we can use this and compare it to $S - K$ to show that the call will not be exercised prematurely. Moreover, all other observations made about formula (1) also apply to formula (2), except that the number of periods n remaining to expiration now emerges clearly as an additional determinant of the call value. For formula (2), $n = 2$. That is, the full list of variables determining C is S, n, K, u, d, and r.

Working backward in time deductively, we can write down the general pricing formula for any n:

$$(3) \qquad C = \left\{ \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max[0, u^j d^{n-j} S - K] \right\} \div r^n .$$

Since $d^n S < ud^{n-1} S < u^2 d^{n-2} S < \ldots < u^n S$, presuming the option will not expire in-the-money for sure, or out-of-the-money for sure, there must exist an integer $0 < a < n$, such that

$$u^{a-1} d^{n-(a-1)} S < K \leq u^a d^{n-a} S .$$

For all $j < a$, $\max[0, u^j d^{n-j} S - K] = 0$ and for all $j \geq a$, $\max[0, u^j d^{n-j} S - K] = u^j d^{n-j} S - K$. Therefore,

$$C = \left\{ \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} [u^j d^{n-j} S - K] \right\} \div r^n .$$

We can solve for a by taking natural logarithms of the above inequality. Thus, a is an integer such that

$$\frac{\ln(K/Sd^n)}{\ln(u/d)} \leq a < 1 + \frac{\ln(K/Sd^n)}{\ln(u/d)} .$$

Breaking up C into two terms

$$C = S\left[\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{r^n} \right] - Kr^{-n}\left[\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \right] .$$

Now, the latter bracketed expression is the complementary binomial distribution function[1] $B[a; n,p]$. The first bracketed expression can also be interpreted as a complementary binomial distribution function $B[a; n, p']$, where

$$p' \equiv (u/r)p \quad \text{and} \quad 1 - p' \equiv (d/r)(1 - p)$$

p' is a probability measure, since $0 < p' < 1$. To see this,

$$p^j(1-p)^{n-j}\,\frac{u^j d^{n-j}}{r^n} = \left[\frac{u}{r}\,p\right]^j \left[\frac{d}{r}\,(1-p)\right]^{n-j} = p'^j(1-p')^{n-j}\,.$$

In summary:

BINOMIAL OPTION PRICING FORMULA

$$C = SB[a;n,p'] - Kr^{-n}B[a;n,p]$$

where

$$p \equiv (r-d)/(u-d) \quad \text{and} \quad p' \equiv (u/r)p$$

$$\frac{\ln(K/Sd^n)}{\ln(u/d)} \leq a < 1 + \frac{\ln(K/Sd^n)}{\ln(u/d)}$$

where a is an integer.

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[1] Caution: Despite this, p is only a probability measure and not to be interpreted necessarily as a subjective probability.

Before analyzing this formula in detail, we should mention an alternative shortcut derivation. Suppose the call value C depends on the concurrent stock price S_n and the number of periods n remaining to expiration. We express the call value as $C(S_n, n)$. As before, we can set up a riskless hedge and derive

$$C(S_n,n) = [pC(uS_n,n-1) + (1-p)C(dS_n,n-1)] \div r ,$$

where, at $n=0$, $C(S_0,0) = \max[0,S_0 - K]$. Since this relationship does not involve investor attitudes toward risk, it must be true regardless of what attitudes we assume. For computational purposes, the most convenient choice is risk neutrality. In this case, present values are just expected future values discounted back to the present at the riskless interest rate. Since the stock is assumed not to pay dividends, it would never be optimal to exercise the call before its expiration date. Therefore, with risk neutrality, its present value must be its expected value on its expiration date discounted back to the present. Recall from our previous discussion, $S_0 = u^j d^{n-j} S$ and, with risk neutrality, $q = p$. Since

$$C = \frac{E\{\max[0 , S_0 - K]\}}{r^n} ,$$

we can then derive equation (3). The proof then proceeds as before.

For some readers, an alternative "complete markets" interpretation of our binomial approach may be instructive. Suppose that π_u and π_d represent the state-contingent discount rates to states u and d,

respectively. Therefore, π_u would be the current price of one dollar received at the end of the period, if and only if state u occurs. Each security--a riskless bond, the stock, and the option must all have returns discounted to the present by π_u and π_d if no riskless arbitrage opportunities are available. Therefore,

$$1 = \pi_u r + \pi_d r ,$$

$$S = \pi_u(uS) + \pi_d(dS) , \text{ and}$$

$$C = \pi_u C_u + \pi_d C_d .$$

The first two equations, for the bond and the stock, imply

$$\pi_u = \left(\frac{r-d}{u-d}\right)\frac{1}{r} \quad \text{and} \quad \pi_d = \left(\frac{u-r}{u-d}\right)\frac{1}{r} .$$

Substituting these equalities for the state-contingent prices in the last equation for the option yields equation (1).

It is important to realize that we are not assuming that the riskless bond and the stock and the option are the only three securities in the economy, or that other securities must follow a binomial process. Rather, however these securities are priced in relation to others in equilibrium, among themselves they must conform to the above relationships.

From either the hedging or complete markets approaches, it should be clear that three-state or trinomial stock price movements will not lead to an option pricing formula solely based on arbitrage considerations.

Suppose, for example, over each period the stock price either moved to uS, dS, or remained the same at S. A hedge ratio which would equate the returns in two states could not in the third. That is, a riskless arbitrage position could not be taken. Under the complete markets interpretation, with three equations in now three unknown state-contingent prices, we would lack the redundant equation necessary to price one security in terms of the other two.

IV. RISKLESS TRADING STRATEGIES

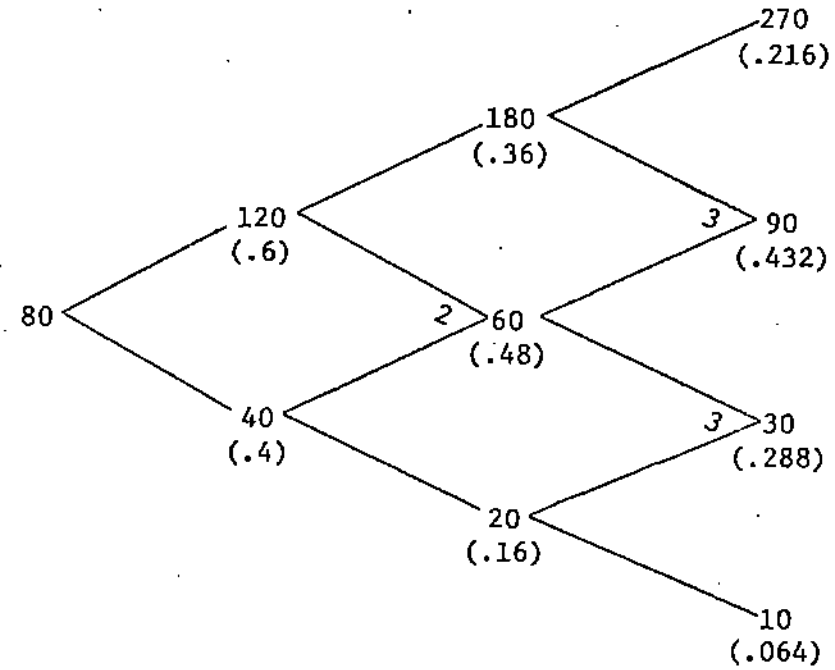
As an exercise, let's see how we could use the formula if the current market price M ever diverged from its formula value C. If $M > C$, we would hedge, and if $M < C$, "reverse hedge" to try and lock in a profit. However, suppose things got worse before they got better, so that the market price of an option we thought was mispriced moves even farther away from the formula value after we take a position. Couldn't we end up losing money?

Consider the following example:

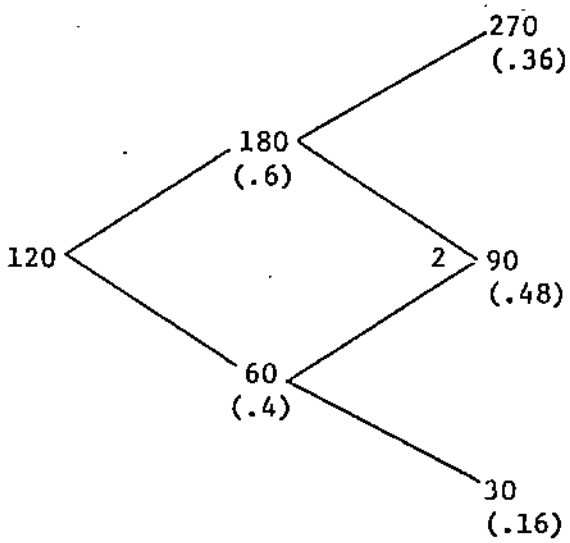
$$S = 80, \quad n = 3, \quad K = 80, \quad u = 1.5, \quad d = .5, \quad r = 1.1$$

In this case, $p = (r - d)/(u - d) = .6$. The paths the stock price may follow and their corresponding probabilities are:

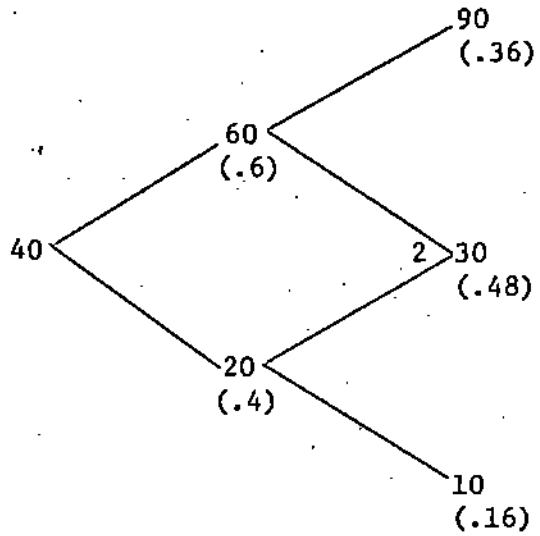
As of $n = 3$, $r^{-3} = .751$ and



As of $n = 2$, when $uS = 120$, then $r^{-2} = .826$ and



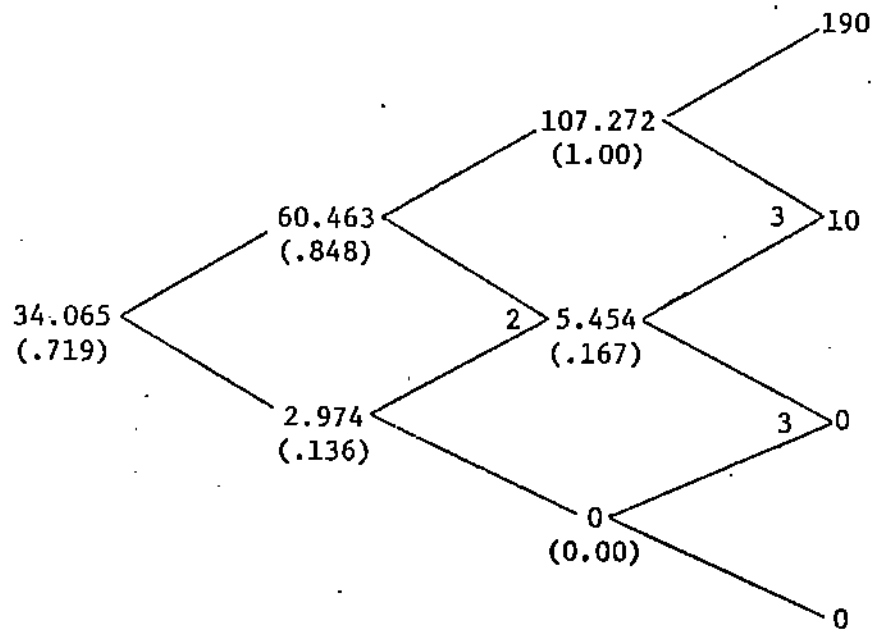
As of $n = 2$, when $dS = 40$, then $r^{-2} = .826$ and



Using the formula, the current value of the call would be

$$C = [.064(0) + .288(0) + .432(90-80) + .216(270-80)] \times .751 = 34.065$$

Recall that to form a riskless hedge, for each written call we buy α shares of stock such that $\alpha = (C_u - C_d) \div [(u-d)S]$. The following tree diagram gives the paths the call value may follow and its corresponding neutral hedge ratios.



With this preliminary analysis, we are prepared to use the formula to take advantage of mispricing in the market. Suppose that when $n = 3$, the market price of the call is 36. Our formula tells us the call should be worth 34.065. The option is overpriced, so we could plan to sell it and assure ourselves of a profit equal to the mispricing differential. Here are the steps you could take for a typical path the stock might follow.

Step 1 ($n = 3$): Sell the call for 36. Take 34.065 of this and invest it in a portfolio containing $\alpha = .719$ shares of stock by borrowing $.719(80) - 34.065 = 23.455$. Take the remainder, $36 - 34.065$, and put it in the bank.

Step 2 ($n = 2$): Suppose the stock goes to 120 so that the new $\alpha = .848$. Buy $.848 - .719 = .129$ more shares of stock at 120 per share for a total expenditure of 15.480. Borrow to pay the bill. With an interest rate of .1, you already owe $23.455(1.1) = 25.801$. Thus, your total current indebtedness is $25.801 + 15.480 = 41.281$.

Step 3 ($n = 1$): Suppose the stock price now goes to 60. The new $\alpha = .167$. Sell $.848 - .167 = .681$ shares at 60 per share, taking in $.681(60) = 40.860$. Use this to pay back part of your borrowings. Since you now owe $41.281(1.1) = 45.409$, after the repayment, you instead owe $45.409 - 40.860 = 4.549$.

Step 4d ($n = 0$): Suppose the stock price now goes to 30. The call you sold has expired worthless. You own .167 shares of stock selling at 30 per share, for a total value of $.167(30) = 5$. Sell the stock and repay the $4.549(1.1) = 5$

that you now owe on the borrowing. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3 = 2.575$.

Step 4u ($n = 0$): Suppose, instead, the stock price goes to 90. The call you sold is in the money at the expiration date. Buy back the call, or buy one share of stock and let it be exercised, incurring a loss of $90 - 80 = 10$ either way. Borrow to cover this, bringing your current indebtedness to $5 + 10 = 15$. You own .167 shares of stock selling at 90 per share, for a total value of $.167(90) = 15$. Sell the stock and repay the borrowing. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3 = 2.575$.

In summary, if we were correct in our original analysis about stock price movements (which did not involve the unenviable task of predicting q, whether the stock price would go up or down), and if we faithfully adjust our portfolio as prescribed by the formula, then we can be assured of walking away in the clear at the expiration date, while still keeping that original differential and the interest it has accumulated. It is true that closing out the position before the expiration date, which involves buying back the option at its then current market price, might produce a loss which would more than offset our profit, but this loss could always be avoided by waiting until the expiration date. Moreover, if the market price comes into line with the formula value before the expiration date, we can close out the position then with no loss and be rid of the concern of keeping the portfolio adjusted.

This answers our original question. The return on a perfectly hedged portfolio, when evaluated at prevailing market prices at intermediate times, may be risky. But over a period ending no later than the expiration date, it will be certain.

Throughout the hedging operation, observe that we always adjusted the hedge ratio by buying or selling the *stock*. As a result, our profit was independent of the market price of the call between the time we initiated the hedge and the expiration date. If things got worse before they got better, it did not matter to us.

We could have adjusted the hedge ratio by buying or selling the *call* instead of the stock. However, this could be dangerous. Suppose that after initiating the position, to maintain neutrality we needed to *increase the hedge ratio*. This can be achieved in two ways:

a) buy more stock, or

b) buy back some written calls.

If we adjust through the stock, there is no problem. If we insist on adjusting through the calls, not only is the hedge no longer riskless, but it could even end up losing money! This can happen if the call has become even more overpriced. We would then be closing out part of our position in calls at a loss. To remain neutral, the number of calls we would need to buy back depends on their value, not their price. Therefore, since we are uncertain about their price, we then become uncertain about the return from the hedge. Worse yet, if the call price gets high enough, the loss on the closed portion of our position could throw the hedge operation into an overall loss.

To see how this could happen, let us rerun the hedging operation, where we adjust the hedge ratio by buying and selling calls.

Step 1 ($n = 3$): Same as before.

Step 2 ($n = 2$): Suppose the stock goes to 120, so that the new $\alpha = .848$. The call price has gotten further out of line and is now selling for 75. Since its value is 60.463, it is now overpriced by 14.537. With .719 shares, you must buy back $1 - .848 = .152$ calls to produce a hedge ratio of $.848 = .719/.848$. This costs $75(.152) = 11.40$. Borrow to pay the bill. With the interest rate of .1, you already owe $23.455(1.1) = 25.801$. Thus, your total current indebtedness is $25.801 + 11.40 = 37.201$.

Step 3 ($n = 1$): Suppose the stock goes to 60 and the call is selling for 5.454. Since the call is now fairly valued, no further excess profits can be made by continuing to hold the position. Therefore, liquidate by selling your .719 shares for $.719(60) = 43.14$ and close out the call position by buying back .848 calls for $.848(5.454) = 4.625$. This nets $43.14 - 4.625 = 38.515$. Use this to pay back part of your borrowing. Since you now owe $37.20(1.1) = 40.921$, after repayment you owe 2.406. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^2 = 2.341$. Unfortunately, after using this to repay your remaining borrowings, you still owe .065.

Since we adjusted our position at step 2 by buying overpriced calls, our profit is reduced. Indeed, since the calls were considerably overpriced, we actually lost money despite apparent profitability of the position at step 1. We can draw the following adjustment rule from our

experiment: *To adjust a neutral position, never buy an overpriced option or sell an underpriced option.* As a corollary, whenever we can adjust a neutral position by buying more of an underpriced option or selling more of an overpriced option, our profit will be enhanced if we do so. For example, at step 3 in the original hedging illustration, had the call still been overpriced, it would have been better to adjust the position by selling more calls rather than selling stock. In summary, by choosing the right side of the position to adjust at intermediate dates, *at a minimum* we can be assured of earning the original differential and its accumulated interest, and we may earn considerably more.[1]

V. LIMITING CASES

In reading the previous sections, there is a natural tendency to associate with each period some particular length of calendar time, perhaps a day. With this in mind, you may have had two objections. In the first place, prices a day from now may take on many more than just two possible values. Furthermore, the market is not open for trading only once a day, but, instead, trading takes place more or less continuously.

These objections are certainly valid. Fortunately, our option pricing approach has the flexibility to meet them. Although it might have been natural to think of a period as one day, there was nothing that

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[1] In principle, if we were prepared to take arbitrarily large positions (i.e., "doubling-up") as the call became more mispriced, we could be assured of a profit even if we were committed to trading in the call. However, practical considerations which might limit our capital would seem to favor revising the hedge with stock.

forced us to do so. We could have taken a much shorter interval--say an hour--or even a minute. By doing so, we have met both objections simultaneously. Not only would trading take place far more frequently, but the stock price could also take on hundreds of values by the end of the day.

However, if we do this, we have to make some other adjustment to keep the probability small that the stock price will change by a large amount over a minute. We do not want the stock to have the same percentage up and down moves for one minute as it did before for one day. Instead, we must think of the stock price as making only a very small percentage change over each minute.

To make this more precise, suppose that (Δt) represents the elapsed time between successive stock price changes. That is, if t is the fixed length of calendar time to expiration, and n is the number of periods of length Δt prior to expiration, then

$$\Delta t \equiv t/n \ .$$

As trading takes place more and more frequently, Δt gets closer and closer to zero. Likewise, the interval-dependent variables r, u, and d in the binomial option pricing formula also must get smaller and smaller as Δt nears zero, or, equivalently, as $n \to \infty$.

As we turn from discrete-time to continuous-time results, it will help if we redefine the symbol r . From this point, the reader is warned that it will take on a related but different meaning. When we have occasion to refer to the discrete one plus rate of interest over the period of length Δt , we will use the symbol $\hat{r}$ in place of r .

Clearly, the size of $\hat{r}$ depends on the number of subintervals n into which t is divided. Over the n periods until expiration, the total return is $\hat{r}^n$, where $n = t/\Delta t$. Now not only do we want $\hat{r}$ to depend on n, but we want it to depend on n in a particular way--so that as n changes the total return $\hat{r}^n$ over the fixed time t remains the same. This is because the interest rate obtainable over some fixed length of calendar time should have nothing to do with how we choose to think of the length of the time interval Δt.

If r (without the "hat") denotes one plus the rate of interest over a *fixed* unit of calendar time, then over elapsed time t, r^t is the total return. Observe that this measure of total return does not depend on n. As we have argued, we want to choose the dependence of $\hat{r}$ on n, so that

$$\hat{r}^n = r^t ,$$

for any choice of n. Therefore, $\hat{r} = r^{t/n}$. This last equation shows how $\hat{r}$ must depend on n for the total return over elapsed time t to be independent of n.

We also need to define u and d in terms of n. At this point, there are two significantly different paths we can take. Depending on the definitions we choose, as $n \to \infty$ (or, equivalently, as $\Delta t \to 0$), we can either have a continuous or a jump stochastic process. The first corresponds to a situation where each successive stock price is different than, but very close to, the previous price, and the second to a situation where each successive stock price is almost always close to the previous price but occasionally significantly different. Both can be

derived from our binomial process simply by choosing how u and d depend on n. We only follow in detail the continuous path which leads to the canonical option pricing formula originally derived by Fischer Black and Myron Scholes. Subsequently, we indicate how to develop the jump path formula originally derived by John Cox and Stephen Ross.

Recall that we supposed over each period that the stock price would experience a one plus rate of return of u with probability q and d with probability $1-q$. It will be easier and clearer to work, instead, with the natural logarithms of the one plus rate of return, $\ln u$ or $\ln d$. This gives the continuously compounded rate of return on the stock over each period. It is a random variable which, in each period, will be equal to $\ln u$ with probability q and $\ln d$ with probability $1-q$.

Consider a typical sequence of five moves, say u, d, u, u, d. Then, $S^* = uduudS$. $S^*/S = u^3d^2$, and $\ln(S^*/S) = 3 \ln u + 2 \ln d$. More generally, over n periods,

$$\ln(S^*/S) = j \ln u + (n-j)\ln d = j \ln(u/d) + n \ln d ,$$

where j is the (random) number of upward moves occurring during the n periods to expiration. Therefore, the expected value of $\ln(S^*/S)$ is

$$E[\ln(S^*/S)] = \ln(u/d) \cdot E(j) + n \ln d ,$$

and its variance is

$$Var[\ln(S^*/S)] = [\ln(u/d)]^2 \cdot Var(j) .$$

Now j is the total number of upward moves that will occur over the next n periods, and each of these moves has probability q. Thus, $E(j) = nq$. Also, since the variance each period is $q(1-q)^2 + (1-q)(0-q)^2 = q(1-q)$, then $Var(j) = nq(1-q)$. Combining all of this, we have:

$$E[\ln(S^*/S)] = [q \ln(u/d) + \ln d]n \equiv \hat{\mu}n$$

$$Var[\ln(S^*/S)] = q(1-q)[\ln(u/d)]^2 n \equiv \hat{\sigma}^2 n .$$

Let us go back to our discussion. We were considering dividing up our original longer time period (a day) into many shorter periods (a minute or even less). Our procedure calls for, over fixed length of calendar time t, making n larger and larger. Now if we held everything else constant while we let n become large, we would be faced with the problem we talked about earlier. In fact, we would certainly not reach a reasonable conclusion if either $\hat{\mu}n$ or $\hat{\sigma}^2 n$ went to zero or infinity as n became large. Since t is a fixed length of time, in searching for a realistic result, we must make the appropriate adjustments in u, d, and q. In doing that, we would at least want the mean and variance of the assumed future stock price and its continuously compounded rate of return to coincide with that of the actual stock price as $n \to \infty$. Suppose we label the actual empirical values of $\hat{\mu}n$ and $\hat{\sigma}^2 n$ as μt and $\sigma^2 t$, respectively. Then we would want to choose u, d, and q, so that

$$\left.\begin{aligned} [q\ \ln(u/d) + \ln d]n &\to \mu t \\ q(1-q)[\ln(u/d)]^2 n &\to \sigma^2 t \end{aligned}\right\} \quad \text{as} \quad n \to \infty .$$

A little algebra shows we can accomplish this by letting

$$u = e^{\sigma\sqrt{t/n}} \quad , \quad d = e^{-\sigma\sqrt{t/n}} \quad \text{and} \quad q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{t/n} \ .$$

In this case, for any n,

$$\hat{\mu}n = \mu t \quad \text{and} \quad \hat{\sigma}^2 n = [\sigma^2 - \mu^2(t/n)]t \ .$$

Clearly, as $n \to \infty$, $\hat{\sigma}^2 n \to \sigma^2 t$, while $\hat{\mu}n = \mu t$ for all values of n.

Alternatively, we could have chosen u, d, and q so that the mean and variance of *rates of return* of the discrete binomial process approach the prespecified mean and variance of stock rate of return as $n \to \infty$. Since this would not change our conclusion and it is computationally more convenient to equalize the continuously compounded rates of return, we will proceed in that way.

This satisfies our initial requirement that the limiting means and variances coincide, but we still need to verify that we are arriving at a sensible limiting probability distribution of the continuously compounded rate of return. The mean and variance only describe certain aspects of that distribution.

At this point, we rely on a version of the central limit theorem. The usual version of this theorem states that as more observations are

added, if they are independently and identically distributed (with finite variance), their sum approaches a normally distributed random variable.

However, for our purpose, the usual version of the central limit theorem will not suffice. This is because as we let n approach infinity, we are not simply adding one more random variable to the previous sum, but are, instead, changing the probabilities q and possible outcomes u and d for every member of the sum. The type of central limit result we will need says that, as $n \to \infty$

$$\text{if} \quad \frac{q\left|\ln u - \hat{\mu}\right|^3 + (1-q)\left|\ln d - \hat{\mu}\right|^3}{\hat{\sigma}^3\sqrt{n}} \to 0 \quad \text{and} \quad \hat{z} \to z \ ,$$

$$\text{then} \qquad \text{Prob}\left\{\left[\frac{\ln(S^*/S) - \hat{\mu}n}{\hat{\sigma}\sqrt{n}}\right] \leq \hat{z}\right\} \to N(z) \ .$$

Putting this into words, for fixed time to expiration, as the number of periods into which it is divided approaches infinity, given the initial condition, then the probability that the standardized continuously compounded rate of return of the stock through the expiration date is not greater than the number $\hat{z}$ approaches the probability under a standard normal distribution.

The initial condition says roughly that higher-order properties of the distribution, such as how it is skewed, become less and less important, relative to its standard deviation, as $n \to \infty$. To verify that the condition is satisfied, by making the appropriate substitutions,

$$\frac{q\left|\ln u - \hat{\mu}\right|^3 + (1-q)\left|\ln d - \hat{\mu}\right|^3}{\hat{\sigma}^3\sqrt{n}} = \frac{q(1-q)^2 + q^2(1-q)}{\sqrt{nq(1-q)}},$$

which goes to zero as $n \to \infty$ since $q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{t/n}$. Thus, the binomial option pricing formula includes the lognormal distribution for stock prices as a limiting case.

Black and Scholes began directly with continuous trading and the assumption of a lognormal distribution for stock prices. Their approach relied on some quite advanced mathematics. However, since our approach contains the continuous trading and lognormal distribution as a limiting case, the two resulting formulas should then coincide. We will see shortly that this is indeed true, and we will have the advantage of using a much simpler method. It is important to remember, however, that the economic arguments we used to link the option price and the stock price are exactly the same as those advanced by Black and Scholes.

The formula derived by Black and Scholes, rewritten in terms of our notation, is:

CANONICAL OPTION PRICING FORMULA

$$C = SN(h) - Kr^{-t}N(h - \sigma\sqrt{t})$$

$$\text{where} \quad h \equiv \frac{\ln(S/Kr^{-t})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

We now wish to confirm that our binomial formula converges to the canonical formula when t is divided into more and more subintervals, and $\hat{r}$, u, d, and q are chosen in the way we described--that is, in a way

such that the binomial probability distribution of stock prices goes to the lognormal distribution.

For easy reference, let us recall our binomial option pricing formula:

$$C = SB[a;n,p'] - K\hat{r}^{-n}B[a;n,p] .$$

The similarities are readily apparent. $\hat{r}^{-n}$ is, of course, always equal to r^{-t}. Therefore, to show the two formulas converge, we need only show that, as $n \to \infty$

$$B[a;n,p'] \to N(h) \quad \text{and} \quad B[a;n,p] \to N(h - \sigma\sqrt{t}) .$$

We will consider only $B[a;n,p]$, since the argument is exactly the same for $B[a;n,p']$.

The complementary binomial distribution function $B[a;n,p]$ is the probability that the number of up moves j is greater than or equal to a. j, itself, represents the sum of n random variables, each of which is either equal to 1 in the case of an up move or 0 in the case of a down move. p is the probability of an up move, and $1-p$ is the probability of a down move. From our previous discussion, we know the mean and standard deviation of j are np and $\sqrt{np(1-p)}$, respectively. Therefore,

$$1 - B[a;n,p] = \text{Prob}[j \leq a-1] = \text{Prob}\left[\frac{j - np}{\sqrt{np(1-p)}} \leq \frac{a-1-np}{\sqrt{np(1-p)}}\right] .$$

As before, $\ln(S^*/S) = j \ln(u/d) + n \ln d$, and the mean and variance of the continuously compounded rate of return of the stock are:

$$\hat{\mu}_p = p \ln(u/d) + \ln d \quad \text{and} \quad \hat{\sigma}_p^2 = p(1-p)[\ln(u/d)]^2 ,$$

where here these are defined in terms of probability p, not q. Using these equalities, a little algebra shows:

$$\frac{j - np}{\sqrt{np(1-p)}} = \frac{\ln(S^*/S) - \hat{\mu}_p n}{\hat{\sigma}_p \sqrt{n}} .$$

Recall from the binomial formula that

$$a - 1 = \ln(K/Sd^n)/\ln(u/d) + o(n) = [\ln(K/S) - n \ln d]/\ln(u/d) + o(n) ,$$

where the term $o(n)$ goes to zero as $n \to \infty$. Using this and the definitions of $\hat{\mu}_p$ and $\hat{\sigma}_p^2$, with a little algebra, we have:

$$\frac{a - 1 - np}{\sqrt{np(1-p)}} = \frac{\ln(K/S) - \hat{\mu}_p n + o(n)}{\hat{\sigma}_p \sqrt{n}} .$$

Putting these results together,

$$1 - B[a;n,p] = \text{Prob}\left[\frac{\ln(S^*/S) - \hat{\mu}_p n}{\hat{\sigma}_p \sqrt{n}} \leq \frac{\ln(K/S) - \hat{\mu}_p n + o(n)}{\hat{\sigma}_p \sqrt{n}}\right].$$

We are now in a position to apply the central limit theorem. First, we must check if the initial condition,

$$\frac{p|\ln u - \hat{\mu}_p|^3 + (1-p)|\ln d - \hat{\mu}_p|^3}{\hat{\sigma}_p \sqrt{n}} = \frac{p(1-p)^2 + p^2(1-p)}{\sqrt{np(1-p)}} \to 0 ,$$

as $n \to \infty$ is satisfied. By first recalling that $p \equiv (\hat{r} - d)/(u - d)$, and

then $\hat{r} = r^{t/n}$, $u = e^{\sigma\sqrt{t/n}}$, and $d = e^{-\sigma\sqrt{t/n}}$, it is possible to show that, as $n \to \infty$,

$$p \to \frac{1}{2} + \frac{1}{2}\left(\frac{\ln r - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\frac{t}{n}} .$$

As a result, the initial condition holds, and we are justified in applying the central limit theorem.

To do so, we need only evaluate[1] $\hat{\mu}_p n$ and $\hat{\sigma}^2_p n$ as $n \to \infty$.

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[1] A surprising feature of this evaluation is that although $p \neq q$ and thus $\hat{\mu}_p \neq \hat{\mu}$, and $\hat{\sigma}_p \neq \hat{\sigma}$, nonetheless $\hat{\sigma}_p\sqrt{n}$ and $\hat{\sigma}\sqrt{n}$ have the same limiting value as $n \to \infty$. By contrast, since $\mu \neq \ln r - (1/2\,\sigma^2)$, $\hat{\mu}_p n$ and $\hat{\mu} n$ do not. This results from the way we needed to specify u and d to obtain convergence to a lognormal distribution. Rewriting this as $\sigma\sqrt{t} = (\ln u)\sqrt{n}$, it is clear that the limiting value σ of the standard deviation does not depend on p or q, and hence must be the same for either. However, at any point before the limit, since

$$\hat{\sigma}^2 n = (\sigma^2 - \mu^2 \frac{t}{n})t \quad \text{and} \quad \hat{\sigma}^2_p n = \left[\sigma^2 - (\ln r - \frac{1}{2}\sigma^2)^2 \frac{t}{n}\right]t ,$$

$\hat{\sigma}$ and $\hat{\sigma}_p$ will generally have different values.

The fact that $\hat{\mu}_p n \to (\ln r - \frac{1}{2}\sigma^2)t$ can also be derived from the property of the lognormal distribution that

$$\ln E[S^*/S] = \mu_p t - \frac{1}{2}\sigma^2 t ,$$

where E and μ_p are measured with respect to probability p. Since $p = (\hat{r} - d)/(u - d)$, it follows that $\hat{r} = pu + (1 - p)d$. For independently distributed random variables, the expectation of a product equals the product of their expectations. Therefore,

$$E[S^*/S] = [pu + (1 - p)d]^n = \hat{r}^n = r^t .$$

Substituting r^t for $E[S^*/S]$ in the previous equation, we have:

$$\mu_p = \ln r - \frac{1}{2}\sigma^2 .$$

Examination of our discussion for parameterizing q shows that, as $n \to \infty$

$$\hat{\mu}_p n \to (\ln r - \tfrac{1}{2}\sigma^2)t \quad \text{and} \quad \hat{\sigma}_p \sqrt{n} \to \sigma\sqrt{t} \ .$$

For this application of the central limit theorem, then, since

$$\hat{z} = \frac{\ln(K/S) - \hat{\mu}_p n + o(n)}{\hat{\sigma}_p \sqrt{n}} \to z = \frac{\ln(K/S) - (\ln r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \ ,$$

we have

$$1 - B[a;n,p] \to N(z) = N\left[\frac{\ln(Kr^{-t}/S)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right] \ .$$

The final step in the argument is to use the symmetry property of the standard normal distribution that $1 - N(z) = N(-z)$. Therefore, as $n \to \infty$

$$B[a;n,p] \to N(-z) = N\left[\frac{\ln(S/Kr^{-t})}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right] = N(h - \sigma\sqrt{t}) \ .$$

Since a similar argument holds for $B[a;n,p']$, this completes our demonstration that the binomial option pricing formula contains the canonical formula as a limiting case.[1]

---

[1] In our original binomial development, our ability to create a riskless hedge led to the following equation (somewhat rewritten):

$$\left[\frac{\hat{r} - d}{u - d}\right] C_u + \left[\frac{u - \hat{r}}{u - d}\right] C_d - \hat{r}C = 0$$

if no riskless arbitrage were to be possible in the first period. By their more difficult methods, Black and Scholes obtained directly a partial differential equation analogous to our discrete-time difference equation. Their equation is:

As we have remarked, the seeds of both the canonical formula and a continuous-time jump process formula are both contained within the binomial formulation. At which end point we arrive depends on how we take limits. Suppose, in place of our former correspondence for u, d, and q, we instead set

$$u = u \quad , \quad d = e^{\zeta\sqrt{t/n}} \quad \text{and} \quad q = \lambda(t/n) \ .$$

This correspondence captures the essence of a pure jump process in which each successive stock price is almost always close to the previous price $(S \to dS)$, but occasionally, with low but continuing probability, significantly different $(S \to uS)$. Observe that as $n \to \infty$, the probability of a change by d becomes larger and larger, while the probability of a change by u approaches zero.

With these specifications, the initial condition of the central limit theorem is no longer satisfied, and it can be shown the stock price movements converge to a Poisson rather than a lognormal distribution as

---

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (\ln r)S\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - (\ln r)C = 0 \ .$$

The value C of the call was then derived by solving this equation subject to the boundary condition $C^* = \text{Max}[0, S^* - K]$.

Based on our previous analysis, we would now suspect that as $n \to \infty$, our binomial hedging equation would approach the Black-Scholes partial differential equation. This can be confirmed by substituting our definitons of $\hat{r}$, u, d in terms of n in the way described earier, expanding C_u, C_d in a Taylor series around $(e^{\sigma\sqrt{\Delta t}}S, t - \Delta t)$ and $(e^{-\sigma\sqrt{\Delta t}}S, t - \Delta t)$, respectively, and then expanding $e^{\sigma\sqrt{\Delta t}}$, $e^{-\sigma\sqrt{\Delta t}}$, and $r^{\Delta t}$ in a Taylor series, substituting these in the equation and collecting terms. All terms higher than order Δt go to zero if we then divide by Δt and let $\Delta t \to 0$. This yields the Black-Scholes equation.

$n \to \infty$. Let us define

$$\phi[h;y] = \sum_{i=h}^{\infty} \frac{e^{-y} y^i}{i!}$$

as the complementary Poisson distribution function. The limiting option pricing formula for the above specifications of u, d, and q is then:

JUMP PROCESS OPTION PRICING FORMULA

$$C = S\phi[h;y] - Kr^{-t}\phi[h;y/u]$$

$$\text{where} \quad y \equiv \frac{(\ln r - \zeta)ut}{u - 1}$$

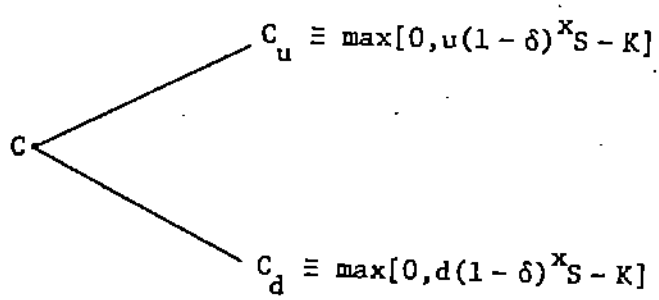
$$\frac{\ln(K/S) - \zeta t}{\ln u} \leq h < 1 + \frac{\ln(K/S) - \zeta t}{\ln u}$$

where h is an integer

VI. DIVIDENDS AND PUT PRICING

We now return to our binomial development of section III and relax the requirement that calls be protected against cash dividends. As we shall see, this produces some surprising complications. With one period remaining before expiration, we suppose the current stock price S will change either to $d(1-\delta)^x S$ or $u(1-\delta)^x S$ by the end of the period. $x = 0$ or 1, depending on whether or not the end of the period is an ex-dividend date. δ is the dividend yield at the end of the period if $x = 1$. Both it and x are assumed known with certainty. When the call expires, its contract and a rational exercise policy imply

that its value must either be $C_u \equiv \max[0, u(1-\delta)^x S - K]$ or $C_d \equiv \max[0, d(1-\delta)^x S - K]$. Therefore,



Again we hedge by writing one call against α shares of stock. The buyer of the call will either retain it until expiration or exercise it immediately. Suppose, first, he does not exercise early. In this case, following a series of formally equivalent steps[1] as in section III, we can show that:

$$C = [pC_u + (1-p)C_d] \div \hat{r} ,$$

where $p \equiv \frac{\hat{r} - d}{u - d}$ and the neutral hedge ratio $\alpha = \frac{C_u - C_d}{(u-d)S}$. In effect, $(1-\delta)^x S$ has simply replaced S in the values for C_u and C_d.

However, considering the possibility of early exercise, the call will be worth the larger of its current exercise value and its retention value. Therefore,

$$C = \max\{S - K, [pC_u + (1-p)C_d] \div \hat{r}\} .$$

---

[1]Remember that the hedger receives the dividend at the end of the period; that is, his stock position is worth either αuS or αdS.

It is easy to see early exercise may be optimal. Suppose S is sufficiently high that $d(1-\delta)^x S > K$. Since $u > d$, then, also, $u(1-\delta)^x S > K$. In this case, $C_u = u(1-\delta)^x S - K$ and $C_d = d(1-\delta)^x S - K$. Therefore, the above equation simplifies to:

$$C = \max\{S-K,\ [pu(1-\delta)^x S - pK + (1-p)d(1-\delta)^x S - (1-p)K] \div \hat{r}\} .$$

Since $(u/\hat{r})p + (d/\hat{r})(1-p) = 1$, this simplifies further to:

$$C = \max\{S-K,\ (1-\delta)^x S - (K/\hat{r})\} .$$

Presuming $\hat{r} \geq 1$, if there is no ex-dividend date prior to expiration (i.e., $x = 0$), then $C = S - (K/\hat{r})$ and early exercise is not optimal. However, if $\delta > 0$ and $x = 1$, then there always exists a critical stock price $\hat{S}$, such that if $S > \hat{S}$, the call should be immediately exercised.[1] This analysis shows that premature exercise of calls is more likely the higher the stock price, the higher the dividend yield, or the lower the interest rate, other things equal.

Unfortunately, unlike our analysis of payout-protected calls, the possibility of premature exercise seems to prohibit a simple formulation for the value of an unprotected call with many periods to go. However, our analysis suggests a sequential numerical procedure that will allow us to calculate the value of an unprotected call to any desired degree of accuracy.

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[1] A little algebra proves $\hat{S} = K(1-\hat{r}^{-1})/\delta$.

Let C be the current value of a call with n periods remaining until expiration. Let $C(n,i,j)$ be its value $n-i$ periods later, given that the current stock price S would have changed to $u^j d^{n-i-j} S$, ignoring dividends, where $j = 0,1,2,\ldots,n-i$. Define

$$\bar{x}(n,i) \equiv \sum_{k=1}^{n-i} x_k ,$$

so that $\bar{x}(n,i)$ is the number of ex-dividend dates occurring prior to period $n-i$.

With this notation, we are prepared to solve for the current value of the call by working backward in time from the expiration date. At expiration, $i = 0$, so that

$$C(n,0,j) = \max[0, u^j d^{n-j} (1-\delta)^{\bar{x}(n,0)} S - K] \quad \text{for} \quad j = 0,1,\ldots,n .$$

One period before the expiration date, $i = 1$ so that

$$C(n,1,j) = \max\{u^j d^{n-1-j} (1-\delta)^{\bar{x}(n,1)} S - K, \; [pC(n,0,j+1) + (1-p)C(n,0,j)] \div \hat{r}\}$$

$$\text{for} \quad j = 0,1,\ldots,n-1 .$$

More generally, i periods before expiration

$$C(n,i,j) = \max\{u^j d^{n-i-j} (1-\delta)^{\bar{x}(n,i)} S - K, \; [pC(n,i-1,j+1) + (1-p)C(n,i-1,j)] \div \hat{r}\}$$

$$\text{for} \quad j = 0,1,\ldots,n-i .$$

Observe that each prior step provides the inputs needed to evaluate the right-hand arguments of each succeeding step. The tree diagram in figure

1 illustrates this process. The number of calculations also reduces as we move backward in time.

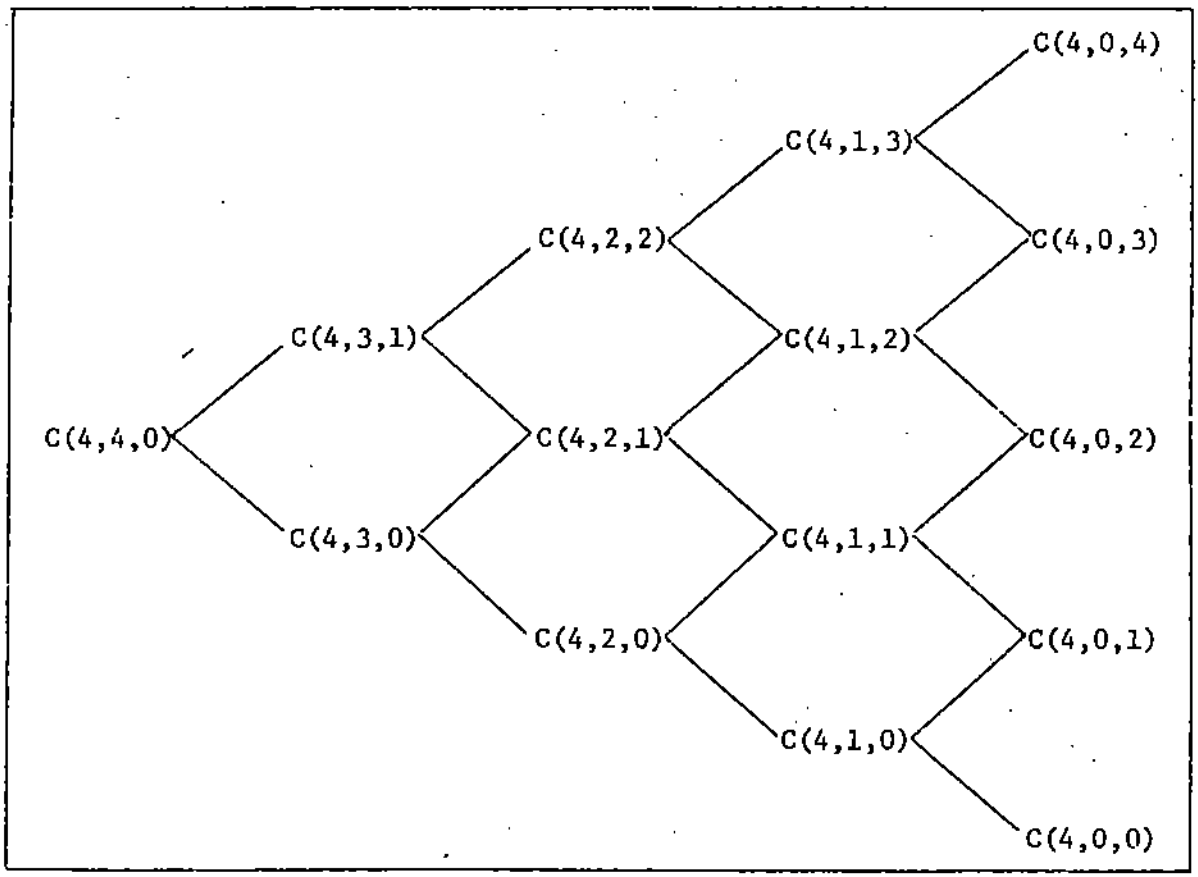


Fig. 1. Illustration of Binomial Numerical Procedure for Valuing Options Where Premature Exercise May Be Desirable

Finally, with n periods before expiration, since $i = n$,

$$C = C(n,n,0) = \max\{S - K, [pC(n,n-1,1) + (1-p)C(n,n-1,0)] \div \hat{r}\} ,$$

and the neutral hedge ratio is:

$$\alpha = \frac{C(n,n-1,1) - C(n,n-1,0)}{(u-d)S} .$$

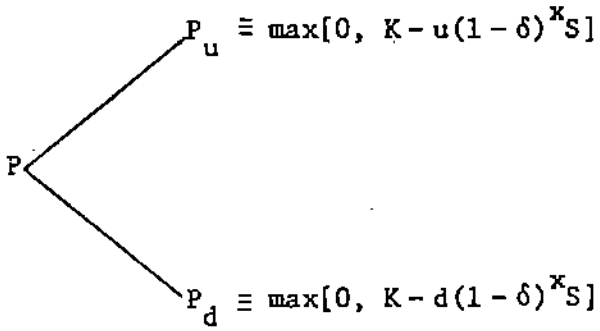
To illustrate, suppose we are given inputs required for the canonical option pricing formula: S, K, t, σ, and r. To convert this information into the inputs d, u, and $\hat{r}$ required for the binomial numerical procedure, we use the relationships:

$$d = 1/u \quad , \quad u = e^{\sigma\sqrt{t/n}} \quad , \quad \hat{r} = r^{t/n} .$$

For d, u, and $\hat{r}$ calculated in this manner, as we have shown, for fixed t, the resulting approximation of C improves as n increases, other things equal.

Table 2 gives us a feeling for how rapidly option values approximated by the binomial method approach the corresponding canonical values. At $n = 5$, the values differ by at most \$.25. At $n = 20$, they differ by at most \$.07, and at $n = 50$, the greatest difference is less than \$.03. Although not shown, at $n = 150$, the values are identical to the penny.

To derive a method for valuing puts, we return again to the binomial argument of section III. With one period remaining before expiration, with P denoting the present put price,



This time we hedge by writing one put against a *short* position of α shares of stock. This will "cost" $\alpha S - P$. Actually, since both terms

Table 2. Binomial Approximation of Canonical Call Values[a]

S = 40 r = 1.05

σ	K	n = 5			n = 20			n = 50		
		Expiration Month								
		JAN[b]	APR	JUL	JAN	APR	JUL	JAN	APR	JUL
.2	35	5.14	5.77	6.45	5.15	5.77	6.39	5.15	5.76	6.40
	40	1.05	2.26	3.12	.99	2.14	2.97	1.00	2.16	2.99
	45	.02	.54	1.15	.02	.51	1.11	.02	.51	1.11
.3	35	5.21	6.30	7.15	5.22	6.26	7.19	5.22	6.26	7.16
	40	1.53	3.21	4.36	1.44	3.04	4.14	1.45	3.06	4.17
	45	.11	1.28	2.12	.15	1.28	2.23	.16	1.26	2.24
.4	35	5.40	6.87	7.92	5.39	6.91	8.05	5.38	6.88	8.11
	40	2.01	4.16	5.61	1.90	3.93	5.31	1.91	3.96	5.35
	45	.46	1.99	3.30	.42	2.09	3.42	.42	2.11	3.45

[a]No adjustment for dividends.

[b]The January options have one month to expiration, the April's four months, and the July's seven months.

are negative, we initially take in money. The buyer of the put will either retain it until expiration or exercise it immediately. Suppose, first, he does not exercise it early. In this case, following a series of formally equivalent steps as in section III, we can show that

$$P = [pP_u + (1-p)P_d] \div \hat{r} ,$$

where $p \equiv \frac{\hat{r} - d}{u - d}$ and the neutral hedge ratio $\alpha = \frac{P_u - P_d}{(u-d)S}$. Observe that, for puts, since $P_u < P_d$, then $\alpha < 0$.

However, considering the possibility of early exercise, the put will be worth the larger of its exercise value and its retention value. Therefore, for a put,

$$P = \max\{K - S, [pP_u + (1-p)P_d] \div \hat{r}\} .$$

If $\hat{r} > 1$, it is easy to see early exercise may be optimal. Suppose that S is sufficiently low that $K > u(1-\delta)^x S$. Since $u > d$, then, also, $K > d(1-\delta)^x S$. In this case, $P_u = K - u(1-\delta)^x S$ and $P_d = K - d(1-\delta)^x S$. Therefore, since $(u/\hat{r})p + (d/\hat{r})(1-p) = 1$, the above equation simplifies to:

$$P = \max\{K - S, (K/\hat{r}) - (1-\delta)^x S\} .$$

If there are no dividends (i.e., $x = 0$), then, since $\hat{r} > 1$, $P = K - S$ and early exercise is optimal. In general, early exercise becomes more likely if the put is deep-in-the-money and the interest rate is high. It can be shown there always exists a critical stock price $\hat{S}$, such that if $S < \hat{S}$, the put should be exercised immediately. The effect of dividends yet to

be paid diminishes the advantages of immediate exercise, since the put buyer will be reluctant to sacrifice the forced declines in the stock price on future ex-dividend dates.

Our analysis also indicates that, with slight modification, we can value puts with the same numerical technique we use for calls. Reversing the difference between the stock price and the striking price at each stage is the only change.[1]

VII. CONCLUSION

It should now be clear that whenever stock price movements conform to a discrete binomial process, or to a limiting form of such a process, options can be priced solely on the basis of arbitrage considerations. Indeed, we could have significantly complicated the simple binomial process while still retaining this property. For example, u and d could

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[1] Michael Parkinson [1977] has suggested a similar numerical procedure based on a trinomial process, where the stock price can either increase, decrease, or remain unchanged. In fact, given the theoretical basis for the binomial numerical procedure provided, the numerical method can be generalized to permit $m + 1 \leq n$ jumps to new stock prices in each period. We can consider exercise only every m periods, using the binomial formula to leap across intermediate periods. In effect, this means permitting $m + 1$ possible new stock prices before exercise is again considered. That is, instead of considering exercise n times, we would only consider it about n/m times. For fixed t and m, as $n \to \infty$, option values will approach their canonical values.

This alternative procedure is interesting, since it may enhance computer efficiency. At one extreme, for dividend protected calls, setting $m + 1 = n$ gives the most efficient results. However, the more important the effect of potential early exercise and the greater the accuracy required, the most efficient results are achieved by setting $m = 1$, as in our description above.

have been deterministic functions of time. More significantly, the size of the relative up and down stock price movements over each period could have depended on the stock price at the beginning of the period.[1] However, if the up and down movements were to depend on any other random variable, not itself a deterministic function of the stock price, then our arbitrage argument would break down.

It is also possible to incorporate certain types of imperfections into the binomial option pricing approach, such as differential borrowing and lending rates and margin requirements. These can be shown to produce upper and lower bounds on option prices, outside of which riskless profitable arbitrage would be possible.

Since all existing preference-free option pricing results can be derived as limiting forms of a discrete binomial process, we might suspect that binomial stock price movements must be in some sense necessary, as well as sufficient, to derive option pricing formulas based solely on arbitrage considerations. Indeed, our earlier observation that a discrete three-state or trinomial process would cause our arbitrage argument to break down suggests just such a conclusion. While we have stopped short of supplying the formal arguments, it can be shown that for an option pricing formula to be derived solely from arbitrage

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[1] Of course, different option pricing formulas would result from these more complex stochastic processes. See Cox and Ross [1976] and Geske [forthcoming]. Nonetheless, all option pricing formulas in these papers can be derived as limiting forms of a properly specified discrete binomial process.

considerations,[1] the stock price must either follow discrete binomial movements or follow a limiting form of such a process.

This rounds out the principal conclusion of this paper: The simple binomial process is really the essential ingredient of option pricing by arbitrage. This is surprising, perhaps, given the mathematical complexities of some of the current models in this field. But it is reassuring to find such simple economic arguments at the heart of this new theory.

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[1] Merton's [1976] multi-state model, with both continuous and jump components, is a good example of a stock price process for which no exact option pricing formula is obtainable purely from arbitrage considerations. To obtain an exact formula, it is necessary to impose restrictions on investor preferences or on stochastic movements of other securities. For example, Rubinstein [1976] has been able to derive the canonical option pricing formula, under circumstances that do not admit arbitrage, by suitably restricting investor preferences.

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