体积形态连续介质有限变形理论—守恒律方程

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1　知识要素

1.1　质量守恒

按物质体输运定理, 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}}\rho\mathrm{d}\tau=\int_{\overset{t}{V}}(\dot{\rho}+\theta\rho)\,\mathrm{d}\tau=0,$$

则有 **Euler 型质量守恒微分方程**

$$\dot{\rho}+\theta\rho=0.$$

进一步考虑分量表示, 可有

$$\begin{aligned}&\frac{\partial\rho}{\partial t}(\boldsymbol{x},t)+\dot{x}^s\frac{\partial\rho}{\partial x^s}(\boldsymbol{x},t)+\rho\nabla_sV^s\\&=\frac{\partial\rho}{\partial t}(\boldsymbol{x},t)+\left(V^s-\left(\frac{\partial X}{\partial t}(\boldsymbol{x},t),\boldsymbol{g}^s\right)_{\mathbb{R}^3}\right)\frac{\partial\rho}{\partial x^s}(\boldsymbol{x},t)+\rho\nabla_sV^s\\&=\frac{\partial\rho}{\partial t}(\boldsymbol{x},t)+\nabla_s(\rho V^s)-\left(\frac{\partial X}{\partial t}(\boldsymbol{x},t),\boldsymbol{g}^s\right)_{\mathbb{R}^3}\frac{\partial\rho}{\partial x^s}(\boldsymbol{x},t)=0.\end{aligned}$$

另一方面, 可考虑

$$\int_{\overset{t}{V}}\rho\mathrm{d}\tau=\int_{\overset{\circ}{V}}\rho|\boldsymbol{F}|\mathrm{d}\tau=\int_{\overset{\circ}{V}}\overset{\circ}{\rho}\mathrm{d}\tau,$$

式中 $\overset{\circ}{\rho}(\boldsymbol{\xi})$ 为相对于初始物理构型的密度分布. 则有 **Lagrange 型质量守恒微分方程**

$$\rho|\boldsymbol{F}|=\overset{\circ}{\rho}(\boldsymbol{\xi},t).$$

1.2　动量守恒

首先考虑如下引理.

引理 1.1. 对 $\forall\boldsymbol{\Phi}\in\mathscr{T}^p(\mathbb{R}^3)$, 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}}\rho\boldsymbol{\Phi}\mathrm{d}\tau=\int_{\overset{t}{V}}\rho\dot{\boldsymbol{\Phi}}\mathrm{d}\tau.$$

证明

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}}\rho\boldsymbol{\Phi}\mathrm{d}\tau&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{\circ}{V}}\rho\boldsymbol{\Phi}|\boldsymbol{F}|\mathrm{d}\tau=\int_{\overset{\circ}{V}}\dot{\overline{\rho\boldsymbol{\Phi}|\boldsymbol{F}|}}\mathrm{d}\tau=\int_{\overset{\circ}{V}}\left[(\dot{\rho}+\theta\rho)\,\boldsymbol{\Phi}|\boldsymbol{F}|+\rho\dot{\boldsymbol{\Phi}}|\boldsymbol{F}|\right]\mathrm{d}\tau\\&=\int_{\overset{\circ}{V}}\rho\dot{\boldsymbol{\Phi}}|\boldsymbol{F}|\mathrm{d}\tau=\int_{\overset{t}{V}}\rho\dot{\boldsymbol{\Phi}}\mathrm{d}\tau.\end{aligned}$$

式中利用了 Euler 型质量守恒方程. □

由动量守恒关系式

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}} \rho \boldsymbol{V} \mathrm{d}\tau = \oint_{\partial \overset{t}{V}} \boldsymbol{t} \cdot \boldsymbol{n} \mathrm{d}\sigma + \int_{\overset{t}{V}} \rho \boldsymbol{f}_m \mathrm{d}\tau,$$

此处 $\boldsymbol{t} \in \mathscr{T}^2(\mathbb{R}^3)$ 为应力张量, $\boldsymbol{f}_m$ 为单位质量物质所受的体积力. 由上述引理以及 Gauss-Ostrogradskii 公式, 可有

$$\int_{\overset{t}{V}} \rho \boldsymbol{a} = \int_{\overset{t}{V}} \boldsymbol{t} \cdot \Box \mathrm{d}\tau + \int_{\overset{t}{V}} \rho \boldsymbol{f}_m \mathrm{d}\tau,$$

即有 **Euler 型动量守恒微分方程**

$$\rho \boldsymbol{a} = \Box \cdot \boldsymbol{t} + \rho \boldsymbol{f}_m.$$

另一方面, 考虑

$$\begin{aligned}
\oint_{\partial \overset{t}{V}} \boldsymbol{t} \cdot \boldsymbol{n} \mathrm{d}\sigma &= \int_{D_{\lambda\mu}} \boldsymbol{t} \cdot \left(\frac{\partial \overset{t}{\boldsymbol{\Sigma}}}{\partial \lambda} \times \frac{\partial \overset{t}{\boldsymbol{\Sigma}}}{\partial \mu} \right) (\lambda, \mu) \mathrm{d}\tau \\
&= \int_{D_{\lambda\mu}} \boldsymbol{t} \cdot \left[|\boldsymbol{F}| \boldsymbol{F}^{-*} \cdot \left(\frac{\partial \mathring{\boldsymbol{\Sigma}}}{\partial \lambda} \times \frac{\partial \mathring{\boldsymbol{\Sigma}}}{\partial \mu} \right) (\lambda, \mu) \right] \mathrm{d}\tau \\
&= \oint_{\partial \mathring{V}} (|\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}) \cdot \boldsymbol{N} \mathrm{d}\tau = \int_{\mathring{V}} (|\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}) \cdot \mathring{\Box} \mathrm{d}\tau,
\end{aligned}$$

故可有 **Lagrange 型动量守恒微分方程**

$$\mathring{\rho} \boldsymbol{a} = (|\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}) \cdot \mathring{\Box} + \mathring{\rho} \boldsymbol{f}_m =: \begin{cases} \boldsymbol{\tau} \cdot \mathring{\Box} + \mathring{\rho} \boldsymbol{f}_m, \\ (\boldsymbol{F} \cdot \boldsymbol{T}) \cdot \mathring{\Box} + \mathring{\rho} \boldsymbol{f}_m, \end{cases}$$

式中 $\boldsymbol{\tau} := |\boldsymbol{F}| \boldsymbol{t} \cdot \boldsymbol{F}^{-*}$ 称为**第一类 Piola-Kirchhoff 应力张量**, $\boldsymbol{T} := \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} = |\boldsymbol{F}| \boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-*}$ 称为**第二类 Piola-Kirchhoff 应力张量**.

1.3 动量矩守恒

由动量矩守恒关系式

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}} \boldsymbol{r} \times (\rho \boldsymbol{V}) \mathrm{d}\tau = \oint_{\partial \overset{t}{V}} \boldsymbol{r} \times (\boldsymbol{t} \cdot \boldsymbol{n}) \mathrm{d}\sigma + \int_{\overset{t}{V}} \boldsymbol{r} \times (\rho \boldsymbol{f}_m) \mathrm{d}\tau + \int_{\overset{t}{V}} \rho \boldsymbol{m} \mathrm{d}\tau,$$

此处 $\boldsymbol{m}$ 为单位质量连续介质所受的内力偶. 考虑到

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}} \boldsymbol{r} \times (\rho \boldsymbol{V}) \mathrm{d}\tau = \int_{\overset{t}{V}} \rho \dot{\overline{\boldsymbol{r} \times \boldsymbol{V}}} \mathrm{d}\tau = \int_{\overset{t}{V}} \rho \boldsymbol{r} \times \boldsymbol{a} \mathrm{d}\tau, \\
&\oint_{\partial \overset{t}{V}} \boldsymbol{r} \times (\boldsymbol{t} \cdot \boldsymbol{n}) \mathrm{d}\sigma = \oint_{\partial \overset{t}{V}} (\boldsymbol{r} \times \boldsymbol{t}) \cdot \boldsymbol{n} \mathrm{d}\sigma = \int_{\overset{t}{V}} (\boldsymbol{r} \times \boldsymbol{t}) \cdot \Box \mathrm{d}\tau,
\end{aligned}$$

则有 **Euler 型动量矩守恒微分方程**

$$\rho \boldsymbol{r} \times \boldsymbol{a} = (\boldsymbol{r} \times \boldsymbol{t}) \cdot \Box + \rho \boldsymbol{r} \times \boldsymbol{f}_m + \rho \boldsymbol{m}.$$

考虑到

$$
\begin{aligned}
(\boldsymbol{r}\times\boldsymbol{t})\cdot\Box &= (\boldsymbol{r}\times\boldsymbol{t})\cdot\left(\boldsymbol{g}^l\frac{\partial}{\partial x^l}\right)(\boldsymbol{x},t) \triangleq \frac{\partial}{\partial x^l}(\boldsymbol{r}\times\boldsymbol{t})(\boldsymbol{x},t)\cdot\boldsymbol{g}^l \\
&= \left[\frac{\partial\boldsymbol{r}}{\partial x^l}(\boldsymbol{x},t)\times\boldsymbol{t}+\boldsymbol{r}\times\frac{\partial\boldsymbol{t}}{\partial x^l}(\boldsymbol{x},t)\right]\cdot\boldsymbol{g}^l \\
&= (\boldsymbol{g}_l\times\boldsymbol{t})\cdot\boldsymbol{g}^l+\boldsymbol{r}\times\left[\frac{\partial\boldsymbol{t}}{\partial x^l}(\boldsymbol{x},t)\cdot\boldsymbol{g}^l\right] = (\boldsymbol{g}_l\times\boldsymbol{t})\cdot\boldsymbol{g}^l+\boldsymbol{r}\times(\boldsymbol{t}\cdot\Box),
\end{aligned}
$$

结合动量守恒方程, 则有动量矩守恒方程的最终形式

$$
\begin{aligned}
(\boldsymbol{g}_l\times\boldsymbol{t})\cdot\boldsymbol{g}^l+\rho\boldsymbol{m} &= \left[\boldsymbol{g}_l\times\left(t^{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j\right)\right]\cdot\boldsymbol{g}^l+\rho\boldsymbol{m} \\
&= \varepsilon_{lik}t^{il}\boldsymbol{g}^k+\rho\boldsymbol{m} = -\varepsilon_{ijk}t^{ij}\boldsymbol{g}^k+\rho\boldsymbol{m} = \boldsymbol{0}\in\mathbb{R}^3.
\end{aligned}
$$

上述表示, 当不考虑内力偶情形, 亦即 $\boldsymbol{m}=\boldsymbol{0}\in\mathbb{R}^3$, 则动量矩守恒等价于 $t^{ij}=t^{ji}$, 亦即应力张量为对称仿射量.

1.4 能量守恒

由能量守恒关系式

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overset{t}{V}}\rho\left(\frac{|\boldsymbol{V}|^2}{2}+e\right)\mathrm{d}\tau &= \oint_{\partial\overset{t}{V}}\boldsymbol{V}\cdot(\boldsymbol{t}\cdot\boldsymbol{n})\,\mathrm{d}\sigma+\int_{\overset{t}{V}}\boldsymbol{V}\cdot(\rho\boldsymbol{f}_m)\,\mathrm{d}\tau+\oint_{\partial\overset{t}{V}}(-k\Box T)\cdot\boldsymbol{n}\mathrm{d}\sigma \\
&\quad+\int_{\overset{t}{V}}\rho q_m\mathrm{d}\tau,
\end{aligned}
$$

此处 T 表示温度, k 表示传热系数, q_m 表示单位质量上的热源强度, 则有 **Euler 型能量守恒微分方程**

$$
\rho\left(\frac{\dot{\overline{|\boldsymbol{V}|^2}}}{2}+\dot{e}\right) = (\boldsymbol{V}\cdot\boldsymbol{t})\cdot\Box+\rho\boldsymbol{V}\cdot\boldsymbol{f}_m-\Box\cdot(k\Box T)+\rho q_m.
$$

进一步考虑

$$
\begin{aligned}
\rho\frac{\dot{\overline{|\boldsymbol{v}|^2}}}{2} &= \rho\boldsymbol{V}\cdot\boldsymbol{a}, \\
(\boldsymbol{V}\cdot\boldsymbol{t})\cdot\Box &= (\boldsymbol{V}\cdot\boldsymbol{t})\cdot\left(\boldsymbol{g}^l\frac{\partial}{\partial x^l}\right)(\boldsymbol{x},t) \\
&= \left[\frac{\partial\boldsymbol{V}}{\partial x^l}(\boldsymbol{x},t)\cdot\boldsymbol{t}\right]\cdot\boldsymbol{g}^l+\left[\boldsymbol{V}\cdot\frac{\partial\boldsymbol{t}}{\partial x^l}(\boldsymbol{x},t)\right]\cdot\boldsymbol{g}^l \\
&= (\nabla_l V_i)\,t^{il}+\boldsymbol{V}\cdot\left[\frac{\partial\boldsymbol{t}}{\partial x^l}(\boldsymbol{x},t)\cdot\boldsymbol{g}^l\right] \\
&= (\boldsymbol{V}\otimes\Box):\boldsymbol{t}+\boldsymbol{V}\cdot(\boldsymbol{t}\cdot\Box),
\end{aligned}
$$

综合 Euler 型动量守恒微分方程, 可有

$$
\rho\dot{e} = (\boldsymbol{V}\otimes\Box):\boldsymbol{t}-\Box\cdot(k\Box T)+\rho q_m.
$$

另考虑

$$
\begin{aligned}
&\int_{\overset{t}{V}} \rho\, \dot{\overline{\left(\frac{|\boldsymbol{V}|^2}{2}+e\right)}}\mathrm{d}\tau=\int_{\overset{\circ}{V}} \overset{\circ}{\rho}\,(\boldsymbol{V}\cdot\boldsymbol{a}+\dot{e})\,\mathrm{d}\tau,\\
&\int_{\overset{t}{V}} \boldsymbol{V}\cdot(\rho\boldsymbol{f}_m)\,\mathrm{d}\tau=\int_{\overset{\circ}{V}} \boldsymbol{V}\cdot\left(\overset{\circ}{\rho}\boldsymbol{f}_m\right)\mathrm{d}\tau,\\
&\int_{\overset{t}{V}} \rho q_m \mathrm{d}\tau=\int_{\overset{\circ}{V}} \overset{\circ}{\rho} q_m \mathrm{d}\tau,
\end{aligned}
$$

上述利用了 Lagrange 质量守恒微分方程 $\rho|\boldsymbol{F}|=\overset{\circ}{\rho}$, 则

$$
\begin{aligned}
\oint_{\partial\overset{t}{V}} \boldsymbol{V}\cdot(\boldsymbol{t}\cdot\boldsymbol{n})\,\mathrm{d}\sigma &=\int_{D_{\lambda\mu}} \boldsymbol{V}\cdot\boldsymbol{t}\cdot\left(\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu)\mathrm{d}\sigma=:\oint_{\partial\overset{\circ}{V}} (\boldsymbol{V}\cdot\boldsymbol{\tau})\cdot\boldsymbol{N}\mathrm{d}\sigma\\
&=\int_{\overset{\circ}{V}} (\boldsymbol{V}\cdot\boldsymbol{\tau})\cdot\overset{\circ}{\Box}\mathrm{d}\tau,\\
-\oint_{\partial\overset{t}{V}} (k\Box T)\cdot\boldsymbol{n}\mathrm{d}\sigma &=-\int_{D_{\lambda\mu}} (k\Box T)\cdot\left(\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{t}{\boldsymbol{\Sigma}}}{\partial\mu}\right)\mathrm{d}\sigma\\
&=-\int_{D_{\lambda\mu}} (k\Box T)\cdot\left(|\boldsymbol{F}|\boldsymbol{F}^{-*}\right)\cdot\left(\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\lambda}\times\frac{\partial\overset{\circ}{\boldsymbol{\Sigma}}}{\partial\mu}\right)(\lambda,\mu)\mathrm{d}\sigma\\
&\quad-\oint_{\partial\overset{\circ}{V}} \left[k|\boldsymbol{F}|\boldsymbol{F}^{-1}\cdot(\Box T)\right]\cdot\boldsymbol{N}\mathrm{d}\sigma=-\int_{\overset{\circ}{V}} \left[k|\boldsymbol{F}|\boldsymbol{F}^{-1}\cdot(\Box T)\right]\cdot\overset{\circ}{\Box}\mathrm{d}\tau\\
&=-\int_{\overset{\circ}{V}} \left[k|\boldsymbol{F}|\left(\boldsymbol{F}^*\boldsymbol{F}\right)^{-1}\cdot\left(\overset{\circ}{\Box}T\right)\right]\cdot\overset{\circ}{\Box}\mathrm{d}\tau,
\end{aligned}
$$

式中

$$
\begin{aligned}
\Box T&=\frac{\partial T}{\partial x^i}(\boldsymbol{x},t)\boldsymbol{g}^i=\frac{\partial T}{\partial \xi^A}(\boldsymbol{\xi},t)\frac{\partial \xi^A}{\partial x^i}(\boldsymbol{x},t)\boldsymbol{g}^i\\
&=\left[\frac{\partial T}{\partial \xi^A}(\boldsymbol{\xi},t)\boldsymbol{G}^A\right]\cdot\left[\frac{\partial \xi^B}{\partial x^i}\boldsymbol{G}_B\otimes\boldsymbol{g}^i\right]=\left(\overset{\circ}{\Box}T\right)\cdot\boldsymbol{F}^{-1}.
\end{aligned}
$$

综上, 则有

$$
\boldsymbol{V}\cdot\left(\overset{\circ}{\rho}\boldsymbol{a}\right)+\overset{\circ}{\rho}\dot{e}=(\boldsymbol{V}\cdot\boldsymbol{\tau})\cdot\overset{\circ}{\Box}+\boldsymbol{V}\cdot\left(\overset{\circ}{\rho}\boldsymbol{f}_m\right)-\left[k|\boldsymbol{F}|\left(\boldsymbol{F}^*\boldsymbol{F}\right)^{-1}\cdot\left(\overset{\circ}{\Box}T\right)\right]\cdot\overset{\circ}{\Box}+\overset{\circ}{\rho}q_m.
$$

再由

$$
\begin{aligned}
(\boldsymbol{V}\cdot\boldsymbol{\tau})\cdot\overset{\circ}{\Box}&=\left[\frac{\partial\boldsymbol{V}}{\partial\xi^L}(\boldsymbol{\xi},t)\cdot\boldsymbol{\tau}+\boldsymbol{V}\cdot\frac{\partial\boldsymbol{\tau}}{\partial\xi^L}(\boldsymbol{\xi},t)\right]\cdot\boldsymbol{G}^L\\
&=\left(\boldsymbol{V}\otimes\overset{\circ}{\Box}\right):\boldsymbol{\tau}+\boldsymbol{V}\cdot\left(\boldsymbol{\tau}\cdot\overset{\circ}{\Box}\right),
\end{aligned}
$$

结合 Lagrange 型动量守恒微分方程, 可有

$$
\overset{\circ}{\rho}\dot{e}=\left(\boldsymbol{V}\otimes\overset{\circ}{\Box}\right):\boldsymbol{\tau}-\left[k|\boldsymbol{F}|\left(\boldsymbol{F}^*\boldsymbol{F}\right)^{-1}\cdot\left(\overset{\circ}{\Box}T\right)\right]\cdot\overset{\circ}{\Box}+\overset{\circ}{\rho}q_m.
$$

由

$$\begin{aligned}\boldsymbol{V}\otimes\overset{\circ}{\square}=\frac{\partial\boldsymbol{V}}{\partial\xi^L}(\boldsymbol{\xi},t)\otimes\boldsymbol{G}^L&=\left[\frac{\partial\boldsymbol{V}}{\partial x^l}(\boldsymbol{x},t)\frac{\partial x^l}{\partial\xi^L}(\boldsymbol{\xi},t)\right]\otimes\boldsymbol{G}^L\\&=\left[\frac{\partial\boldsymbol{V}}{\partial x^l}(\boldsymbol{x},t)\otimes\boldsymbol{g}^l\right]\cdot\left[\frac{\partial x^i}{\partial\xi^L}(\boldsymbol{\xi},t)\boldsymbol{g}_i\otimes\boldsymbol{G}^L\right]\\&=(\boldsymbol{V}\otimes\square)\cdot\boldsymbol{F}=\boldsymbol{L}\cdot\boldsymbol{F},\end{aligned}$$

可有

$$\begin{aligned}\left(\boldsymbol{V}\otimes\overset{\circ}{\square}\right):\boldsymbol{\tau}=(\boldsymbol{LF}):(\boldsymbol{FT})&=\mathrm{tr}\left(\boldsymbol{F}^*\boldsymbol{L}^*\boldsymbol{FT}\right)\\&=\mathrm{tr}\left[\left(\boldsymbol{F}^*\boldsymbol{L}^*\boldsymbol{F}\right)^*\boldsymbol{T}^*\right]=\mathrm{tr}\left(\boldsymbol{F}^*\boldsymbol{LFT}\right)\\&=\mathrm{tr}\left(\boldsymbol{F}^*\boldsymbol{DFT}\right)=\left(\boldsymbol{F}^*\boldsymbol{DF}\right):\boldsymbol{T}.\end{aligned}$$

计算 Almansi 应变张量 $\boldsymbol{E}\triangleq\frac{1}{2}\left(\boldsymbol{F}^*\boldsymbol{F}-\boldsymbol{I}\right)$, 有

$$\dot{\boldsymbol{E}}=\frac{1}{2}\left(\dot{\boldsymbol{F}}^*\boldsymbol{F}+\boldsymbol{F}^*\dot{\boldsymbol{F}}\right)=\frac{1}{2}\left(\boldsymbol{F}^*\boldsymbol{L}^*\boldsymbol{F}+\boldsymbol{F}^*\boldsymbol{LF}\right)=\boldsymbol{F}^*\boldsymbol{DF},$$

故有

$$\left(\boldsymbol{V}\otimes\overset{\circ}{\square}\right):\boldsymbol{\tau}=\dot{\boldsymbol{E}}:\boldsymbol{T}.$$

综上, 有 **Lagrange 型能量守恒微分方程**

$$\overset{\circ}{\rho}\dot{e}=\dot{\boldsymbol{E}}:\boldsymbol{T}-\left[k|\boldsymbol{F}|\left(\boldsymbol{F}^*\boldsymbol{F}\right)^{-1}\cdot\left(\overset{\circ}{\square}T\right)\right]\cdot\overset{\circ}{\square}+\overset{\circ}{\rho}q_m.$$

此处 $\boldsymbol{T}\triangleq|\boldsymbol{F}|\boldsymbol{F}^{-1}\cdot\boldsymbol{t}\cdot\boldsymbol{F}^{-*}$ 为第二类 Piola-Kirchhoff 应力张量. $\overset{\circ}{\rho}e$ 可理解为单位体积固定质点系统所具有的内能.

2 应用事例

3 建立路径

- 守恒律方程的推导, 首先按自然界中的守恒律列出物质体上的积分关系式, 然后结合物质体输运定理获得积分型及微分型关系式. 本讲稿推导了质量守恒, 动量守恒, 动量矩守恒以及能量守恒的 Euler 型以及 Lagrange 型微分方程.

- 值得指出, 质量守恒因为同介质的物性完全无关, 故可隶属运动学, 而其它形式的守恒律方程则隶属动力学.