Discrete Mathematics

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Outline

- Review of partial order set
- Review of abstract algebra
- Lattice and Sublattice

Introduction

- Intensively explored area
  1. By 1960s, 1,500 papers and books
  2. By 1970s, 2,700 papers and books
  3. By 1980s, 3,200 papers and books
  4. By 1990s, 3,600 papers and books
- History
  1. By 1850, George Boole's attempt to formalize proposition logic.
  2. At the end of 19th century, Charles S. Pierce and Ernst Schröder
  3. Independently, Richar Dedekind.
  4. Until mid-1930's, Garrett Birkhoff developed general theory on lattice.

Partial order set(Poset)

Definition

Given a set A and a relation R on it, $< A, R >$ is called a partially ordered set(**poset** in brief) if R is *reflexive*, *antisymmetric* and *transitive*.

Poset

Definition

Given a poset $< A, \leq >$, we can define:

1. a is maximal if there does not exist $b \in A$ such that $a \leq b$.
2. a is minimal if there does not exist $b \in A$ such that $b \leq a$.
3. a is greatest if for every $b \in A$, we have $b \leq a$.
4. a is least if for every $b \in A$, we have $a \leq b$.

Poset

Definition

Given a poset $< A, \leq >$ and a set $S \subseteq A$.

1. $u \in A$ is a *upper bound* of S if $s \leq u$ for every $s \in S$.
2. $l \in A$ is a *lower bound* of S if $l \leq s$ for every $s \in S$.

Poset

Definition

Given a poset $< A, \leq >$ and a set $S \subseteq A$.

1. u is a *least upper bound* of S, $(LUB(S))$, if u is the upper bound of S and $u \leq u'$ for any other upper bound u' of S.
2. l is a *greatest lower bound* of S, $(GLB(S))$, if l is the upper bound of S and $l' \leq l$ for any other lower bound l' of S.

Theorem

A poset has at most one LUB or GLB.

Lattice

Definition

A *lattice* (structure) is a poset $< A, \leq >$ in which any two elements a, b have a $LUB(a, b)$ and a $GLB(a, b)$.

From now on, we define $a \cup b = LUB(a, b)$ and $a \cap b = GLB(a, b)$ in brief. We also call them join and meet respectively.

Representation Lattice

1. Hasse diagram
2. Joint/meet table

Lattice

Property

The Lattice has the following properties:

1. *Commutative:* $a \cap b = b \cap a, a \cup b = b \cup a.$
2. *Associative:* $(a \cap b) \cap c = a \cap (b \cap c), (a \cup b) \cup c = a \cup (b \cup c).$
3. *Idempotent:* $a \cap a = a, a \cup a = a.$
4. *Absorption:* $(a \cup b) \cap a = a, (a \cap b) \cup a = a.$

Property

A lattice could be divided into a join-semilattice and a meet-semilattice.

Semilattice

Definition

A semilattice is an algebra $\mathcal{S} = (S, *)$ satisfying, for all $x, y, z \in S$,

1. $x * x = x$,
2. $x * y = y * x$,
3. $x * (y * z) = (x * y) * z$.

Lattice

Definition

Given a algebra $\mathcal{L} = (L, \cap, \cup)$, it is a lattice if it subjects to:

1. $(L, \cap)$ and $(L, \cap)$ are two semilattices.
2. $(a \cup b) \cap a = a, (a \cap b) \cup a = a$.

Theorem

If L is any set in which there are two operation defined as $\cup$ and $\cap$ satisfying the last four properties, then L is a lattice.

Sublattice and extension

Definition

A subset S of a lattice L is called sublattice if it is closed under the operation $\cup$ and $\cap$.

Definition

If S is a sublattice of L, L is an extension of S.

Definition

The subset S of the lattice L is called *convex* if $a, b \in S, c \in L$, and $a \leq c \leq b$ imply that $c \in S$.

Sublattice

Theorem

Given two lattice L *and* L', *a bijection* $f : L \to L'$ *from* L *to* L' *is an isomorphism if and only if* $a \leq b$ *in* L *implies* $f(a) \leq f(b)$ *in* L'.

Next Class

- Special lattices
- Boolean algebra