Discrete Mathematics

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Review

- Deduction from promises
- Compactness theorem
- Application

Outline

- Application of compactness theorem
- Limits of propositional logic
- Predicates and quantifiers

Application of compactness theorem

Example

Given an infinite planar graph. If its every finite subgraph is k-colorable, then the graph itself is also k-colorable.

Application of compactness theorem

Solution: Let $p_{a,i}$ represent vertex a is colored with i. We can formulate a graph which is k-colorable with the following propositions.

1. $p_{a,1} \vee p_{a,2} \vee \cdots \vee p_{a,k}$, for every $a \in V$. It means every vertex could be colored with at least one of k colors.
2. $\neg(p_{a,i} \wedge p_{a,j}), 1 \leq i < j \leq k$ for all $a \in V$. It means $C_i \cap C_j = \emptyset$.
3. $\neg(p_{a,i} \wedge p_{b,i}), i = 1, \ldots, k$ for all aEb. It means no neighbors have the same color .

Application of compactness theorem

Example

Every set S can be (totally) ordered.

Application of compactness theorem

Theorem

An infinite tree with finite branch has an infinite path.

Expressive Power of PL

1. not.
2. and.
3. or.
4. if . . . then . . . .
5. Declarative sentences.

Expressive Power of PL

Example

If Socrates is a man then Socrates is mortal.

Solution (Propositional Logic)

1. *A: "Socrates is a man".*
2. *B: "Socrates is mortal".*
3. *We can represent the previous statement as $A \rightarrow B$.*
4. *If A is true, then we know B is true.*

Limits of PL

Example

Given two statements:"All men are mortal" and "Socrates is a man". What can we do?

Solution

1. *We all know the following statement holding, "Socrates is mortal".*
2. *If they are formalized as two proposition, nothing can be implied.*

Limits of PL

Example

The previous example can be abstracted further like

1. P: All As have property B.
2. Q: C is one of As.

Remark

1. *There is no relation between P and Q.*
2. *The flaw is that "is/in" relationship can not be expressed by PL.*
3. *Consider "there exists ...", "all ...", "among ...", and "only ...".*
4. *A richer language is needed.*

Limits of PL

Example

Consider a simple sentence, every student is younger than some instructor.

Solution

It is a declarative statement, which can be expressed by a proposition letter, say P.

Remark

1. *However, it means being a students, being a instructor and being younger than somebody else.*
2. *P fails to reflect the finer logic structure of it.*

Limits of PL

Solution (Continue)

Consider a special instance of this statement. Suppose Andy is a student and Paul is a instructor.
Let's introduce some predicates,which asserts something has some property.
Now we have

1. $S(Andy)$: *Andy is a student.*
2. $I(Paul)$:*Paul is an instructor.*
3. $Y(Andy, Paul)$: *Andy is younger than Paul.*

Limits of PL

Remark

1. *There are many instances. Too many symbols are needed.*
2. *Introduce variable, which can represent any students or instructors.*
3. *How about "every/all" and "some"?*
    1. $\exists$ *means "there exists", which is called existential quantifier.*
    2. $\forall$ *means "for all", which is called universal quantifiers.*

Solution (Continue)

Now it can be written as
$\forall x(S(x) \to \exists y(I(y) \land Y(x, y)))$.

Limits of PL

Example

Consider this sentence "Bobby's father can beat up the father of any other kid on the block".

Solution

Here the point is that how to express the connection between child and father. Function is proper than predicate because a child can determine his father. Let

1. $K(x)$: *x is a child on the block and b means Bobby.*
2. $f(x)$: *x's father,*
3. $B(x, y)$: *x can beat up y,*
4. *Finally,* $\forall x(K(x) \rightarrow (\neg(x = b) \rightarrow B(f(b), f(x))))$.

Predicates and Quantifiers

- *predicates*
- *variables*
- *constants*
- *functions*
- *universal quantifier*: $\forall$, "for all".
- *existential quantifier*: $\exists$, there exists

The Language

Definition (Language)

A *language* $\mathcal{L}$ consists of the following disjoint sets of distinct primitive symbols:

1. Variables: $x, y, x_0, x_1, \ldots, y_0, y_1, \ldots$ (an infinite set)
2. Constants: $c, d, c_0, d_0, \ldots$ (any set of them).
3. Connectives: $\wedge, \neg, \vee, \rightarrow, \leftrightarrow$
4. Quantifiers: $\forall, \exists$
5. Predicate symbols: $P, Q, R, P_1, P_2, \ldots$
6. Function symbols: $f, g, h, f_0, f_1, \ldots$
7. Punctuation: the comma , and (right and left) parentheses), (

Term

Definition (Term)

Terms.

1. Every variable is a term
2. Every constants symbol is a term.
3. If f is an n-ary function symbol ($n = 1, 2, \ldots$) and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is also a term.

Definition (Ground term)

Terms with no variables are called *variable-free terms* or *ground terms*.

Formula

Definition (Atomic formula)

An *atomic formula* is an expression of the form $R(t_1, \ldots, t_n)$ where R is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms.

Formula

Definition (Formula)

Formulas.

1. Every atomic formula is a formula.
2. If α, β are formulas, then so are $(\alpha \wedge \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta), (\neg\alpha), (\alpha \vee \beta)$.
3. If v is a variable and α is a formula, then $((\exists v)\alpha)$ and $((\forall v)\alpha)$ are also formulas.

Examples

Example

Let the domain consist of all relational numbers $\mathcal{Q}$.
Again $\varphi(x, y) = (x < y), f(x, y) = x + y$,
$g(x, y) = x \div y$ and $a = 0, b = 1, c = 2$ are constants.

1. $(\varphi(x, y) \wedge \varphi(y, z))$
2. $((\exists y)(\varphi(x, y) \wedge \varphi(y, z)))$
3. $((\forall x)(\varphi(x, z) \rightarrow ((\exists y)(\varphi(x, y) \wedge \varphi(y, z))))$
4. $((\forall x)((\forall y)(\varphi(x, y) \rightarrow (\varphi(x, g(f(x, y), c)) \wedge \varphi(g(f(x, y), c), y)))))$
5. $\varphi(y, f(y, y))$

Next Class

- Syntax of Predicate Logic