Discrete Mathematics

Yi Li

Software School
Fudan University

June 5, 2012

Review

- Semantics: Meaning and Truth
- Structure
- Relation between Predicate Logic and Propositional Logic
- Some Application

Outline

- Atomic tableaux
- Tableau proof
- Property of CST

Tableaux

- Signed sentence
- Entries of a tableaux
- How to deal with quantifiers?

Truth

Definition (Truth)

The *truth* of a sentence φ of $\mathcal{L}$ in a structure $\mathcal{A}$ in which every $a \in A$ is named by a ground term of $\mathcal{L}$ is defined by induction.

7. $\mathcal{A} \models \exists v \varphi(v) \Leftrightarrow$ for some ground term t, $\mathcal{A} \models \varphi(t)$.
8. $\mathcal{A} \models \forall v \varphi(v) \Leftrightarrow$ for all ground term t, $\mathcal{A} \models \varphi(t)$.

Quantifiers: Atomic Tableaux

$\mathcal{A} \models \exists v \varphi(v) \Leftrightarrow$ for some ground term t, $\mathcal{A} \models \varphi(t)$.

$$\begin{array}{c} T(\exists x)\varphi(x) \\ | \\ T\varphi(c) \end{array}$$

for a new constant c

Quantifiers: Atomic Tableaux

$\mathcal{A} \models \forall v \varphi(v) \Leftrightarrow$ for all ground term t, $\mathcal{A} \models \varphi(t)$.

$$\begin{array}{c} T(\forall x)\varphi(x) \\ | \\ T\varphi(t) \\ \text{for any ground term } t \text{ of } \mathcal{L}_{\mathcal{C}} \end{array}$$

Quantifiers: Atomic Tableaux

$$\begin{array}{c} F(\forall x)\varphi(x) \\ | \\ F\varphi(c) \end{array}$$

for a new constant c

$$\begin{array}{c} F(\exists x)\varphi(x) \\ | \\ F\varphi(t) \end{array}$$

for any ground term t of $\mathcal{L}_{\mathcal{C}}$

Tableaux: definition

We define *tableaux* as binary trees labeled with signed sentence(of $\mathcal{L}_{\mathcal{C}}$) called entries by induction.
Base step:

- All atomic tableaux are tableaux.
- In cases 7b and 8a, c is new simply means that c is one of the constants c_i added on to $\mathcal{L}$ to get $\mathcal{L}_{\mathcal{C}}$(which therefore does not appear in φ).

Tableaux: definition

Induction step:
if τ is a finite tableau, P a path on τ, E and entry of τ occurring on P.

- $\tau\prime$ is obtained from τ by adjoining an atomic tableau with root entry E to τ at the end of the path P, then $\tau\prime$ is also a tableau.
- Here the requirement that c be new in Case 7b and 8a means that it is one of the c_i that do not appear in any entries on P.
- In actual practice it is simpler in terms of bookkeeping to choose one not appearing at any nod of τ.

Tableaux: definition

If we have

1. τ_0 is a finite tableau.
2. $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ is a sequence of tableaux such that, for every $n \geq 0, \tau_{n+1}$ is constructed from τ_n by an application of induction step,

$\tau = \cup \tau_n$ is also a tableau.

Tableaux from S: definition

The definition for tableaux from S is the same as for ordinary tableaux except that we include an additional formation rule:

2. If τ is a finite tableau from S, φ a sentence from S, P a path on τ and $\tau\prime$ is obtained from τ by adjoining $T\varphi$ to the end of the path P, then $\tau\prime$ is also a tableau from S.

Tableau Proof

Definition

Tableau proofs (from S): Let τ be a tableau and P a path in τ.

1. P is *contradictory* if, for some sentence α, $T\alpha$ and $F\alpha$ both appear as labels of nodes of P.
2. τ is *contradictory* if every path on τ is contradictory.

Tableau Proof(Cont.)

Definition

3. τ is a *proof of* α (*from* S) if τ is a finite contradictory tableau (from S) with its root node labeled $F\alpha$. If there is proof τ of α (from S), we say α is provable (from S) and write $\vdash \alpha$ ($S \vdash \alpha$).
4. S is *inconsistent* if there is a proof of $\alpha \wedge \neg\alpha$ from S for some sentence α.

Tableau proof(Cont.)

Example

Check the formula $((\forall x)\varphi(x) \to (\exists x)\varphi(x))$.

Example

Check the formula
$(\forall x)(P(x) \to Q(x)) \to ((\forall x)P(x) \to (\forall x)Q(x))$

Example

Check the statement
$(\psi(x) \to (\exists x)\varphi(x)) \Rightarrow (\exists x)(\psi(x) \to \varphi(x))$

Tableau(Cont.)

Definition

Let $\tau = \cup \tau_n$ be a tableau (from S), P a path in τ, E an entry on P and ω the i^{th} occurrence of E on P (i.e., the i^{th} node on P labeled with E).

- ω is *reduced on P* if
  1. E is neither of the form $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$ and , for some j, τ_{j+1} is gotten from τ_j by an application Rule (ii) of Definition 1 to E and a path on τ_j which is an initial segment of P. (In this case, we say that E occurs on P as the root entry of an atomic tableau.)

Tableau(Cont.)

Definition

or

2. E is of the form $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, $T\varphi(t_i)$ or $F\varphi(t_i)$, respectively, is an entry on P and there is an $(i+1)^{st}$ occurrence of E on P.

Tableau(Cont.)

Definition

- τ is *finished* if every occurrence of every entry on τ is reduced on every noncontradictory path containing it (and $T\varphi$ appears on every noncontradictory path of τ for every φ in S). It is *unfinished* otherwise.

Complete Systematic Tableau(Cont.)

Definition

Suppose T is a tree with a left-right ordering on the nodes at each of its levels. Recall that if T is, for example, a tree of binary sequence, the left-right ordering is given by the usual lexicographic ordering. We define the *level-lexicographic ordering* on $\leq_{LL}$ on the nodes ν, μ of T as follows:

$\nu \leq_{LL} \Leftrightarrow$ the level of ν in T is less than that of μ or ν and μ are on the same level of T and ν is to the left of μ.

Complete Systematic Tableau

Definition

We construct the CST, the *complete systematic tableau*, with any given signed sentence as the label of its root, by induction.

1. We begin with τ_0 an atomic tableau with root the given signed sentence. This atomic tableau is uniquely specified by requiring that in Cases 7a and 8b we use the term t_i and that in Cases 7b and 8a we use c_i for the least allowable i.

Complete Systematic Tableau(Cont.)

Definition

2. If E is not of the form $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, we adjoin the atomic tableau with apex E to the end of every noncontradictory path in τ that contains ω. For E of the form $T(\exists x)\varphi(x)$ or $F(\forall x)\varphi(x)$, we use the least constant c_j not yet appearing in the tableau.

Complete Systematic Tableau(Cont.)

Definition

3. If E is of the form $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ and ω is the i^{th} occurrence of E on P we adjoin

$$\begin{array}{c} E \\ | \\ T\varphi(t_i) \end{array} \quad \text{or} \quad \begin{array}{c} E \\ | \\ F\varphi(t_i) \end{array}$$

respectively, to the end of every noncontradictory path in τ containing ω.

Property of CST

Proposition

Every CST is finished.

Next Class

- Soundness
- Completeness