Discrete Mathematics

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Review

- Formation tree
- Parsing algorithm

Outline

- Truth assignment
- Truth valuation
- Tautology
- Consequence

Truth Assignment

How we discuss the truth of propositional letters?

Definition (Assignment)

A *truth assignment* $\mathcal{A}$ is a function that assigns to each **propositional letter** A a unique truth value $\mathcal{A}(A) \in \{T, F\}$.

Truth Valuation

How we discuss the truth of propositions?

Example

Truth assignment of α and β and valuation of $(\alpha \vee \beta)$.

α	β	$(\alpha \vee \beta)$
T	T	T
T	F	T
F	T	T
F	F	F

Assignment and Valuation

Definition (Valuation)

A *truth valuation* $\mathcal{V}$ is a function that assigns to each **proposition** α a unique truth value $\mathcal{V}(\alpha)$ so that its value on a compund proposition is determined in accordance with the appropriate truth tables.

Specially, $\mathcal{V}(\alpha)$ determines one possible *truth assignment* if α is a propositional letter.

Assignment and Valuation

Theorem

Given a truth assignment $\mathcal{A}$ there is a unique truth valuation $\mathcal{V}$ such that $\mathcal{V}(\alpha) = \mathcal{A}(\alpha)$ for every propositonal letter α.

Proof.

The proof can be divided into two step.

1. Construct a $\mathcal{V}$ from $\mathcal{A}$ by induction on the depth of the associated formation tree.
2. Prove the uniqueness of $\mathcal{V}$ with the same $\mathcal{A}$ by induction bottom-up.

$\square$

Assignment and Valuation

Corollary

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two valuations that agree on the support of α, the finite set of propostional letters used in the construction of the proposition of the proposition α, then $\mathcal{V}_1(\alpha) = \mathcal{V}_2(\alpha)$.

Tautology

Definition

A proposition σ of propostional logic is said to be *valid* if for any valuation $\mathcal{V}, \mathcal{V}(\sigma) = T$. Such a proposition is also called a *tautology*.

Tautology

Example

$\alpha \vee \neg\alpha$ is a tautology.

Solution:

α	$\neg\alpha$	$\alpha \vee \neg\alpha$
T	F	T
F	T	T

$\square$

Logical Equivenlence

Definition

Two proposition α and β such that, for every valuation $\mathcal{V}, \mathcal{V}(\alpha) = \mathcal{V}(\beta)$ are called *logically equivalent*. We denote this by $\alpha \equiv \beta$.

Logical Equivenlence(Cont.)

Example

$\alpha \to \beta \equiv \neg\alpha \vee \beta$.

Proof.

Prove by truth table.

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

α	β	$\neg\alpha$	$\neg\alpha \vee \beta$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	T	T

$\square$

Consequence

Definition

Let Σ be a (possibly infinite) set of propositions. We say that σ is a *consequence* of Σ (and write as $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,

$$(\mathcal{V}(\tau) = T \text{ for all } \tau \in \Sigma) \Rightarrow \mathcal{V}(\sigma) = T.$$

Consequence

Example

1. Let $\Sigma = \{A, \neg A \vee B\}$, we have $\Sigma \models B$.
2. Let $\Sigma = \{A, \neg A \vee B, C\}$, we have $\Sigma \models B$.
3. Let $\Sigma = \{\neg A \vee B\}$, we have $\Sigma \not\models B$.

Model

Definition

We say that a valuation $\mathcal{V}$ is a *model* of Σ if $\mathcal{V}(\sigma) = T$ for every $\sigma \in \Sigma$. We denote by $\mathcal{M}(\Sigma)$ the set of all models of Σ.

Model

Example

Let $\Sigma = \{A, \neg A \vee B\}$, we have models:

1. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T$
2. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T, \mathcal{A}(C) = T$.
3. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T, \mathcal{A}(C) = F, \mathcal{A}(D) = F, \ldots$.

Model

Definition

We say that propositions Σ is *satisfiable* if it has some model. Otherwise it is called *unsatisfiable*. To a proposition, it is called *invalid*.

Properties

Proposition

Let $\Sigma, \Sigma_1, \Sigma_2$ *be sets of propositions. Let* $Cn(\Sigma)$ *denote the set of consequence of* Σ *and* $Taut$ *the set of tautologies.*

1. $\Sigma_1 \subseteq \Sigma_2 \Rightarrow Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$.
2. $\Sigma \subseteq Cn(\Sigma)$.
3. $Taut \subseteq Cn(\Sigma) = Cn(Cn(\Sigma))$.
4. $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \mathcal{M}(\Sigma_2) \subseteq \mathcal{M}(\Sigma_1)$.
5. $Cn(\Sigma) = \{\sigma | \mathcal{V}(\sigma) = T \text{ for all } \mathcal{V} \in \mathcal{M}(\Sigma)\}$.
6. $\sigma \in Cn(\{\sigma_1, \ldots, \sigma_n\}) \Leftrightarrow \sigma_1 \to (\sigma_2 \ldots \to (\sigma_n \to \sigma) \ldots) \in Taut$.

Deduction Theorem

Theorem

For any propostions φ, ψ, $\Sigma \cup \{\psi\} \models \varphi \Leftrightarrow \Sigma \models \psi \to \varphi$ *holds.*

Proof.

Prove by the definition of consequence. $\square$

Next Class

- Tableau proof system