Discrete Mathematics

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Review

- Tableau Proof
- Complete Systematic Tableaux

Outline

- Soundness
- Completeness
- Compactness

Tableau Proof

Example

Consider a sentence $(\exists y)(\neg R(y, y) \vee P(y, y)) \vee (\forall x)R(x, x)$. There is a model $\mathcal{A}$.

Soundness

Lemma

If $\tau = \cup\tau_n$ is a tableau from a set of sentences S with root $F\alpha$, then any $\mathcal{L}$-structure $\mathcal{A}$ that is a model of $S \cup \{\neg\alpha\}$ can be extended to one agreeing with every entry on some path P through τ.(Recall that $\mathcal{A}$ agrees with $T\alpha(F\alpha)$ if α is true(false) in $\mathcal{A}$.)

Theorem (Soundness)

If there is a tableau proof τ of α from S, then $S \models \alpha$.

Completeness

Theorem

Suppose P is a noncontradictory path through a complete systematic tableau τ from S with root $F\alpha$. There is then a structure $\mathcal{A}$ in which α is false and every sentence in S is true.

Completeness(Cont.)

Lemma

Let the notation be as above

1. *If* $F\beta$ *occurs on* P*, then* β *is false in* $\mathcal{A}$*.*
2. *If* $T\beta$ *occurs on* P*, then* β *is true in* $\mathcal{A}$*.*

Property of CST

Proposition

If every path of a complete systematic tableau is contradictory, then it is a finite tableau.

Property of CST

Corollary

For every sentence α and set of sentences S of $\mathcal{L}$, either

1. *the CST from S with root $F\alpha$ is a tableau proof of α from S.*

or

2. *there is a noncontradictory branch through the complete systematic tableau that yields a structure in that α is false and every element of S is true.*

Completeness and Soundness

Theorem (Completeness and Soundness)

1. *α is a tableau provable from $S \Leftrightarrow \alpha$ is a logical consequence of S.*
2. *If we take α to be any contradiction such as $\beta \wedge \neg\beta$ in 1, we see that S is inconsistent if and only if S is unsatisfiable.*

Size of model

Definition

The *size* of a model is the cardinality of the universe A in the structure $\mathcal{A}$.

Example

Let $\mathcal{A} =< \{c_0, c_1, \ldots, c_n\}, \{P = \{< c_0, c_0 >\}, R = \{< c_0, c_0 >, < c_1, c_1 >, \ldots, < c_n, c_n >\} >$. It is easy to check it is a model of

$$\alpha = (\exists y)(\neg R(y, y) \vee P(y, y)) \vee (\forall x)R(x, x)$$

.

Size of model

Example

Suppose we have a language $\mathcal{L} =< \{P,\}, \{f(x,y)\}, \{c,d\} >$. Given two sentences $(\forall x)P(c,x)$ and $(\forall x)(P(x,c) \to P(x,d))$.
We know that the structure $\mathcal{A} =< \mathcal{N}, \{P = \leq\}, \{f(x,y)\}, \{c = 0, d = 2\} >$ is a infinite model of them.

Size of model

Theorem (Löwenheim-Skolem)

If a countable set of sentences S is satisfiable, then it has a countable model.

Proof.

Consider the tableau proof with the root $F\alpha \wedge \neg\alpha$. □

Compactness

Theorem

Let $S = \{\alpha_1, \alpha_2, \ldots\}$ be a set of sentences of predicate logic. S is satisfiable if and only if every finite subset of S is satisfiable.

Proof.

Consider the tableau proof with the root $F(\alpha \wedge \neg\alpha)$.
The tree should not be finite □

Compactness

Theorem

Let L be a first-order language. Any set S of sentences of L that is satisfiable in arbitrarily large finite models is satisfiable in some infinite model.

Sketch Idea:

Suppose S is satisfiable in arbitrary large finite models.
Let R be a 2-ary relation symbol that is not part of the language L, and enlarge L to L' by adding R.
We can modify the interpretation of R without affecting the truth values of members of S, since R does not occur in members of S. So we can write a sentence A_n that asserts there are at least n thing.