Discrete Mathematics

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Review

- Soundness and Completeness Theorem
- Compactness Theorem
- Size of model
- Compactness theorem

Outline

- Application of Logic
- Limitation of First Order Logic

Application

Example (linear order)

A structure $\mathcal{A} =< A, <>$ is called an ordering if it is a model of the following sentences:

Solution.

$$\Phi_{ord} = \begin{cases} (\forall x)(\neg x < x), \\ (\forall x)(\forall y)(\forall z)((x < y \wedge y < z) \rightarrow x < z), \\ (\forall x)(\forall y)(x < y \vee x = y \vee y < x). \end{cases}$$

□

Application(Cont.)

Example (dense order)

In order to describe dense linear orders, we could add into linear order the following sentence

$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$$

Application(Cont.)

Example (Graphs)

Let $\mathcal{L} = \{R\}$ where R is a binary relation. We can characterize undirected irreflexive graphs with

1. $\forall x \neg R(x, x)$,
2. $\forall x \forall y (R(x, y) \rightarrow R(y, x))$.

Application(Cont.)

Example (Groups)

Let $\mathcal{L} = \{\cdot, e\}$ where $\cdot$ is a binary relation and e is a constant symbol. The class of group is described as

1. $\forall x e \cdot x = x \cdot e = x$,
2. $\forall x \forall y \forall z x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
3. $\forall x \exists y x \cdot y = y \cdot x = e$.

Application(Cont.)

Example (Equivalence relation)

The equivalence relation can be formalized with a single binary relation symbols as follows:

Solution.

$$\Phi_{equ} = \begin{cases} (\forall x)R(x,x), \\ (\forall x)(\forall y)(R(x,y) \to R(y,x), \\ (\forall x)(\forall y)(\forall z)((R(x,y) \wedge R(y,z)) \to R(x,z)). \end{cases}$$

$\square$

Application(Cont.)

Example

Suppose R is a binary relation. If it is non-trival, which means nonempty, transitive and symmetric, then it must be reflexive.

Solution.

We can represent these properties as:

1. $nontriv = (\forall x)(\exists y)R(x, y)$.
2. $sym = (\forall x)(\forall y)(R(x, y) \to R(y, x))$.
3. $ref = (\forall x)R(x, x)$.
4. $trans = (\forall x)(\forall y)(\forall z)((R(x, y) \wedge R(y, z)) \to R(x, z))$.

Then $\{trans, sym, nontriv\} \models ref$. □

Compactness

Theorem

Let L be a first-order language. Any set S of sentences of L that is satisfiable in arbitrarily large finite models is satisfiable in some infinite model.

Sketch Idea:

Suppose S is satisfiable in arbitrary large finite models. Let R be a 2-ary relation symbol that is not part of the language L, and enlarge L to L' by adding R.

We can modify the interpretation of R without affecting the truth values of members of S, since R does not occur in members of S. So we can write a sentence A_n that asserts there are at least n thing.

We can imply Theorem by applying Compactness Theorem. $\square$

Expressibility

Example

Let $\mathcal{L} = \{\cdot, +, <, 0, 1\}$ and $Th(\mathcal{N})$ be the full theory of $\mathcal{N}$. There is $M \models Th(\mathcal{N})$ and $a \in M$ such that a is larger than every member.

Proof.

Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. We can construct a set of sentence

$$S = \{\varphi_n = \underbrace{1 + 1 + \cdots + 1}_{n} < c, n \geq 1\}.$$

Then apply compactness theorem. □

Limitation

Example

The property of being strongly-connected is not a first-order property of directed graphs.

Proof.

Assume that sentence Φ_{SC} represents the property of being strongly-connected. Define sentences Φ_{SL}, Φ_{IN} and Φ_{out} as follows.

- Let $\Phi_{SL} = (\forall x)(\neg E(x, x))$.
- Let $\Phi_{OUT} = (\forall x)(\forall y)(\forall z)(E(x, y) \land E(x, z) \to y = z)$.
- Let $\Phi_{IN} = (\forall x)(\forall y)(\forall z)(E(y, x) \land E(z, x) \to y = z)$.

$\square$

Limitation(continue)

Proof.

(Continued) Let $\Phi = \Phi_{SC} \wedge \Phi_{SL} \wedge \Phi_{OUT} \wedge \Phi_{IN}$. Thus it describes the class of graphs that are strongly connected, have no self loops and have all vertices of in-degree and out-degree 1.

This is clearly the class of cycle graphs (of finite size). By the previous theorem, there must be a infinite graph satisfying Φ. But it is impossible.

The problem must be something wrong with Φ_{SC}. So it can not described by predict logic. $\square$

Upward Skolem-Löwenheim theorem

Theorem

If S has an infinite model. Then for every set A there is a model of S which contains at least as many elements as A.

Idea.

For each $a \in A$ let c_a be a new constant (i.e. $c_a \notin \mathcal{L}$) such that for distinct $a, b \in A$. We show that the set

$$S' = S \cup \{\neg(c_a = c_b)\}$$

of $\mathcal{L}_C$ where $C = \{c_a | a \in A\}$ is satisfiable. $\square$

About Examination

1. Propositional logic and predicate logic.
2. Open exam with two pieces of A4 cheat manuscript.