Discrete Mathematics

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Review

- Language
- Truth table
- Connectives

Outline

- Formation tree
- Parsing algorithm

Ambiguity

Example

Consider the following sentences:

1. The lady hit the man with an umbrella.
2. He gave her cat food.
3. They are looking for teachers of French, German and Japanese.

Ambiguity

Example

Consider the following proposition

$$A_1 \vee A_2 \wedge A_3.$$

We have two possible different propositions

1. $(A_1 \vee A_2) \wedge A_3$
2. $A_1 \vee (A_2 \wedge A_3)$

Of course, they have different abbreviated truth tables.

Count of Parentheses

Theorem

Every well-formed proposition has the same number of left as right parentheses.

Proof.

1. Consider the symbols without parentheses first.
2. And then prove it by induction with more complicated propositions according to the Definition.

□

Prefix

Theorem

Any proper initial segement of a well-defined proposition contains an excess of left parenthesiss. Thus no proper initial segement of a well defined propositon can itself be a well defined propositions.

Proof.

Prove it by induction from simple to complicated propositions. □

Formation Tree

Example

The formation tree of $(A \vee B), ((A \wedge B) \rightarrow C)$.

How?

Form a tree bottom-up while constructing the proposition according to the Definition.

Formation Tree

Definition (Top-down)

A *formation tree* is a finite tree T of binary sequences whose nodes are all labeled with propositions. The labeling satisfies the following conditions:

1. The leaves are labeled with propositional letters.
2. if a node σ is labeled with a proposition of the form $(\alpha \vee \beta), (\alpha \wedge \beta), (\alpha \rightarrow \beta)$ or $(\alpha \leftrightarrow \beta)$, its immediate successors, $\sigma\hat{\ }0$ and $\sigma\hat{\ }1$, are labeled with α and β (in that order).
3. if a node σ is labeled with a proposition of the form $(\neg\alpha)$, its unique immediate successor, $\sigma\hat{\ }0$, is labeled with α.

Formation Tree

Definition

1. The *depth* of a proposition is the depth of associated formation tree.
2. The *support* of a proposition is the set of propositional letters that occur as labels of the leaves of the associated formation tree.

Formation Tree

Theorem

Each well-defined proposition has a unique formation tree associated with it.

Proof.

1. Existence of the formation tree by induction on depth.
2. Uniqueness of the formation tree by induction on depth.

□

Parsing algorithm

1. Is an expression a well defined proposition?
2. How to check?
3. Build the formation tree.

Parsing algorithm

step 1 If all leaf nodes are labeled with proposition letters, stop it. Otherwise select a leaf node having expressions other than letter and examine it.

step 2 The first symbol must be (. if the second symbol is $\neg$, jump to step 4. Otherwise go to step 3.

step 3 Next slide.

step 4 The first two symbols are now known to be ($\neg$. The remainder of the expression, β) must consist of a an expression β and). Then we extend the tree by adding β as its immediate successor. Goto step 1.

Parsing algorithm

Step 3:

1. Scan the expression from the left until first reaching (α, where α is a nonempty expression having a balance between (and).
2. The α is the first of the two constituents.
3. The next symbol must be $\wedge, \vee, \rightarrow$, or $\leftrightarrow$.
4. The remainder of the expression, β) must consist of a an expression β and).
5. Extend the tree by adding α and β as left and right immediate successor respectively.

Parentheses

Method 1: define the following conventions:

1. The outermost parenthesis need not be explicitly mentioned.
2. The negation symbol applies to as little as possible.
3. The conjuction and disjunction symbols apply to as little as possible.
4. Where one connective symbols is used repeatedly, grouping is to the right.

Parentheses

Method 2: Polish notation.

1. $\mathcal{D}_{\neg}(\alpha) = \neg\alpha$.
2. $\mathcal{D}_{\vee}(\alpha, \beta) = \vee\alpha\beta$.
3. $\mathcal{D}_{\wedge}(\alpha, \beta) = \wedge\alpha\beta$.
4. $\mathcal{D}_{\rightarrow}(\alpha, \beta) = \rightarrow \alpha\beta$.
5. $\mathcal{D}_{\leftrightarrow}(\alpha, \beta) = \leftrightarrow \alpha\beta$.

Example

Determine the well defined proposition of

$$\rightarrow \wedge AD \vee \neg B \leftrightarrow CB.$$

Inductive definition

Definition

A set S is *closed* under a single operation $f(s_1, \ldots, s_n)$ if and only if every $s_1, \ldots, s_n \in S$, $f(s_1, \ldots, s_n) \in S$

Definition

The *closure* of a set S under (all) the operations in a set T is the smallest C such that

1. $S \subseteq C$ and
2. if $f \in T$ is n-ary and $s_1, \ldots, s_n \in C$, then $f(s_1, \ldots, s_n) \in C$.

Inductive definition

Theorem

Suppose that S is closed under the operations of T, there is a smallest closure C such that $S \subseteq C$.

Proof.

Given a set D, $S \subseteq D$ and D is closed under the operations of T. Consider the set

$$C = \cap\{D|S \subseteq D \wedge D \text{ is closed under the operations of } T\}.$$

It is obvious that $S \subseteq C$. And C is the smallest set. □

Inductive definition

Theorem

If S is a set of well defined propositions and is closed under all five connectives, then S is the set of all well defined propositions

Next Class

- Assignment
- Truth value
- Deduction theorem