Discrete Mathematics

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March 20, 2012

Review

- Introduction
- Tree
- König lemma

Outline

- Propositions
- Truth table
- Adequacy

Connectives

Example

Consider the following statements:

1. I am a student.
2. I am not a student.
3. I am a student and I study computer science.
4. I am a boy or I am a girl.
5. If I am a student, I have a class in a week.
6. I am student if and only if I am a member of some university.

Connectives

We don't care about the following:

- Are you a student?
- Sit down please.
- What are you doing?

Connectives

A summary of connectives:

Symbol	Verbose name	Remark
$\vee$	disjunction	or
$\wedge$	conjunction	and
$\neg$	negation	not
$\rightarrow$	conditional	if ..., then ...
$\leftrightarrow$	biconditional	if and only if

Language

- Symbols of propositional logic:
  1. Connectives: $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$
  2. Parentheses:), (
  3. Propositional Letters: $A, A_1, A_2, \cdots, B, B_1, B_2, \cdots$.
- A propositional letter is the most elementary object.

Propositions

Definition (Proposition)

1. Propositional letters are propositions.
2. if α and β are propositions, then $(\alpha \vee \beta), (\alpha \wedge \beta), (\neg\alpha), (\alpha \to \beta)$ *and* $(\alpha \leftrightarrow \beta)$ are propositions.
3. A string of symbols is a proposition if and only if it can be obtained by starting with propositional letters (1) and repeatedly applying (2).

Propositions

Definition

The proposition constructed according to the definition of Proposition is *well-defined* or *well-formed*.

Example

Check the following strings:

1. $(A \vee B), ((A \wedge B) \rightarrow C)$.
2. $A \vee \neg, (A \wedge B$

Truth Tables

α	β	$\alpha \vee \beta$
T	T	T
T	F	T
F	T	T
F	F	F

α	β	$\alpha \wedge \beta$
T	T	T
T	F	F
F	T	F
F	F	F

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Tables

α	β	$\alpha \leftrightarrow \beta$
T	T	T
T	F	F
F	T	F
F	F	T

α	$\neg\alpha$
T	F
F	T

Truth Tables

Example

Why do we let $\alpha \to \beta$ true when α is false?
Figure out what would happen if man can fly like a bird!

Truth Tables

Example

Consider the proposition, if $n > 2$, then $n^2 > 4$.

Solution.

We first all know that the statement is correct. Let $n = 3, 1, -3$. Consider the truth of the statement:

1. $n = 3$, true and true.
2. $n = 1$, false and false.
3. $n = -3$, false and true.

□

Connectives

Definition

A k-place *Boolean function* is a function from $\{F, T\}^k$ to $\{T, F\}$. We let F and T themselves to be 0-place Boolean functions.

Example

x_1	x_2	$x_1 \to x_2$	$f_\to(x_1, x_2)$
T	T	T	$f_\to(T, T) = T$
T	F	F	$f_\to(T, F) = F$
F	T	T	$f_\to(F, T) = T$
F	F	T	$f_\to(F, F) = T$

Connectives

Let $I_i(x_1, x_2, \ldots, x_n) = x_i$, which is a projection function of i-th parameter.

1. For each n, there are 2^{2^n} n-place Boolean functions.
2. 0-ary connectives: T and F.
3. Unary connectives: $\neg$, I , T and F.
4. Binary connectives: 10 of 16 are real binary functions.

Adequacy

Definition (Adequate connectives)

A set S of truth functional connectives is *adequate* if, given any truth function connective σ, we can find a proposition built up from the connectives is S with the same abbreviated truth table as σ.

Adequacy

Definition (Truth functional)

An n-ary connective is *truth functional* if the truth value for $\sigma(A_1, \ldots, A_n)$ is uniquely determined by the truth value of $A_1, \ldots, A_n$.

Theorem (Adequacy)

$\{\neg, \vee, \wedge\}$ *is adequate(complete).*

Proof.

Construct the truth table of any connective $\sigma(A_1, \ldots, A_k)$. □

Adequacy

Corollary

$\{\neg, \vee\}$ *is adequate.*

Normal Form

Definition (DNF)

α is called *disjunctive normal form* (abbreviated DNF). If α is a disjunction

$$\alpha = \gamma_1 \vee \cdots \vee \gamma_k,$$

where each γ_i is a conjunction

$$\gamma_i = \beta_{i1} \wedge \cdots \wedge \beta_{in_i}$$

and each β_{ij} is a proposition letter of the negation of a proposition letter.

Normal Form

Example

$\alpha = (A_1 \wedge A_2 \wedge A_3) \vee (\neg B_1 \wedge B_2) \vee (\neg C_1 \wedge \neg C_2 \wedge \neg C_3)$ is a DNF.

Normal Form

Definition (CNF)

α is called *conjunctive normal form* (abbreviated CNF). If α is a conjunction

$$\alpha = \gamma_1 \wedge \cdots \wedge \gamma_k,$$

where each γ_i is a disjunction

$$\gamma_i = \beta_{i1} \vee \cdots \vee \beta_{in_i}$$

and each β_{ij} is a proposition letter of the negation of a proposition letter.

Normal Form

Example

$\alpha = (A_1 \vee A_2 \vee A_3) \wedge (\neg B_1 \vee B_2) \wedge (\neg C_1 \vee \neg C_2 \vee \neg C_3)$ is a CNF.

Normal Form

Theorem

Any proposition can be reformed as a DNF and a CNF.

How?

Proof.

According to adequacy theorem. □

Next Class

- Formation tree
- Proposition parsing