Discrete Mathematics

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Review

- Truth assignment
- Truth valuation
- Tautology
- Consequence

Outline

- Tableau proof system

Terminologies

- signed proposition
- entries of the tableau
- atomic tableau

Tableau

Definition (Tableaux)

A *finite tableau* is a binary tree, labeled with signed propositions called entries, such that:

1. All atomic tableaux are finite tableaux.
2. If τ is a finite tableau, P a path on τ, E an entry of τ occurring on P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also a finite tableau.

If $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ is a (finite or infinite) sequence of the finite tableaux such that, for each $n \geq 0, \tau_{n+1}$ is constructed from τ_n by an application of (2), then $\tau = \cup \tau_n$ is a *tableau*.

Tableau

Example

A tableau with the signed proposition
$F(((\alpha \to \beta) \vee (\gamma \vee \delta)) \wedge (\alpha \vee \beta))$.

Tableau

Definition

Let τ be a tableau, P a path on τ and E an entry occurring on P.

1. E has been *reduced* on P if all the entries on one path through the atomic tableau with root E occur on P.
2. P is *contradictory* if, for some proposition α, $T\alpha$ and $F\alpha$ are both entries on P. P is *finished* if it is contradictory or every entry on P is reduced on P.
3. τ is *finished* if every path through τ is finished.
4. τ is *contradictory* if every path through τ is contradictory.

Proof

Definition

1. A *tableau proof* of a proposition α is a contradictory tableau with root entry $F\alpha$. A proposition is *tableau provable*, written $\vdash \alpha$, if it has a tableau proof.
2. A *tableau refutation* for a proposition α is a contradictory tableau starting with $T\alpha$. A proposition is *tableau refutable* if it has a tableau refutation.

Complete Systematic Tableaux

Definition (Complete systematic tableaux)

Let R be a signed proposition. We define the *complete systematic tableau*(CST) with root entry R by induction.

1. Let τ_0 be the unique atomic tableau with R at its root.
2. Assume that τ_m has been defined. Let n be the smallest level of τ_m and let E be the leftmost such entry of level n.
3. Let τ_{m+1} be the tableau gotten by adjoining the unique atomic tableau with root E to the end of every noncontradictory path of τ_m on which E is unreduced.

The union of the sequence τ_m is our desired complete systematic tableau.

Properties of CST

Theorem

Every CST is finished.

Proof.

Reduce the E level by level and there is no E unreduced for any fixed level. □

Properties of CST

Theorem

If $\tau = \cup\tau_n$ *is a contradictory tableau, then for some* m, τ_m *is a finite contradictory tableau. Thus, in particular, if a CST is a proof, it is a finite tableau.*

Proof.

By König lemma. □

Properties of CST

Definition

We define $d(\alpha)$, the degree of a proposition α by induction.

1. if α is a propositional letter, then $d(\alpha) = 0$.
2. if α is $\neg\beta$, then $d(\alpha) = d(\beta) + 1$.
3. if α is $\beta \vee \gamma, \beta \wedge \gamma,$ or $\beta \rightarrow \gamma$, then $d(\alpha) = d(\beta) + d(\gamma) + 1$.

The degree of a signed proposition $T\alpha$ or $F\alpha$ is the degree of α. If P is a path in a tableau τ, then $d(P)$ the degree of P is the sum of the degree of the signed propositions on P that are not reduced on P.

Properties of CST

Theorem

Every CST is finite.

Proof.

Every path is finite with $d(P_{m+1}) < d(P_m)$. □

Next Class

- Soundness theorem
- Completeness theorem