张量代数—置换运算及其应用

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1　知识要素

1.1　置换运算的定义

定义 1.1 (置换)**.** 置换可定义为一种改变有序元素组排列顺序的映照, 可有两种记法:

$$P_r \ni \sigma := \begin{bmatrix} 1 & 2 & \cdots & r \\ \sigma(1) & \sigma(2) & \cdots & \sigma(r) \end{bmatrix} = \begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ \sigma(i_1) & \sigma(i_2) & \cdots & \sigma(i_r) \end{pmatrix},$$

其中 $\{i_1, \cdots, i_r\}$ 为有序元素组的初始排列, $\{1, \cdots, r\}$ 为每一元素对应的初始序号.

$$\begin{aligned} \sigma : \{1, \cdots, r\} &\mapsto \{\sigma(1), \cdots, \sigma(r)\}, \\ \sigma : \{i_1, \cdots, i_r\} &\mapsto \{\sigma(i_1), \cdots, \sigma(i_r)\} \end{aligned}$$

$\{\sigma(i_1), \cdots, \sigma(i_r)\}$ 表示排序后的有序元素组, $\{\sigma(1), \cdots, \sigma(r)\}$ 表示排序后的原有序元素组的序号; 前者可称为置换的**元素定义**, 后者称为置换的**序号定义**. σ 称为r **阶置换**[①], 记作 $\sigma \in P_r$. 此外, 定义 $\operatorname{sgn}\sigma$ 称为置换 σ 的**符号**如下:

$$\operatorname{sgn}\sigma \triangleq \begin{cases} +1, & 将\sigma(1), \cdots, \sigma(r)恢复原本顺序需偶数次操作; \\ -1, & 将\sigma(1), \cdots, \sigma(r)恢复原本顺序需奇数次操作. \end{cases}$$

此处, 每次交换两个数字称为一次 “操作”.

以下以 7 阶置换为例, 给出

$$\begin{aligned}
P_7 \ni \sigma &= \begin{bmatrix} 3 & 5 & 8 & 2 & 6 & 9 & 4 \\ 8 & 4 & 2 & 9 & 5 & 3 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 4 & 6 & 2 & 1 & 5 \end{pmatrix}, & \operatorname{sgn}\sigma &= -1; \\
P_7 \ni \sigma^{-1} &= \begin{bmatrix} 8 & 4 & 2 & 9 & 5 & 3 & 6 \\ 3 & 5 & 8 & 2 & 6 & 9 & 4 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 1 & 3 & 7 & 4 & 2 \end{pmatrix}, & \operatorname{sgn}\sigma^{-1} &= -1; \\
P_7 \ni \tau &= \begin{bmatrix} 8 & 4 & 2 & 9 & 5 & 3 & 6 \\ 5 & 8 & 6 & 9 & 2 & 4 & 3 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 4 & 3 & 2 & 6 \end{pmatrix}, & \operatorname{sgn}\tau &= -1; \\
P_7 \ni \tau^{-1} &= \begin{bmatrix} 5 & 8 & 6 & 9 & 2 & 4 & 3 \\ 8 & 4 & 2 & 9 & 5 & 3 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 5 & 4 & 1 & 7 & 3 \end{pmatrix}, & \operatorname{sgn}\tau^{-1} &= -1; \\
P_7 \ni \tau \circ \sigma &= \begin{bmatrix} 3 & 5 & 8 & 2 & 6 & 9 & 4 \\ 5 & 8 & 6 & 9 & 2 & 4 & 3 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 6 & 4 & 7 & 1 \end{pmatrix}, & \operatorname{sgn}\tau \circ \sigma &= +1.
\end{aligned}$$

① 用方括号表示置换的序号定义, 用圆括号表示置换的元素定义.

1.2 置换运算的基本性质

性质 1.1 (置换运算的基本性质[①]). 置换运算的基本性质可归纳如下:

1. 对 $\forall\sigma\in P_r$, 有

$$A_{i_1j_1k_1}\cdots A_{i_rj_rk_r}=A_{\sigma(i_1)\sigma(j_1)\sigma(k_1)}\cdots A_{\sigma(i_r)\sigma(j_r)\sigma(k_r)};$$

2. 对 $\forall\tau\in P_r$, 有

$$\begin{aligned}\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}&=\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|\forall\,\sigma\in P_r\right\}\\&=\{(\sigma\circ\tau(i_1),\cdots,\sigma\circ\tau(i_r))\,|\forall\,\sigma\in P_r\}\\&=\{(\tau\circ\sigma(i_1),\cdots,\tau\circ\sigma(i_r))\,|\forall\,\sigma\in P_r\};\end{aligned}$$

3. 对 $\forall\sigma\in P_r$, 有

$$\begin{aligned}\{(i_1,\cdots,i_r)\,|i_1,\cdots,i_r=1,\cdots,m\}&=\{(\sigma(i_1),\cdots,\sigma(i_r))\,|i_1,\cdots,i_r=1,\cdots,m\}\\&=\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|i_1,\cdots,i_r=1,\cdots,m\right\}.\end{aligned}$$

证明 按证明集合相等的方法, 易于证明置换的基本性质.

1. 实际考虑

$$\begin{aligned}\sigma=\begin{bmatrix}1&2&3&4&5&6&7\\3&7&4&6&2&1&5\end{bmatrix}&\sim\begin{pmatrix}i_1&i_2&i_3&i_4&i_5&i_6&i_7\\i_3&i_7&i_4&i_6&i_2&i_1&i_5\end{pmatrix}\\&\sim\begin{pmatrix}j_1&j_2&j_3&j_4&j_5&j_6&j_7\\j_3&j_7&j_4&j_6&j_2&j_1&j_5\end{pmatrix}\sim\begin{pmatrix}k_1&k_2&k_3&k_4&k_5&k_6&k_7\\k_3&k_7&k_4&k_6&k_2&k_1&k_5\end{pmatrix},\end{aligned}$$

那么就有

$$A_{\sigma(i_1)\sigma(j_1)\sigma(k_1)}\cdots A_{\sigma(i_r)\sigma(j_r)\sigma(k_r)}=A_{i_3j_3k_3}A_{i_7j_7k_7}A_{i_4j_4k_4}A_{i_6j_6k_6}A_{i_2j_2k_2}A_{i_1j_1k_1}A_{i_5j_5k_5}.$$

实际上此性质即为改变相乘的次序.

2. 考虑到 $\sigma^{-1}\in P_r$, 故有

$$\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|\forall\,\sigma\in P_r\right\}\subset\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}.$$

另考虑到对 $\forall\sigma\in P_r$, 有

$$\begin{aligned}(\sigma(i_1),\cdots,\sigma(i_r))&=\left((\sigma^{-1})^{-1}(i_1),\cdots,(\sigma^{-1})^{-1}(i_r)\right)\\&\in\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|\forall\,\sigma\in P_r\right\},\end{aligned}$$

① 相关结果发表于：谢锡麟. 基于郭仲衡先生现代张量分析及有限变形理论知识体系的相关研究. 力学季刊, 2013, 34(2):337-351.

所以

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}\subset\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|\forall\,\sigma\in P_r\right\},$$

即有

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}=\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|\forall\,\sigma\in P_r\right\}.$$

考虑到 $\sigma\circ\tau\in P_r$, 故

$$\{(\sigma\circ\tau(i_1),\cdots,\sigma\circ\tau(i_r))\,|\forall\,\sigma\in P_r\}\subset\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}.$$

另考虑到对 $\forall\,\tau\in P_r$, 有

$$(\sigma(i_1),\cdots,\sigma(i_r))=\left(\sigma\circ\tau^{-1}\circ\tau(i_1),\cdots,\sigma\circ\tau^{-1}\circ\tau(i_r)\right),$$

可有

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}\subset\{(\sigma\circ\tau(i_1),\cdots,\sigma\circ\tau(i_r))\,|\forall\,\sigma\in P_r\},$$

故

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}=\{(\sigma\circ\tau(i_1),\cdots,\sigma\circ\tau(i_r))\,|\forall\,\sigma\in P_r\}.$$

考虑到 $\tau\circ\sigma\in P_r$, 于是

$$\{(\tau\circ\sigma(i_1),\cdots,\tau\circ\sigma(i_r))\,|\forall\,\sigma\in P_r\}\subset\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}.$$

另考虑到对 $\forall\,\sigma\in P_r$, 有

$$(\sigma(i_1),\cdots,\sigma(i_r))=\left(\tau\circ\tau^{-1}\circ\sigma(i_1),\cdots,\tau\circ\tau^{-1}\circ\sigma(i_r)\right),$$

则

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}\subset\{(\tau\circ\sigma(i_1),\cdots,\tau\circ\sigma(i_r))\,|\forall\,\sigma\in P_r\},$$

故

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\forall\,\sigma\in P_r\}=\{(\tau\circ\sigma(i_1),\cdots,\tau\circ\sigma(i_r))\,|\forall\,\sigma\in P_r\}.$$

3. 显然有, 对 $\forall\,\sigma\in P_r$, 满足

$$(\sigma(i_1),\cdots,\sigma(i_r))\in\{(i_1,\cdots,i_r)\,|i_1,\cdots,i_r=1,\cdots,m\},$$

即

$$\{(\sigma(i_1),\cdots,\sigma(i_r))\,|i_1,\cdots,i_r=1,\cdots,m\}\subset\{(i_1,\cdots,i_r)\,|i_1,\cdots,i_r=1,\cdots,m\}.$$

另考虑到对 $i_1,\cdots,i_r=1,\cdots,m$, 有

$$(i_1,\cdots,i_r)=\left(\sigma\circ\sigma^{-1}(i_1),\cdots,\sigma\circ\sigma^{-1}(i_r)\right),$$

由此即有

$$\{(i_1,\cdots,i_r)\,|i_1,\cdots,i_r=1,\cdots,m\}\subset\{(\sigma(i_1),\cdots,\sigma(i_r))\,|i_1,\cdots,i_r=1,\cdots,m\},$$

故

$$\{(i_1,\cdots,i_r)\,|\,i_1,\cdots,i_r=1,\cdots,m\}=\{(\sigma(i_1),\cdots,\sigma(i_r))\,|\,i_1,\cdots,i_r=1,\cdots,m\}.$$

同理可得

$$\{(i_1,\cdots,i_r)\,|\,i_1,\cdots,i_r=1,\cdots,m\}\subset\left\{\left(\sigma^{-1}(i_1),\cdots,\sigma^{-1}(i_r)\right)|\,i_1,\cdots,i_r=1,\cdots,m\right\}.\ \square$$

值得指出, 实际分析/计算中涉及置换运算的操作不外乎上述基本性质.

2 应用事例

定义 2.1 (矩阵)**.** 下面的结构被称为**矩阵**

$$\boldsymbol{A}:=\begin{pmatrix}A_{i_1j_1} & A_{i_1j_2} & \cdots & A_{i_1j_n}\\ A_{i_2j_1} & A_{i_2j_2} & \cdots & A_{i_2j_n}\\ \vdots & \vdots & & \vdots\\ A_{i_mj_1} & A_{i_mj_2} & \cdots & A_{i_mj_n}\end{pmatrix}\in\mathbb{R}^{m\times n},$$

其中 $i_1,\cdots,i_m\in\mathbb{N}$, $j_1,\cdots,j_n\in\mathbb{N}$. 矩阵 $\boldsymbol{A}$ 的行数和列数分别为 m 和 n, 相应的矩阵可以表示为 $\boldsymbol{A}\in\mathbb{R}^{m\times n}$.

定义 2.2 (矩阵的行列式[①])**.** 对于任意方阵 $\boldsymbol{A}\in\mathbb{R}^{m\times m}$,

$$\boldsymbol{A}:=\begin{pmatrix}A_{i_1j_1} & \cdots & A_{i_1j_m}\\ \vdots & & \vdots\\ A_{i_mj_1} & \cdots & A_{i_mj_m}\end{pmatrix}\in\mathbb{R}^{m\times m},$$

它的行列式记作

$$\det\boldsymbol{A}=|\boldsymbol{A}|=\begin{vmatrix}A_{i_1i_1} & \cdots & A_{i_1i_m}\\ \vdots & & \vdots\\ A_{i_mi_1} & \cdots & A_{i_mi_m}\end{vmatrix}.$$

可有以下 3 种等价性定义

$$\det\boldsymbol{A}=\begin{cases}\displaystyle\sum_{\sigma\in P_m}\operatorname{sgn}\sigma A_{i_1\sigma(j_1)}\cdots A_{i_m\sigma(j_m)}, & \text{(行置换)}\\ \displaystyle\sum_{\sigma\in P_m}\operatorname{sgn}\sigma A_{\sigma(i_1)j_1}\cdots A_{\sigma(i_m)j_m}, & \text{(列置换)}\\ \displaystyle\frac{1}{m!}\sum_{\sigma,\tau\in P_m}(\operatorname{sgn}\sigma\cdot\operatorname{sgn}\tau)A_{\sigma(i_1)\tau(j_1)}\cdots A_{\sigma(i_m)\tau(j_m)}. & \text{(行列置换)}\end{cases}$$

① 矩阵的行列式即隐含了此矩阵是方阵的含义, 即该矩阵的行数和列数是相等的.

如考虑 $\det \boldsymbol{A} \triangleq \sum\limits_{\sigma\in P_m} \operatorname{sgn}\sigma A_{i_1\sigma(j_1)}\cdots A_{i_m\sigma(j_m)}$, 可有

$$\begin{aligned}\det \boldsymbol{A} &\triangleq \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{i_1\sigma(j_1)}\cdots A_{i_m\sigma(j_m)} = \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma^{-1}(i_1)j_1}\cdots A_{\sigma^{-1}(i_m)j_m}\\ &= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma^{-1} A_{\sigma^{-1}(i_1)j_1}\cdots A_{\sigma^{-1}(i_m)j_m} = \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(i_1)j_1}\cdots A_{\sigma(i_m)j_m}.\end{aligned}$$

进一步, 可有

$$\begin{aligned}\det \boldsymbol{A} &= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(i_1)j_1}\cdots A_{\sigma(i_m)j_m}\\ &= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\tau\circ\sigma(i_1)\tau(j_1)}\cdots A_{\tau\circ\sigma(i_m)\tau(j_m)}, \quad \forall\,\tau\in P_m\\ &= \sum_{\sigma\in P_m} \operatorname{sgn}\tau(\operatorname{sgn}\tau\cdot\operatorname{sgn}\sigma) A_{\tau\circ\sigma(i_1)\tau(j_1)}\cdots A_{\tau\circ\sigma(i_m)\tau(j_m)}, \quad \forall\,\tau\in P_m\\ &= \sum_{\sigma\in P_m} \operatorname{sgn}\tau\cdot\operatorname{sgn}\sigma A_{\sigma(i_1)\tau(j_1)}\cdots A_{\sigma(i_m)\tau(j_m)}, \quad \forall\,\tau\in P_m\\ &= \frac{1}{m!}\sum_{\sigma\in P_m}\sum_{\tau\in P_m} (\operatorname{sgn}\sigma\cdot\operatorname{sgn}\tau) A_{\sigma(i1)\tau(j1)}\cdots A_{\sigma(im)\tau(jm)}.\end{aligned}$$

根据行列式的置换运算定义, 可方便地得到如下行列式的基本性质.

性质 2.1. 交换方阵的行 (或列) 奇数次, 其行列式变号；交换方阵的行 (或列) 偶数次, 其行列式不变号. 即

$$\begin{vmatrix} A_{\tau(1)1} & \cdots & A_{\tau(1)m}\\ \vdots & & \vdots\\ A_{\tau(m)1} & \cdots & A_{\tau(m)m}\end{vmatrix} = \begin{vmatrix} A_{1\tau(1)} & \cdots & A_{1\tau(m)}\\ \vdots & & \vdots\\ A_{m\tau(1)} & \cdots & A_{m\tau(m)}\end{vmatrix} = \operatorname{sgn}\tau \begin{vmatrix} A_{11} & \cdots & A_{1m}\\ \vdots & & \vdots\\ A_{m1} & \cdots & A_{mm}\end{vmatrix}.$$

证明 根据行列式的定义, 有

$$\begin{aligned}
\begin{vmatrix} A_{\tau(1)1} & \cdots & A_{\tau(1)m} \\ \vdots & & \vdots \\ A_{\tau(m)1} & \cdots & A_{\tau(m)m} \end{vmatrix} &= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(\tau(1))1}\cdots A_{\sigma(\tau(m))m} \\
&= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma\circ\tau(1)1}\cdots A_{\sigma\circ\tau(m)m} \\
&= \operatorname{sgn}\tau \sum_{\sigma\in P_m} (\operatorname{sgn}\sigma\cdot\operatorname{sgn}\tau) A_{\sigma\circ\tau(1)1}\cdots A_{\sigma\circ\tau(m)m} \\
&= \operatorname{sgn}\tau \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots A_{\sigma(m)m} = \operatorname{sgn}\tau \det \boldsymbol{A};
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} A_{1\tau(1)} & \cdots & A_{1\tau(m)} \\ \vdots & & \vdots \\ A_{m\tau(1)} & \cdots & A_{m\tau(m)} \end{vmatrix} &= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)\tau(1)}\cdots A_{\sigma(m)\tau(m)} \\
&= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(\tau^{-1}(1))1}\cdots A_{\sigma(\tau^{-1}(m))m} \\
&= \operatorname{sgn}\tau^{-1} \sum_{\sigma\in P_m} (\operatorname{sgn}\sigma\cdot\operatorname{sgn}\tau^{-1}) A_{\sigma\circ\tau^{-1}(1)1}\cdots A_{\sigma\circ\tau^{-1}(m)m} \\
&= \operatorname{sgn}\tau \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots A_{\sigma(m)m} = \operatorname{sgn}\tau \det \boldsymbol{A}.
\end{aligned}$$

□

利用此性质, 可得下面的性质.

性质 2.2. 有两行 (或列) 完全相同的行列式值为零.

$$\begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{i1} & \cdots & A_{im} \\ \vdots & & \vdots \\ A_{i1} & \cdots & A_{im} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix} = \begin{vmatrix} A_{11} & \cdots & A_{1i} & \cdots & A_{1i} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & A_{mi} & \cdots & A_{mi} & \cdots & A_{mm} \end{vmatrix} = 0.$$

性质 2.3. 方阵的某一行 (或列) 乘以一系数所得方阵之行列式等于原方阵的行列式乘以该系数.

$$\begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ \lambda A_{i1} & \cdots & \lambda A_{im} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix} = \begin{vmatrix} A_{11} & \cdots & \lambda A_{1j} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & \lambda A_{mj} & \cdots & A_{mm} \end{vmatrix} = \lambda \begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix}.$$

证明 根据行列式的定义, 有

$$\begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ \lambda A_{i1} & \cdots & \lambda A_{im} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix} = \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots(\lambda A_{\sigma(i)i})\cdots A_{\sigma(m)m}$$

$$= \lambda \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots A_{\sigma(i)i}\cdots A_{\sigma(m)m} = \lambda \det \boldsymbol{A}.$$

对于列的情况类似可证. □

性质 2.4. 将方阵的某一行 (或列) 整体乘以一系数加到其他行 (或列) 上, 其行列式不变. 即当 $i \neq j$ 时, 有

$$\begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{i1}+\lambda A_{j1} & \cdots & A_{im}+\lambda A_{jm} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix} = \begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix},$$

$$\begin{vmatrix} A_{11} & \cdots & A_{1i}+\lambda A_{1j} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & A_{mi}+\lambda A_{mj} & \cdots & A_{mm} \end{vmatrix} = \begin{vmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{vmatrix}.$$

证明 根据行列式的定义, 有

$$\begin{vmatrix} A_{11} & \cdots & A_{1i}+\lambda A_{1j} & \cdots & A_{1j} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ii}+\lambda A_{ij} & \cdots & A_{ij} & \cdots & A_{im} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & A_{mi}+\lambda A_{mj} & \cdots & A_{mj} & \cdots & A_{mm} \end{vmatrix}$$

$$\begin{aligned}
&= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots(A_{\sigma(i)i}+\lambda A_{\sigma(i)j})\cdots A_{\sigma(j)j}\cdots A_{\sigma(m)m} \\
&= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots A_{\sigma(i)i}\cdots A_{\sigma(j)j}\cdots A_{\sigma(m)m} \\
&\quad + \lambda \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots A_{\sigma(i)j}\cdots A_{\sigma(j)j}\cdots A_{\sigma(m)m} \\
&= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{\sigma(1)1}\cdots A_{\sigma(i)i}\cdots A_{\sigma(j)j}\cdots A_{\sigma(m)m}.
\end{aligned}$$

对于行的情况, 类似可证. □

性质 2.5. 对 $\forall \boldsymbol{A} \in \mathbb{R}^{m\times m}, \boldsymbol{B} \in \mathbb{R}^{n\times n}, \boldsymbol{U} \in \mathbb{R}^{m\times n}, \boldsymbol{V} \in \mathbb{R}^{n\times m}$, 有

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{U} \\ \boldsymbol{O} & \boldsymbol{B} \end{vmatrix} = \begin{vmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{V} & \boldsymbol{B} \end{vmatrix} = \det \boldsymbol{A} \det \boldsymbol{B},$$

此处 $\boldsymbol{O}$ 表示相应行数和列数的零矩阵. 上面的结果可以推广至一般情况, 即

$$\begin{aligned}\prod_{k=1}^{p} \det \boldsymbol{M}_{kk} &= \begin{vmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \cdots & \boldsymbol{M}_{1(p-1)} & \boldsymbol{M}_{1p} \\ \boldsymbol{O} & \boldsymbol{M}_{22} & \cdots & \boldsymbol{M}_{2(p-1)} & \boldsymbol{M}_{2p} \\ \vdots & \vdots & & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{M}_{(p-1)(p-1)} & \boldsymbol{M}_{(p-1)p} \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{M}_{pp} \end{vmatrix} \\ &= \begin{vmatrix} \boldsymbol{M}_{11} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & & \vdots & \vdots \\ \boldsymbol{M}_{(p-1)1} & \boldsymbol{M}_{(p-1)2} & \cdots & \boldsymbol{M}_{(p-1)(p-1)} & \boldsymbol{O} \\ \boldsymbol{M}_{p1} & \boldsymbol{M}_{p2} & \cdots & \boldsymbol{M}_{p(p-1)} & \boldsymbol{M}_{pp} \end{vmatrix},\end{aligned}$$

其中 $\boldsymbol{M}_{ij} \in \mathbb{R}^{m_i \times m_j}$, $1 \leqslant i, j \leqslant p$.

证明 记 $\boldsymbol{M} =: \begin{vmatrix} \boldsymbol{A} & \boldsymbol{U} \\ \boldsymbol{O} & \boldsymbol{B} \end{vmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$, 则 $\boldsymbol{M}$ 的行列式可以计算如下

$$\begin{aligned}\det \boldsymbol{M} &= \sum_{\gamma \in P_{m+n}} \operatorname{sgn} \gamma \left[M_{1\gamma(1)} \cdots M_{m\gamma(m)}\right] \cdot \left[M_{m+1\gamma(m+1)} \cdots M_{m+n\gamma(m+n)}\right] \\ &= \sum_{\sigma \in P_m} \sum_{\tau \in P_n} \operatorname{sgn} \sigma \operatorname{sgn} \tau \left[M_{1\sigma(1)} \cdots M_{m\sigma(m)}\right] \cdot \left[M_{m+1\tau(m+1)} \cdots M_{m+n\tau(m+n)}\right] \\ &= \left[\sum_{\sigma \in P_m} \operatorname{sgn} \sigma M_{1\sigma(1)} \cdots M_{m\sigma(m)}\right] \cdot \left[\sum_{\tau \in P_n} \operatorname{sgn} \tau M_{m+1\tau(m+1)} \cdots M_{m+n\tau(m+n)}\right] \\ &= \left[\sum_{\sigma \in P_m} \operatorname{sgn} \sigma A_{1\sigma(1)} \cdots A_{m\sigma(m)}\right] \cdot \left[\sum_{\tau \in P_n} \operatorname{sgn} \tau B_{m+1\tau(m+1)} \cdots B_{m+n\tau(m+n)}\right] \\ &= \det \boldsymbol{A} \det \boldsymbol{B}.\end{aligned}$$

另一种情况可以用类似的方法证明. 考虑一般情况, 相对应的结果也可以使用类似的方法或者利用归纳法来证明. □

藉此性质, 可得行列式降阶公式.

性质 2.6 (行列式降阶公式)**.** 设 $\boldsymbol{A} \in \mathbb{R}^{m\times m}, \boldsymbol{D} \in \mathbb{R}^{n\times n}, \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{n\times m}$, 则有

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = \begin{cases} \det \boldsymbol{A} \det(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}), & \text{当} \det \boldsymbol{A} \neq 0; \\ \det \boldsymbol{D} \det(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}), & \text{当} \det \boldsymbol{D} \neq 0. \end{cases}$$

证明 如果 $\det \boldsymbol{A} \neq 0$, 即 $\boldsymbol{A}$ 非奇异, 则有

$$\begin{pmatrix} \boldsymbol{I}_m & \boldsymbol{O} \\ -\boldsymbol{C}\boldsymbol{A}^{-1} & \boldsymbol{I}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{O} & \boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \end{pmatrix}.$$

另一方面, 也可以考虑

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_m & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{O} & \boldsymbol{I}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{C} & \boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \end{pmatrix}.$$

另一种情况可以用完全类似的方法证明. □

性质 2.7 (行列式展开定理)**.** 对任意方阵 $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, 它的行列式可以计算如下:

$$\begin{aligned} \det \boldsymbol{A} &= \sum_{j=1}^{m} (-1)^{i+j} A_{ij} \varDelta_{ij}, \quad \forall\, i = 1, \cdots, m \quad (\text{按行展开}) \\ &= \sum_{i=1}^{m} (-1)^{i+j} A_{ij} \varDelta_{ij}, \quad \forall\, j = 1, \cdots, m \quad (\text{按列展开}), \end{aligned}$$

其中 $\varDelta_{ij}$ 即为方阵 A 去掉第 i 行和第 j 列之后的方阵的行列式, 即

$$\varDelta_{ij} \triangleq \begin{vmatrix} A_{11} & \cdots & A_{1(j-1)} & A_{1(j+1)} & \cdots & A_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{(i-1)1} & \cdots & A_{(i-1)(j-1)} & A_{(i-1)(j+1)} & \cdots & A_{(i-1)m} \\ A_{(i+1)1} & \cdots & A_{(i+1)(j-1)} & A_{(i+1)(j+1)} & \cdots & A_{(i+1)m} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{m1} & \cdots & A_{m(j-1)} & A_{m(j+1)} & \cdots & A_{mm} \end{vmatrix},$$

$\varDelta_{ij}$ 称为元素 A_{ij} 的余子式, $(-1)^{i+j}\varDelta_{ij}$ 称为元素 A_{ij} 的代数余子式.

证明 根据行列式的定义, 有

$$\begin{aligned} \det \boldsymbol{A} = \sum_{\sigma\in P_m} & \operatorname{sgn}\sigma A_{1\sigma(1)} \cdots A_{i-1\sigma(i-1)} A_{i\sigma(i)} A_{i+1\sigma(i+1)} \cdots A_{j-1\sigma(j-1)} A_{j\sigma(j)} \\ & \cdot A_{j+1\sigma(j+1)} \cdots A_{m\sigma(m)}. \end{aligned}$$

显然, 对于任意满足 $\sigma(i) = j$ 的置换 $\sigma \in P_m$, 有

$$\sigma = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & m \\ \sigma(1) & \cdots & \sigma(i-1) & \sigma(i) & \sigma(i+1) & \cdots & \sigma(j-1) & \sigma(j) & \sigma(j+1) & \cdots & \sigma(m) \\ \hline \sigma(1) & \cdots & \sigma(i-1) & j & \sigma(i+1) & \cdots & \sigma(j-1) & \sigma(j) & \sigma(j+1) & \cdots & \sigma(m) \end{pmatrix},$$

都唯一存在另一个 $m-1$ 阶置换 $\tilde{\sigma} \in P_{m-1}$:

$$\tilde{\sigma} = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j+1 & \cdots & m \\ \tilde{\sigma}(1) & \cdots & \tilde{\sigma}(i-1) & \tilde{\sigma}(i) & \tilde{\sigma}(i+1) & \cdots & \tilde{\sigma}(j-1) & \tilde{\sigma}(j+1) & \cdots & \tilde{\sigma}(m) \end{pmatrix},$$

满足

$$\sigma = \left(\begin{array}{cccccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & m \\ \sigma(1) & \cdots & \sigma(i-1) & \sigma(i) & \sigma(i+1) & \cdots & \sigma(j-1) & \sigma(j) & \sigma(j+1) & \cdots & \sigma(m) \\ \hline \tilde{\sigma}(1) & \cdots & \tilde{\sigma}(i-1) & j & \tilde{\sigma}(i) & \cdots & \tilde{\sigma}(j-2) & \tilde{\sigma}(j-1) & \tilde{\sigma}(j+1) & \cdots & \tilde{\sigma}(m) \end{array} \right).$$

另有 $\operatorname{sgn}\sigma = (-1)^{j-i}\operatorname{sgn}\tilde{\sigma} = (-1)^{i+j}\operatorname{sgn}\tilde{\sigma}$. 所以有

$$\begin{aligned} \det \boldsymbol{A} &= \sum_{\sigma\in P_m} \operatorname{sgn}\sigma A_{1\sigma(1)}\cdots A_{i-1\sigma(i-1)}A_{i\sigma(i)}A_{i+1\sigma(i+1)} \\ &\quad \cdots A_{j-1\sigma(j-1)}A_{j\sigma(j)}A_{j+1\sigma(j+1)}\cdots A_{m\sigma(m)} \\ &= \sum_{j=1}^{m}(-1)^{i+j}A_{ij}\sum_{\tilde{\sigma}\in P_{m-1}} \operatorname{sgn}\tilde{\sigma} A_{1\tilde{\sigma}(1)}\cdots A_{i-1\tilde{\sigma}(i-1)}A_{i+1\tilde{\sigma}(i)} \\ &\quad \cdots A_{j-1\tilde{\sigma}(j-2)}A_{j\tilde{\sigma}(j-1)}A_{j+1\tilde{\sigma}(j+1)}\cdots A_{m\tilde{\sigma}(m)} \\ &= \sum_{j=1}^{m}(-1)^{i+j}A_{ij}\Delta_{ij}. \end{aligned}$$

类似可以证明 $\det \boldsymbol{A} = \sum_{i=1}^{m}(-1)^{i+j}A_{ij}\Delta_{ij}$.

□

3 建立路径

- 置换运算时非常初等的代数运算, 本讲稿归纳了其基本性质 (证明也十分简单), 后续的各种应用中都不离这些基本性质.
- 本讲稿应用部分, 给出了置换运算在线性代数中的作用.