高维微分学——向量值映照的极限

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1　知识要素

1.1　$\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_0 \in \mathbb{R}^n$ 的意义

定义 1.1 (向量值映照极限)**.** 向量值映照极限为向量值映照的一种局部行为, 记为 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_0 \in \mathbb{R}^n$, 可按以下二种叙述理解

- Cauchy 叙述

$$\forall\,\varepsilon > 0, \quad \exists\,\delta_\varepsilon > 0, \quad \text{满足：} |\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}_0|_{\mathbb{R}^n} < 0, \quad \forall \boldsymbol{x} \in \mathring{B}_{\delta_\varepsilon}(\boldsymbol{x}_0) \cap \mathscr{D}_f;$$

- Heine 叙述

$$\forall\,\{\boldsymbol{x}_n\}_{n\in\mathbb{N}} \subset \mathscr{D}_f \backslash \{\boldsymbol{x_0}\}, \quad \boldsymbol{x}_n \to \boldsymbol{x}_0 \in \mathbb{R}^m, \quad \text{有：} \boldsymbol{f}(\boldsymbol{x}_n) \to \boldsymbol{y}_0 \in \mathbb{R}^n.$$

定理 1.1 (向量值映照极限的等价性叙述)**.** Cauchy 叙述等价于 Heine 叙述.

证明　(1) 证明由 Cauchy 叙述得出 Heine 叙述. 考虑 $\forall\,\{\boldsymbol{x}_p\}_{p\in\mathbb{N}} \subset \mathbb{R}^m$, $\boldsymbol{x}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m$, 需证 $\boldsymbol{f}(\boldsymbol{x}_p) \to \boldsymbol{y}_0 \in \mathbb{R}^n$, 则按 Cauchy 叙述, 有

$$\forall\,\varepsilon > 0, \exists\,\delta_\varepsilon > 0, \text{成立}\, \boldsymbol{f}(\boldsymbol{x}) \in B_\varepsilon(\boldsymbol{y}_0), \forall\,\boldsymbol{x} \in \mathring{B}_{\delta_\varepsilon}(\boldsymbol{x}_0) \cap \mathscr{D}_x.$$

由 $\boldsymbol{x}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m$, 即

$$\exists\, N_{\delta_\varepsilon} \in \mathbb{N}, \text{成立}\, \boldsymbol{x}_p \in \mathring{B}_{\delta_\varepsilon}(\boldsymbol{x}_0) \cap \mathscr{D}_x, \forall\, p > N_{\delta_\varepsilon},$$

故有 $\boldsymbol{f}(\boldsymbol{x}_p) \in B_\varepsilon(\boldsymbol{y}_0), \forall\, p > N_{\delta_\varepsilon}$, 亦即 $\boldsymbol{f}(\boldsymbol{x}_p) \to \boldsymbol{y}_0 \in \mathbb{R}^n$.

(2) 证明由 Heine 叙述得出 Cauchy 叙述. 利用反证法, 假设 Cauchy 叙述不成立, 即有

$$\exists\,\varepsilon_* > 0, \forall\,\delta > 0, \exists\,\boldsymbol{x}_\delta \in \mathring{B}_\delta(\boldsymbol{x}_0) \cap \mathscr{D}_x \text{ 满足 } \boldsymbol{f}(\boldsymbol{x}_\delta) \notin B_{\varepsilon_*}(\boldsymbol{y}_0),$$

取 $\delta_p = \frac{1}{p}$, 则 $\exists\, x_p \in \mathring{B}_{\delta_p}(\boldsymbol{x}_0) \cap \mathscr{D}_x$ 满足 $\boldsymbol{f}(\boldsymbol{x}_p) \notin B_{\varepsilon_*}(\boldsymbol{y}_0)$. 现有 $\mathscr{D}_x \backslash \{\boldsymbol{x}_0\} \ni \boldsymbol{x}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m$, 按 Heine 叙述有 $\boldsymbol{f}(\boldsymbol{x}_p) \to \boldsymbol{y}_0 \in \mathbb{R}^n$, 故产生矛盾. □

定理 1.2 (映照极限的 Cauchy 收敛原理)**.** 已有 $(\mathbb{R}^n, |\cdot|_{\mathbb{R}^n})$ 为完备的赋范线性空间, 则有

$$\exists \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_0 \in \mathbb{R}^n$$

等价于

$$\forall\, \varepsilon > 0, \quad \exists\, \delta_\varepsilon > 0, \quad \text{满足 } |\boldsymbol{f}(\tilde{\boldsymbol{x}}) - \boldsymbol{f}(\hat{\boldsymbol{x}})|_{\mathbb{R}^n} < \varepsilon, \quad \forall\, \tilde{\boldsymbol{x}}, \hat{\boldsymbol{x}} \in \overset{\circ}{B}_{\delta_\varepsilon}(\boldsymbol{x}_0) \cap \mathscr{D}_f$$

证明 (1) 充分性. 现有 $\exists \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_0 \in \mathbb{R}^n$, 按 Cauchy 叙述有

$$\forall\, \varepsilon > 0, \exists\, \delta_\varepsilon > 0, \text{成立}\, |\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}_0|_{\mathbb{R}^n} < \varepsilon, \forall\, \boldsymbol{x} \in \overset{\circ}{B}_{\delta_\varepsilon}(\boldsymbol{x}_0) \cap \mathscr{D}_f,$$

故有对 $\forall\, \widetilde{\boldsymbol{x}}, \widehat{\boldsymbol{x}} \in \overset{\circ}{B}_{\delta_\varepsilon}(\boldsymbol{x}_0) \cap \mathscr{D}_f$, 有

$$|\boldsymbol{f}(\widetilde{\boldsymbol{x}}) - \boldsymbol{f}(\widehat{\boldsymbol{x}})|_{\mathbb{R}^n} \leqslant |\boldsymbol{f}(\widetilde{\boldsymbol{x}}) - \boldsymbol{y}_0|_{\mathbb{R}^n} + |\boldsymbol{f}(\widehat{\boldsymbol{x}}) - \boldsymbol{y}_0|_{\mathbb{R}^n} < \varepsilon + \varepsilon = 2\varepsilon.$$

(2) 必要性. 现有 $\forall\, \varepsilon > 0, \exists\, \delta_\varepsilon > 0$ 成立 $|\boldsymbol{f}(\widetilde{\boldsymbol{x}}) - \boldsymbol{f}(\widehat{\boldsymbol{x}})|_{\mathbb{R}^n} < \delta_\varepsilon, \forall\, \widetilde{\boldsymbol{x}}, \widehat{\boldsymbol{x}} \in \overset{\circ}{B}_{\delta_\varepsilon}(x_0) \cap \mathscr{D}_f$. 考虑 $\forall\, \{x_p\}_{\mathbb{R}^m} \subset \mathscr{D}_x \backslash \{\boldsymbol{x}_0\}, \boldsymbol{x}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m$, 则有

$$\exists\, N_{\delta_\varepsilon} \in \mathbb{N}, \text{成立}\, 0 < |\boldsymbol{x}_p - \boldsymbol{x}_0|_{\mathbb{R}^m} < \delta_\varepsilon, \forall\, p > N_{\delta_\varepsilon},$$

故有

$$|\boldsymbol{f}(\boldsymbol{x}_p) - \boldsymbol{f}(\boldsymbol{x}_q)|_{\mathbb{R}^n} < \varepsilon, \forall\, p, q > N_{\delta_\varepsilon},$$

亦即 $\{\boldsymbol{f}(\boldsymbol{x}_p)\}_{p\in\mathbb{N}} \subset \mathbb{R}^n$ 为基本点列. 再由 $(\mathbb{R}^n, |\cdot|_{\mathbb{R}^n})$ 为完备的赋范线性空间, 因此 $\{\boldsymbol{f}(\boldsymbol{x}_p)\}_{p\in\mathbb{N}} \subset \mathbb{R}^n$ 收敛. 再考虑

$$\forall\, \{\widetilde{\boldsymbol{x}}_p\} \subset \mathscr{D}_f \backslash \{\boldsymbol{x}_0\}, \widetilde{\boldsymbol{x}}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m \text{ 有 } \boldsymbol{f}(\widetilde{\boldsymbol{x}}_p) \to \widetilde{\boldsymbol{y}}_0 \in \mathbb{R}^n,$$
$$\forall\, \{\widehat{\boldsymbol{x}}_p\} \subset \mathscr{D}_f \backslash \{\boldsymbol{x}_0\}, \widehat{\boldsymbol{x}}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m \text{ 有 } \boldsymbol{f}(\widehat{\boldsymbol{x}}_p) \to \widehat{\boldsymbol{y}}_0 \in \mathbb{R}^n,$$

需证 $\widetilde{\boldsymbol{y}}_0 = \widehat{\boldsymbol{y}}_0 \in \mathbb{R}^n$. 就此作

$$\overline{\boldsymbol{x}}_p = \begin{cases} \widetilde{\boldsymbol{x}}_{2k}, & p = 2k, \\ \widehat{\boldsymbol{x}}_{2k-1}, & p = 2k-1, \end{cases}$$

有 $\{\overline{\boldsymbol{x}}_p\} \subset \mathscr{D}_f \backslash \{\boldsymbol{x}_0\}$, 满足 $\overline{\boldsymbol{x}}_p \to \boldsymbol{x}_0 \in \mathbb{R}^m$. 故有 $\boldsymbol{f}(\overline{\boldsymbol{x}}_p) \to \overline{\boldsymbol{y}}_0 \in \mathbb{R}^n$. 由于收敛点列的所有子列均收敛且极限相同以及点列极限存在的唯一性, 故有 $\widetilde{\boldsymbol{y}}_0 = \widehat{\boldsymbol{y}}_0 = \overline{\boldsymbol{y}}_0$. 按映照极限的 Heine 叙述, 即有 $\exists \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_0 \in \mathbb{R}^n$. □

定义 1.2 (向量值映照的连续性)**.** 如果 $\exists \lim_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}_0) \in \mathbb{R}^n$, 则称 $\boldsymbol{f}(\boldsymbol{x}_0)$ 在 $\boldsymbol{x}_0 \in \mathscr{D}_f$ 点连续.

1.2 向量值映照极限的分析性质

性质 1.3 (基本分析性质)**.** 类比于一元函数极限, 向量值映照极限亦具有如下基本性质.

1. 极限存在唯一性

$$\text{如有}\begin{cases}\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{y}_0\in\mathbb{R}^n\\ \exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{z}_0\in\mathbb{R}^n\end{cases},\quad\text{则有 }\boldsymbol{y}_0=\boldsymbol{z}_0.$$

2. 局部有界性　如果 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m}\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{y}_0\in\mathbb{R}^n$, 则有

$$\exists M,\ \delta_M\in\mathbb{R}^+,\ s.t.\ |\boldsymbol{f}(\boldsymbol{x})|_{\mathbb{R}^m}\leq M,\quad \forall\boldsymbol{x}\in\mathring{B}_{\delta_M}(\boldsymbol{x}_0)\cap\mathscr{D}_f$$

3. 多元函数极限的保号性　保号性具有二个方面: (1) 如果 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m}f(\boldsymbol{x})>0$, 则 $\exists\lambda\in\mathbb{R}^+$, 有 $f(x)>0, \forall\boldsymbol{x}\in\mathring{B}_\lambda(\boldsymbol{x}_0)\cap\mathscr{D}_f$; (2) 如有 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m}f(\boldsymbol{x})\in\mathbb{R}$, 以及 $\exists\lambda\in\mathbb{R}^+$, 有 $f(x)>(\text{或}\geq)\,0, \forall\boldsymbol{x}\in\mathring{B}_\lambda(\boldsymbol{x}_0)\cap\mathscr{D}_f$, 则有 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x_0}\in\mathbb{R}^m}f(\boldsymbol{x})>0$.

4. 多元函数极限的夹逼性　设多元函数 $\phi(\boldsymbol{x})$, $\psi(\boldsymbol{x})$ 和 $\theta(\boldsymbol{x})$ 具有共同的定义域 $\mathscr{D}_x\subset\mathbb{R}^m$,

$$\text{如有}\begin{cases}\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\phi(\boldsymbol{x})=\lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\psi(\boldsymbol{x})=y_0\in\mathbb{R}\\ \text{夹逼性条件}\quad \phi(\boldsymbol{x})\leq\theta(\boldsymbol{x})\leq\psi(\boldsymbol{x}),\quad\forall\boldsymbol{x}\in\mathring{B}_\lambda(\boldsymbol{x}_0)\cap\mathscr{D}_x\end{cases},$$

则有 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\theta(\boldsymbol{x})=y_0$.

定理 1.4 (复合映照极限定理). 如有

$$\begin{cases}\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\boldsymbol{\theta}(\boldsymbol{x})=\boldsymbol{y}_0\in\mathbb{R}^n,\\ \exists \lim\limits_{\boldsymbol{y}\to\boldsymbol{y}_0\in\mathbb{R}^n}\boldsymbol{\Theta}(\boldsymbol{y})=\boldsymbol{z}_0\in\mathbb{R}^l,\end{cases}$$

且满足“非接触性条件”①: $\exists\lambda>0$, 有

$$\theta(\mathring{B}_\lambda(x_0)\cap\mathscr{D}_\theta)\subset\mathscr{D}_\Theta\backslash\{\boldsymbol{y}_0\},$$

则有

1. 存在局部复合, 即有

$$\boldsymbol{\Theta}\circ\boldsymbol{\theta}(\boldsymbol{x}):\mathring{B}_\lambda(\boldsymbol{x}_0)\cap\mathscr{D}_\theta\ni\boldsymbol{x}\mapsto\boldsymbol{\Theta}\circ\boldsymbol{\theta}(\boldsymbol{x})\equiv\boldsymbol{\Theta}(\boldsymbol{\theta}(\boldsymbol{x}));$$

2. $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\boldsymbol{\Theta}\circ\boldsymbol{\theta}(\boldsymbol{x})=\boldsymbol{z}_0=\lim\limits_{\boldsymbol{y}\to\boldsymbol{y}_0\in\mathbb{R}^n}\boldsymbol{\Theta}(\boldsymbol{y})\in\mathbb{R}^l$.

证明 (1) 按非接触性条件 $\boldsymbol{\theta}(\mathring{B}_\lambda(\boldsymbol{x}_0)\cap\mathscr{D}_\theta)\subset\mathscr{D}_\Theta\backslash\{\boldsymbol{y}_0\}$, 显然成立.
(2) 利用 Heine 叙述, 考虑 $\forall\{\boldsymbol{x}_p\}\subset\mathring{B}_\lambda(\boldsymbol{x}_0)\cap\mathscr{D}_\theta$, $\boldsymbol{x}_p\to\boldsymbol{x}_0\in\mathbb{R}^m$.
由 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m}\boldsymbol{\theta}(\boldsymbol{x})=\boldsymbol{y}_0\in\mathbb{R}^n$ 的 Heine 叙述, 以及非接触性条件, 有

$$\mathscr{D}_\Theta\backslash\{\boldsymbol{y}_0\}\ni\boldsymbol{\theta}(\boldsymbol{x}_p)\to\boldsymbol{y}_0\in\mathbb{R}^n.$$

① “非接触性” 指, 当 $\boldsymbol{x}\neq\boldsymbol{x}_0\in\mathbb{R}^m$, 有 $\boldsymbol{\theta}(\boldsymbol{x})\neq\boldsymbol{y}_0\in\mathbb{R}^n$.

又由 $\exists \lim\limits_{\boldsymbol{y}\to\boldsymbol{y}_0\in\mathbb{R}^n} \boldsymbol{\Theta}(\boldsymbol{y}) = \boldsymbol{z}_0 \in \mathbb{R}^l$ 的 Heine 叙述, 有

$$\boldsymbol{\Theta}(\boldsymbol{\theta}(\boldsymbol{x}_p)) = \Theta \circ \theta(\boldsymbol{x}_p) \to \boldsymbol{z}_0 \in \mathbb{R}^l.$$

综上, 有 $\exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \boldsymbol{\Theta} \circ \boldsymbol{\theta}(\boldsymbol{x}) = \boldsymbol{z}_0 \in \mathbb{R}^l$. □

需指出, 按连续性的 Heine 叙述, 如有

$$\exists \lim_{\boldsymbol{y}\to\boldsymbol{y}_0\in\mathbb{R}^n} \boldsymbol{\Theta}(\boldsymbol{y}) = \boldsymbol{\Theta}(\boldsymbol{y}_0) \in \mathbb{R}^l,$$

则上述定理中 "非接触性条件" 可改为 "可接触性条件"

$$\exists \lambda > 0,\ 有\ \theta(\mathring{B}_\lambda(x_0) \cap \mathscr{D}_\theta) \subset \mathscr{D}_\Theta.$$

1.3 向量值映照极限的计算

性质 1.5 (存在向量值映照极限等价于存在各分量极限)**.**

$$\exists \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_0 \in \mathbb{R}^n \quad \Leftrightarrow \quad \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} f^\alpha(\boldsymbol{x}) = y_0^\alpha \in \mathbb{R}$$

证明 仅需到基本关系式

$$\left|a^j - b^j\right| \leqslant |\boldsymbol{a}-\boldsymbol{b}|_{\mathbb{R}^m} \leqslant \sum_{i=1}^m \left|a^i - b^i\right|, \quad j = 1, \cdots, m.$$

□

本性质表明, 计算向量值映照的极限可以归结为计算其分量 (多元函数) 的极限.

性质 1.6 (多元函数极限的四则运算)**.** 设多元函数 $f(\boldsymbol{x})$ 和 $g(\boldsymbol{x})$ 具有相同的定义域 $\mathscr{D}_x$. 如有

$$\exists \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} f(\boldsymbol{x}) = A \in \mathbb{R}, \quad \exists \lim_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} g(\boldsymbol{x}) = B \in \mathbb{R},$$

则有

$$\begin{cases} \exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} (\alpha f + \beta g)(\boldsymbol{x}) = \alpha A + \beta B \in \mathbb{R} \\ \exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} (fg)(\boldsymbol{x}) = AB \in \mathbb{R} \\ \exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \left(\dfrac{f}{g}\right)(\boldsymbol{x}) = \dfrac{A}{B} \in \mathbb{R}, \quad 此处\ B \neq 0 \end{cases}$$

需指出, 上述性质为计算函数极限的充分性方法, 亦即可以把函数视成二个函数的线性组合、乘积或除法, 如果相应的函数都具有极限, 则原函数的极限为相应极限的线性组合、乘积或比值.

2 应用事例

3 拓广深化

值得指出, 本知识点所述的向量值映照的极限定义, Cauchy 叙述与 Heine 叙述的等价性证明以及映照极限的 Cauchy 收敛原理, 都可以 "逐字逐句" 地用于一般赋范线性空间 $(X, |\cdot|_X)$ 与 $(Y, |\cdot|_Y)$ 之间的映照

$$f(x): X \supset \mathscr{D}_f \ni x \to f(x) \in Y$$

的极限, 仅需以 $(\mathbb{R}^m, |\cdot|_{\mathbb{R}^m})$ 替代 $(X, |\cdot|_X)$, $(\mathbb{R}^n, |\cdot|_{\mathbb{R}^n})$ 替代 $(Y, |\cdot|_Y)$.

4 建立路径